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Affine Groups and levels of Stable Rationality

Thm. (*) : If V is an indecomposable representation of $ASL_n(\mathbb{C}) = SL_n(\mathbb{C}) \ltimes \mathbb{C}^n$ of sufficiently large dimension, then $V/ASL_n(\mathbb{C})$ is rational.

Thm. 1 : For any n and generically free irreducible $SL_n(\mathbb{C})$-representation V, $V/SL_n(\mathbb{C})$ is stably rational of level n (i.e. $V/SL_n(\mathbb{C}) \times \mathbb{P}^n$ is rational). If a central quotient $SL_n(\mathbb{C})/(\mathbb{Z}/m\mathbb{Z})$ acts generically freely in V, $(V + \mathbb{C}^n)/SL_n(\mathbb{C})$ is stably rational of level n.

More precisely we prove: if P is the stabilizer of $v \in \mathbb{C}^n$, $v \neq 0$, inside $SL_n(\mathbb{C})$, then V/P is rational.

Thm. 2 : Let V be an irreducible repr. of $Sp_{2n}(\mathbb{C})$ s.t. a central quotient acts gen. freely in V. Then $V/\tilde{P}$ is rational (for $n \geq 4$) where

$\tilde{P}$ is the stabilizer of $v \in \mathbb{C}^{2n} \setminus \{0\}$ inside $Sp_{2n}(\mathbb{C})$.

<u>Thm. 3</u>: a) V an irred. representation of $O_N(\mathbb{C})$ with generic stabilizer contained in the centre. Then $V/O_N(\mathbb{C})$ is stably rational of level N if V is already generically free for $O_N(\mathbb{C})$; otherwise, $(V + \mathbb{C}^N)/O_N(\mathbb{C})$ is stably rational of level N.

b) V an irred. representation of $SO_N(\mathbb{C})$ with gen. stabilizer contained in the centre. Then:

- if $N = 2n$ and $V = V(c\omega_{n-1})$ or $= V(c\omega_n)$,

then $V/SO_N(\mathbb{C})$ (in the case where V is gen. free) or $(V + \mathbb{C}^N)/SO_N(\mathbb{C})$ (otherwise) is stably rational of level $2N$.

- in all other cases, $V/SO_N(\mathbb{C})$ is stably rational of level N (V gen. free) or $(V + \mathbb{C}^N)/SO_N(\mathbb{C})$ is stably rational of level $\cancel{2}N$.

<u>Thm. 4</u>: If V is a gen. free irreducible representation of G_2, then V/G_2 is stably rational of level 7.

Illustrate part of the proof by showing the statement for $V = \Lambda^3 \mathbb{C}^n$, $G = SL_n(\mathbb{C})$.

As G is special, stable rat. of level $n^2-1 = \dim G$ is known.

- $(V + \mathbb{C}^n)/SL_n(\mathbb{C}) \approx V/P$.

- As P-module, V is an extension

$$0 \to \Lambda^2 \mathbb{C}^{n-1} \to V \to \Lambda^3 \mathbb{C}^{n-1} \to 0$$

$\dim \Lambda^2 \mathbb{C}^{n-1} < \dim SL_{n-1}(\mathbb{C})$, so

we try to prove the statement by induction on n.

- note: $\Lambda^3 \mathbb{C}^n$ is not a representation of the type considered in Thm. 1 (namely one with a nontrivial stabilizer in general position) for $n \le 9$. Thus as induction hypothesis we need:

(#) $\Lambda^3 \mathbb{C}^{10} / SL_{10}(\mathbb{C})$ is stably rational of level 10.

The induction step is as follows: for $n \geq 11$, V is an extension

$$0 \to \underbrace{\Lambda^2 \mathbb{C}^{n-1}}_{= S} \to V \to \underbrace{\Lambda^3 \mathbb{C}^{n-1}}_{= Q} \to 0$$

and a central quotient of $SL_n(\mathbb{C})$ acts generically freely in $\Lambda^3 \mathbb{C}^{n-1}$. Moreover, inductively, we know that certainly $(\Lambda^3 \mathbb{C}^{n-1} + \mathbb{C}^{n-1}) / SL_{n-1}(\mathbb{C})$ is stably rational of level $n-1$.

But since $2m-1 \geq m-1$, we obtain that

S_1 is generically a vector bundle over S_2 of

rank at least m. So the function field of

V/P is obtained from that of S_2 by adjoining

at least $m+1$ indeterminates, so rationality

$(\mathbb{C}^*)$

of V/P follows from stable rationality of level

m of $(Q + \mathbb{C}^m)/SL_m(\mathbb{C})$.

So it suffices to prove:

- $\Lambda^3 \mathbb{C}^{10}$ is stably rational of level 10.

Follows from

$$\Lambda^3 \mathbb{C}^{10} / P' \text{ is rational}$$

where: P' = stabilizer of a generic point in $\underline{\underline{(\mathbb{C}^{10})^\vee}}$.

We have $\quad (m := n-1)$

$$\dim \Lambda^2 \mathbb{C}^m = \frac{m(m-1)}{2} \geqslant 3m \quad \text{for}$$

$m \geqslant 7$.

<u>Claim</u>: This implies rationality for V/P.

In fact, $V/_P$ is a $\mathbb{C}^*$-bundle over some (maybe nontrivial) Severi-Brauer variety $\mathcal{S}_1$ over $Q/SL_m(\mathbb{C})$ and the fibre dimension of $\mathcal{S}_1$ is greatther or equal

$$\dim \Lambda^2 \mathbb{C}^{n-1} - 1 - \underset{\substack{\uparrow \\ \text{dimension of} \\ \text{unipotent radical of } P.}}{m} \geqslant 2m-1$$

But $\mathcal{S}_1$ is stably equivalent to $\mathcal{S}_2$ obtained by dividing out homotheties in the fibres of

$$\left(Q + \mathbb{C}^m\right)/SL_m(\mathbb{C}) \longrightarrow Q/SL_m(\mathbb{C}),$$

(one pulls back to a vector bundle over the other).

As P'-representation, V is an extension

$$0 \to \Lambda^3 \mathbb{C}^9 \to V \to \Lambda^2 \mathbb{C}^9 \to 0 .$$

Stabilizer in general position in $\Lambda^2 \mathbb{C}^9$ inside $SL_9(\mathbb{C})$ is $Sp_8(\mathbb{C}) \ltimes \mathbb{C}^8$.

$\rightsquigarrow$ suffices to show rationality of the quotient $W / Sp_8(\mathbb{C}) \ltimes \mathbb{C}^8$ where

$$0 \to (\Lambda^3 \mathbb{C}^8)_0 \to W \to (\Lambda^2 \mathbb{C}^8)_0 \to 0 .$$

traceless tensors

(note : $\Lambda^3 \mathbb{C}^9 = \Lambda^3 (\mathbb{C}^8 + \mathbb{C})$
$= \Lambda^3 \mathbb{C}^8 + \Lambda^2 \mathbb{C}^8$
$= (\Lambda^3 \mathbb{C}^8)_0 + \mathbb{C}^8 + (\Lambda^2 \mathbb{C}^8)_0 + \mathbb{C}$).

Now in $(\Lambda^2 \mathbb{C}^8)_0$ the generic stabilizer is

$H = \prod_{i=1}^{4} SL_2(\mathbb{C})$ with normalizer

$$N(H) = S_4 \ltimes (SL_2(\mathbb{C}))^4 .$$

$$\underset{\text{as } N(H)\text{-repr.}}{\mathbb{C}^8 \simeq} R_1 + R_2 + R_3 + R_4 \ , \ R_i \simeq \mathbb{C}^2$$

S_4 permuting the R_i.

$$\underbrace{(\Lambda^2 \mathbb{C}^8)_0^H}_{H\text{-invariants}} \simeq \mathbb{C}^3 \leftarrow \text{standard repr. of } S_4 .$$

Decomposing $(\Lambda^3 \mathbb{C}^8)_0$ as $N(H)$-module we find that we only have to prove stable rationality of

$$\Big(\sum_{1 \le i < j < k \le 4} R_i \otimes R_j \otimes R_k \Big) / N(H)$$

of level $\le 11 = \dim (R_1 + \ldots + R_4) + \underset{\dim (\Lambda^2 \mathbb{C}^8)_0^H}{\overset{\shortparallel}{3}}$.

We prove indeed rationality of

$$\left(\sum_{1 \le i < j < k \le 4} R_i \otimes R_j \otimes R_k \right) + \left(R_1 + R_2 + R_3 + R_4 \right)$$

mod. $N(H)$ easily by taking again a stabilizer of a generic pt. in $R_1 + R_2 + R_3 + R_4$ which is

$$\Gamma = S_4 \ltimes \left(G_a \times G_a \times G_a \times G_a \right)$$

and noting that

$$\left(\sum_{1 \le i < j < k \le 4} R_i \otimes R_j \otimes R_k \right) / \Gamma \quad \text{is}$$

rational.

So, the principle is that <u>all</u> representations of a given group, small and large, have to be considered simultaneously here, the small ones being important since they are the induction base.