Special Galois coverings and the irreducibility of certain spaces of coverings of curves, with applications to moduli of curves.

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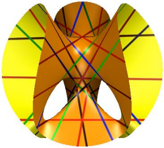

Outline

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From the point of view of moduli it is very important to understand flat families of such covers.
We shall concentrate here only on special cases where

- G is cyclic
- G is abelian
- G is dihedral.
- Y is a stable curve
- Y is a factorial variety of higher dimension.

From fields to schemes

Assume that X and Y are normal varieties: then, once Y is fixed, the Galois covering

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is completely determined by the field extension

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However, making the normalization process explicit is important for two reasons:

- to calculate the invariants of X
- to determine the direct image of basic sheaves on X $(f_*(\mathcal{O}_X), f_*(\Omega^1_X), f_*(\Theta_X),..)$
- to write explicit flat families of such coverings.

The interplay of algebra and topology.

An important transcendental method is the
(Grauert- Remmert's version of Riemann's existence theorem)

Theorem

There is a bijection between
1) the set isomorphism classes of finite coverings $f: X \to Y$ *between normal varieties* X, Y *and*
2) classes of maximal connected unramified coverings $V \to U$*, where* U *is a Zariski open set in* Y*.*
f *is Galois if and only if* $V \to U$ *is associated to the class of a surjection* $\phi : \pi_1(U) \to G$ *(class for the action of* $Aut(G)$*).*

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If $U = Y$, then one says that f is étale (or unramified).
The complement $\mathcal{B} := Y \setminus U$ of the maximal set U over which f is unramified is called the branch locus of f.

Very simple covers

Let us illustrate the simplest examples.

Definition

A simple cyclic cover of a scheme Y is the hypersurface X of a geometric line bundle $\mathbb{L}$ over Y which is defined by

$$X = \{(y,z) | z^d = F(y)\}, \ F \in H^0(Y, \mathcal{O}_Y(dL)).$$

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Here, L is the Cartier divisor whose associated sheaf of sections $\mathcal{O}_Y(L)$ is the sheaf of regular sections of $\mathbb{L} \to Y$.
The branch locus is the Cartier divisor $D := div(F)$, i.e.,
$\mathcal{B} = \{y | F(y) = 0\}$.
The cyclic Galois group G

$$G \cong \mathbb{Z}/d \cong \mu_d := \{\zeta \in \mathbb{C} | \zeta^d = 1\}$$

operates by $z \mapsto \zeta z$.

Very simple dihedral covers

The following new construction was discovered in the course of a very general investigation of dihedral covers, in progress together with Fabio Perroni.

Definition

A very simple dihedral cover of a scheme Y is the subscheme X of a geometric line bundle $\mathbb{L} \oplus \mathbb{L}$ over Y defined by

$$u^n + v^n = 2a, \ a \in H^0(Y, \mathcal{O}_Y(nL)),$$

$$uv = F, F \in H^0(Y, \mathcal{O}_Y(2L)).$$

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The action of the Galois group D_n on X is generated by the cyclic action

$$(u, v) \mapsto (\zeta u, \zeta^{-1} v), \zeta \in \mu_n$$

and by the reflection $(u, v) \mapsto (v, u)$.

Very simple dihedral covers

Here the branch locus is the Cartier divisor $D := div(a^2 - F^n)$, i.e., $\mathcal{B} = \{y | a^2(y) - F^n(y) = 0\}$.
In fact, the covering factors as the double covering provided by (set $w := v^n - u^n$)

$$w^2 = -4(F^n(y) - a^2(y)),$$

followed by the cyclic covering

$$2v^n = 2a + w$$

which is only ramified in codimension 2 (in the points where $a(y) = F(y) = 0$).

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In particular, the fundamental group of $Y \setminus \mathcal{B}$ is non abelian, having a surjection onto D_n.
This shows some advantage of the algebraic set up, where one does not have to preliminarily compute $\pi_1(Y \setminus \mathcal{B})$

The branching data

Returning to the general situation, assume that Y is smooth.
We have a surjection $\mu : \pi_1(Y \setminus \mathcal{B}) \to G$, called the monodromy.
By purity of the branch locus, $\mathcal{B}$ is a divisor, and let D_α be an irreducible component of $\mathcal{B}$.
Let $\gamma_\alpha \in \pi_1(Y \setminus \mathcal{B})$ be a geometrical loop going simply around D_α, so that $\mu(\gamma_\alpha) := g_\alpha \in G$.
Since a different choice of γ_α remains in the same conjugacy class, we attach to D_α a conjugacy class $\mathcal{C}_\alpha := [g_\alpha]$ inside G, called the local monodromy.
Ordering the components D_α, we get a sequence of conjugacy classes $\mathcal{C}_\alpha$ of G, and one defines

Definition

The branching datum of $X \to Y$ is the sequence $(\mathcal{C}_\alpha)$, $D_\alpha \leq \mathcal{B}$ of conjugacy classes of G, taken up to permutation and up to the action of $Aut(G)$.

The branching data

Assume now that Y is a smooth curve.
For each point $p_\alpha \in \mathcal{B}$ we get a conjugacy class $\mathcal{C}_\alpha := [g_\alpha]$ inside G.
In this case the branching datum of $X \to Y$ is the datum of a function defined on the set of conjugacy classes of G:
$k(\mathcal{C}) := card\{\alpha | \mathcal{C}_\alpha = \mathcal{C}\}$, taken up to the action of $Aut(G)$.
For instance, if G is cyclic, $G \cong \mathbb{Z}/d$, then we have a sequence $k_1, \ldots, k_{d-1}$ taken up to the action of $Aut(G) = (\mathbb{Z}/d)^*$ acting on the set of indices.

Outline

Cyclic coverings with Y factorial.

Assume now that Y is a factorial variety and that G is cyclic, $G \cong \mathbb{Z}/d$.
Then $\mathbb{C}(X)$ is a cyclic Galois extension of $\mathbb{C}(Y)$.
Denote by $G \cong \mu_d := \{\zeta \in \mathbb{C} | \zeta^d = 1\}$ the Galois group, by $\mathbb{Z}/d$ the group of characters

$$\mathbb{Z}/d \cong \{\chi | \exists m \in \mathbb{Z}/d, \chi(\zeta) = \zeta^m\}.$$

The extension is given by

$$\mathbb{C}(X) = \mathbb{C}(Y)(w), w^d = f(y) \in \mathbb{C}(Y),$$

where w is an eigenvector for a primitive character.
Since Y is factorial, f admits a unique prime factorization as a fraction of pairwise relatively prime sections of line bundles,

$$w^d = \frac{\prod_i \sigma_i^{n_i}}{\prod_j \tau_j^{m_j}}.$$

Cyclic coverings with Y factorial.

Write now

$$n_i = N_i + dn'_i, \quad r_j = -M_j + dm'_j$$

with $0 \leq N_i, M_j \leq d-1$ and set

$$z := w \cdot \prod_i \sigma_i^{-n'_i} \prod_j \tau_j^{m'_j}.$$

Whence z is a rational section of a line bundle L on Y and we have

$$z^d = \prod_i \sigma_i^{N_i} \prod_j \tau_j^{M_j}.$$

We put together the prime factors which appear with the same exponent and write:

$$z^d = \prod_{i=1}^{d-1} \delta_i^i.$$

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Since X is normal each factor δ_i is reduced, but not irreducible, and corresponds to a Cartier divisor that we shall denote D_i. D_i is exactly the divisorial part of the branch locus $D := \sum_i D_i$ where the local monodromy is exactly the element $i \in \mathbb{Z}/d$. We have then the basic linear equivalence

$$(*)\ dL \equiv \sum_i iD_i.$$

Cyclic coverings with Y factorial.

The following theorem is a special case of the structure theorem for Abelian coverings due to Pardini.

Theorem

i) Given a factorial variety Y, the datum of a pair (X, γ) where X is a normal scheme and γ is an automorphism of order d such that $X/\gamma \cong Y$, is equivalent to the datum of reduced effective divisors $D_1, \dots D_{d-1}$ without common components, and of a divisor class L such that

$$(*)\ dL \equiv \sum_i iD_i.$$

Cyclic coverings with Y factorial.

In the above theorem X is a variety if , setting $m := G.C.D.\{i | D_i \neq 0\}$, either
(**) $m = 1$ or, setting $d = mn$, the divisor class

$$(***)\ L' := \frac{d}{m}L - \sum_i \frac{i}{m} D_i$$

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The scheme structure of X was explicitly given, in the more general case of abelian coverings by Pardini, who also calculated explicitly the basic invariants of the covering.
The explicit description of the scheme structure can then be used to describe flat families of cyclic coverings. The main obstruction to a smooth family of deformation is that deforming Y the space $H^0(\mathcal{O}_Y(D))$ may become smaller, or even vanish.
This fact was used by Fantechi and Pardini to construct moduli spaces whch are not Cohen Macaulay.

Cyclic coverings of curves.

Theorem

The pairs (C, G) where C is a complex projective curve of genus $g \geq 2$, and G is a finite cyclic group of order d acting faithfully on C with a given branching datum $[(k_1, \ldots k_{d-1})]$ are parametrized by a connected complex manifold $\mathcal{T}_{g;d,[(k_1,\ldots k_{d-1})]}$ of dimension $3(h-1)+k$, where $k := \sum_i k_i$ and h is the genus of the quotient curve $C' := C/G$.
The image $\mathfrak{M}_{g;d,[(k_1,\ldots k_{d-1})]}$ of $\mathcal{T}_{g;d,[(k_1,\ldots k_{d-1})]}$ inside the moduli space $\mathfrak{M}_g$ is a closed subset of the same dimension.

Cyclic coverings of curves.

The previous result was proven by Cornalba in the particular case where d is a prime number.
Barbara Fantechi posed the problem to show the result in full generality.
Observe that h is determined by the Hurwitz formula.
The proof shows also that, if $3(h-1)+k$ is strictly positive, then the general curve C inside $\mathfrak{M}_{g;d,[(k_1,\dots k_{d-1})]}$ has G as a maximal cyclic group of automorphisms.

Cyclic coverings of curves.

Idea of the proof:

Such pairs (C, γ) are determined by the following data: a curve C' of genus h, divisors D_i for all $i \in \mathbb{Z}/d$ and a surjective homomorphism $\psi : H_1(C' \setminus D, \mathbb{Z}) \to \mathbb{Z}/d$ such that the image of a small circle around a point $p \in D_i$ maps to the class of i in $\mathbb{Z}/d$.

We have the homology exact sequence, where we write $D_1 = p_1 + \cdots + p_{k_1}$, $D_2 = p_{k_1+1} + \dots p_{k_1+k_2}, \dots,$

$$(**) 0 \to A := (\oplus_j \mathbb{Z} p_j)/\mathbb{Z}(\sum_j p_j) \to H_1(C' \setminus D, \mathbb{Z}) \to H_1(C', \mathbb{Z}) \cong \mathbb{Z}^{2h} \to$$

which admits several splittings: we choose a splitting such that $\mathbb{Z}^{2h}$ maps onto $\mathbb{Z}/d$.

Cyclic coverings of curves.

Idea of the proof, continuation:

Topologically, this means that we choose a special symplectic basis of $H_1(C', \mathbb{Z})$, such that the points p_j lie in the complement of the corresponding canonical dissection of the curve C', and then we take a disk Δ contained in this complement and containing the branch divisor D.

Geometrically, this means that the ramified covering is just obtained glueing together a ramified covering of Δ with an unramified covering of $C' \setminus \Delta$.

Cyclic coverings of curves.

Idea of the proof, continuation:

The theorem clearly holds for genus $h = 0$, since there is only one divisor class L satisfying $(*)$ $dL \equiv \sum_i iD_i$, hence two coverings of the disk with the same branching behaviour are equivalent.

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The unramified coverings correspond to primitive vectors in $H^1(C', \mathbb{Z}/d)$, and recall that the symplectic group $Sp(2h, \mathbb{Z})$ acts transitively on the set of such primitive elements.

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We want to show that for a suitable diffeomorphism of C' which leaves the disk Δ pointwise fixed we can transform the resulting homomorphism Ψ into one in normal form.

To this purpose we take a product of Dehn twists over loops supported in $C' \setminus \Delta$, and we observe that these generate the mapping class group of C'. Since the mapping class group maps onto the symplectic group, we are done.

Outline

Irreducible components of $Sing(\mathfrak{M}_g)$.

$\overline{\mathfrak{M}_g}$ is locally the quotient of the Kuranishi family of a stable curve C by the finite group $Aut(C)$.
Hence $\overline{\mathfrak{M}_g}$ is smooth unless C has an automorphism of prime order.

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$\overline{\mathfrak{M}_g}$ is locally the quotient of the Kuranishi family of a stable curve C by the finite group $Aut(C)$.
Hence $\overline{\mathfrak{M}_g}$ is smooth unless C has an automorphism of prime order.
Moreover, by Chevalley's theorem, the quotient of a smooth manifold by a finite group Γ is smooth if and only if the group is generated by pseudoreflections.
This implies that the singular locus is contained in the union of the irreducible closed subsets $\mathfrak{M}_{g;p,[(k_1,\dots k_{d-1})]}$ which have codimension at least 2.
But these loci have always codimension at least 2 unless γ is the hyperelliptic involution and $g = 2$ or $g = 3$.

Irreducible components of $Sing(\mathfrak{M}_g)$.

Discarding these two hyperelliptic loci in genus $g = 2, 3$ we can write $Sing(\mathfrak{M}_g)$ as a union of irreducible closed subsets $\mathfrak{M}_{g;p,[(k_1,\dots k_{d-1})]}$.
The question whether this is an irredundant irreducible decomposition was completely solved by Cornalba.
We have a shorter approach, based on the following

Proposition

Assume one such component $\mathfrak{M}_{g;p,[(k_1,\dots k_{d-1})]}$ is contained in a bigger one: then, for the general curve C in $\mathfrak{M}_{g;p,[(k_1,\dots k_{d-1})]}$ the normalizer of G in $Aut(C)$ is strictly bigger than G.

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The argument is based on a numerical calculation.
It guarantees that the quotient curve C' has a nontrivial automorphism leaving the branch locus invariant, and this allows the analysis of the possible cases.

Automorphisms of prime order of stable curves.

Since the topological type of stable curves is not constant, the basic idea is contained in the following

Definition

We shall say that a pair (C, γ) is **simplifiable** *if it admits a small deformation to a pair with a smaller number of nodes, whereas we shall say that (C, γ) is* **maximal** *if it is not simplifiable.*

A very simple argument shows when a node fixed by γ is smoothable.

Automorphisms of prime order of stable curves.

If a node p is fixed by γ, then if γ does not exchange the branches there are local holomorphic coordinates (x, y) such that $C = \{xy = 0\}$, and

$$\gamma(x, y) = (\zeta^{m_i} x, \zeta^{m_j} y).$$

The node is then smoothable if and only if $m_i + m_j \equiv 0(d)$.
In the case where γ exchanges the branches we have $d = 2$ (since d is prime) and we have:

$$\gamma(x, y) = (y, x),$$

thus the node is smoothable.
Finally,an easy argument shows that all the nodes which are not fixed by γ are smoothable, and the conclusion is:
If (C, γ) is maximal, then every component of C is left invariant by γ.

Automorphisms of prime order of stable curves.

Since if (C, γ) is maximal, then every component of C is left invariant by γ, we divide the set of components $C_i, i \in I$ into two sets: I_0 is the set of indices such that γ is the identity on C_i for $i \in I_0$, $I_1 := I \setminus I_0$.

Definition

To a maximal pair (C, γ), for d prime, we attach a graph $\mathcal{G}$ with set of vertices $I = I_0 \cup I_1$, and with edges correspond to the nodes p. Each vertex i is labelled by the genus g_i of C_i.
For $i \in I_1$, we associate to i a branching sequence $(k'_1, \dots, k'_{d-1})$ corresponding to the fixed points of $\gamma|_{C_i}$ which are not nodes .
For each edge p connecting i and j, $i \in I_1$, $i \neq j$, we give labels $m(p, i), m(p, j) \in \{0, 1, \dots, d-1\}$ describing the local action at p, while for $i = j$ we obtain an unordered pair $m(p, i), n(p, i)$.

Automorphisms of prime order of stable curves.

Theorem

The pairs (C, G) where C is a stable projective curve of genus $g \geq 2$, and G is a finite cyclic group of prime order d acting faithfully on C with a given topological type associated to an admissible automorphism graph $\mathcal{G}$ are parametrized by a connected complex manifold $\mathcal{T}_{g;d,[\mathcal{G}]}$.
The image $\overline{\mathfrak{M}}_{g;d,[\mathcal{G}]}$ of $\mathcal{T}_{g;d,[\mathcal{G}]}$ inside the compactified moduli space $\overline{\mathfrak{M}}_g$ is a locally closed subset of the same dimension whose closure consists of the coverings whose topological type can be simplified to the topological type of $\mathcal{T}_{g;d,[\mathcal{G}]}$.

Here admissible means that the graph corresponds to a maximal pair.

Automorphisms of prime order of stable curves.

If $\mathcal{T}_{g;d,[\mathcal{G}]}$ contains only stable singular curves, then $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ is not a divisor in the moduli space $\overline{\mathfrak{M}_g}$, unless, for $g \geq 3$ we are in the following case:

1. $C = C_1 \cup C_2$, where $1 \in I_0$, $2 \in I_1$, and $g_2 = 1$ (elliptic tail).

Irreducible components of $Sing(\overline{\mathfrak{M}_g})$.

Theorem

Assume that $g \geq 2$, and consider the closed subvarieties $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ inside the compactified moduli space $\overline{\mathfrak{M}_g}$, such that

1. *d is a prime number*
2. *the cyclic group G either has order $d \neq 2$ or it acts trivially on the elliptic tails.*
3. *the subset I_1 contains exactly one element*
4. *I_0 is not empty (hence $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ contains only singular stable curves).*

The above components $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ are then all distinct, for different d and different topological types, and provide the irreducible components of $Sing(\overline{\mathfrak{M}_g})$ which do not intersect $\mathfrak{M}_g$.

Outline

Coverings of smooth curves

Before we concentrate on the dihedral groups, let us explain the underlying philosophy.
Given a group G acting faithfully on a curve C of genus g, Teichmüller theory shows that, once the topological type is fixed, then one has an irreducible family parametrizing all the coverings having the fixed topological type.
Hence the basic question is to understand geometrically the meaning of topological type.
We let $C' := C/G$ and let g' be the genus of C': this is the basic topological invariant, together with the number k of branch points.

Coverings of smooth curves

Given a curve C' of genus g', and k marked points, the topological type is given by the equivalence classes of the monodromy, which belongs to the set of surjections $\mu : \pi_{g,k} \to G$ where

$$\pi_{g,k} := \pi_1(C' \setminus \{p_1, \dots, p_k\}).$$

The classes are the orbits for the action of the modular group $\mathcal{M}_{g',k}$ and of $Aut(G)$.
In the case $g' = 0$ we have an action of the braid group $\mathcal{B}r_k$, and for the unramified case ($k = 0$) we have an action of the group $\mathcal{M}_{g',0} = Out(\pi_{g'})$.
Hence the problem is reduced to a group theoretical problem: but, as we saw in the case of cyclic coverings, geometry does certainly help.

Dihedral coverings of smooth curves

When considering the dihedral group

$$D_n := \langle x, y | y^2 = xyxy = x^n = 1 \rangle$$

a basic difference occurs for n even, resp. n odd:

- n is odd: then the abelianization of D_n equals $\mathbb{Z}/2$, and there are the following nontrivial conjugacy classes
- $[x^i], 1 \leq i \leq \frac{n-1}{2}$, $[y]$
- n is even: then the abelianization of D_n equals $(\mathbb{Z}/2)^2$, and there are the following nontrivial conjugacy classes
- $[x^i], 1 \leq i \leq \frac{n}{2}$, $[y]$, $[xy]$.

Dihedral coverings of smooth curves

In work in progress together with Michael Lönne and Fabio Perroni we showed:

Theorem

Dihedral coverings with group D_n with $g' = 0$ and a fixed branching datum form an irreducible family when n is odd.

A similar proof holds also for the case $g' = 0$ and n even, but we would like to verify all the details once more.

Conjecture: the same result holds also for n odd and any g'.

Outline

Automorphisms of stable curves

Consider a stable curve consisting of a rational smooth component C_0, intersecting g elliptic tails $C_1, \dots C_g$ in nodes $p_1, \dots p_g$.
Then, if $C_1, \dots C_g$ and $p_1, \dots p_g \in C_0$ are general, $Aut(C)$ has cardinality at least 2^g. If instead the elliptic curves are equianharmonic, and the points $p_1, \dots p_g$. are roots of unity in the complex line $\mathbb{C}$, then $Aut(C)$ has cardinality $(2g) \cdot 6^g$.
This number is by far larger than the Hurwitz bound $84(g-1)$ for the cardinality of $Aut(C)$ for a smooth curve of genus g.

Automorphisms of stable curves

Question 1: which is the Hurwitz bound for stable curves? I.e., which is the maximal cardinality $a_{st}(g)$ of $Aut(C)$ for a stable curve of genus g ?
Question 2: which are the Hurwitz stable curves? I.e., which is the geometrical description of the stable curves C of genus g such that $Aut(C)$ attains the maximal allowed cardinality $a_{st}(g)$?

Abelian coverings of smooth curves

Let $C \to C'$ be an unramified covering with group $(\mathbb{Z}/2)^2$.
To it, one associates a two dimensional subspace V of $H^1(C', \mathbb{Z}/2)$.
There are therefore two different possible cases: V is isotropic or not.
This remark provides further topological invariants for dihedral coverings with group D_{2m} when $g' > 0$.
Are there other ?