Geography on 3-folds of General Type

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A Classical Problem

- Let V be a nonsingular projective variety. Consider the canonical line bundle $\omega_V = \mathcal{O}_V(K_V)$. One task of birational geometry is to study the geometry induced from linear system $|mK_V|$ or $|-mK_V|$, $\forall m \in \mathbb{Z}^+$.

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$$\mathfrak{V}_n := \{\text{n-dimensional variety of general type}\}.$$

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- Post-MMP Problem: how to classify $\mathfrak{V}_n$?

Pluricanonical boundedness

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  (1) $r_3 \leq 73$;

  (2) $Vol(V) \geq 1/2660\ \forall\ V \in \mathfrak{V}_3$.

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• The aim of this talk—geography $\Rightarrow$ to improve the above results.

• Let X be a ($\mathbb{Q}$FT) minimal projective 3-fold of general type. Reid $\Rrightarrow$ $\exists!$ weighted basket $\mathbb{B}_X := \{B_X, P_2, \mathcal{O}_X\}$ such that all the birational invariants of X are uniquely determined by $\mathbb{B}_X$, where $B_X = \{\frac{1}{r_i}(1, -1, b_i) | i = 1, \ldots, t\}$.

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• Open problem: to find exact relations between the sets

$$\mathfrak{V}_3 \leftrightsquigarrow \{\text{weighted baskets}\}$$

Two geographical inequalities

- Miyaoka-Reid inequality:

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- Inequalities of Noether type (Chen-Chen):

$$K_X^3 \geq a_m P_m(X) - b_m$$

where $a_m, b_m \in \mathbb{Q}^+$, $m \geq 1$.

Numerical genus

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- When $p_g(X) \leq 1$, $n_0(X) := \min\{m | P_m(X) \geq 2\}$. Chen-Chen $\Rightarrow$ $2 \leq n_0(X) \leq 18$.

Definition

The numerical genus of X is defined as:

$$g(X) := \begin{cases} p_g(X); & p_g(X) \geq 2 \\ \frac{1}{n_0(X)}; & \text{otherwise.} \end{cases}$$

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- What is the Noether function $\mathcal{N}(g)$?

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• In 2006, Catanese-Chen-Zhang $\Rrightarrow$ $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for nonsingular minimal 3-folds of general type.

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- Conjecture: $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ holds for Gorenstein minimal 3-folds of general type.

Known value of $\mathcal{N}(g)$

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- Chen $\Rrightarrow$

$$\mathcal{N}(2) = \frac{1}{3};$$
$$\mathcal{N}(3) = 1;$$
$$\mathcal{N}(4) = 2;$$
$$\mathcal{N}(g) \geq g - 2 \text{ for } g \geq 5.$$

due to supporting examples of Fletcher-Reid.

- Fletcher-Reid's example:

$X_{46} \subset \mathbb{P}(4,5,6,7,23)$, $K^3 = \frac{1}{420}$.

The strategy to get the lower bound of K_X^3

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- When $p_g(X) \leq 1$, $\frac{1}{18} \leq g \leq \frac{1}{2}$.
- When $K_X^3 < \frac{1}{420}$, Reid's weighted baskets can be completely listed, but the list is too big!
- To find a function $c(g)$ such that $K_X^3 \geq c(g)$ with $g(X) = g$.

The main statements

- Chen-Chen $\Rrightarrow$ $\exists$ a very effective function $v(g)$ $(g < 2)$ satisfying $K_X^3 \geq v(g(X))$.

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- Set $g = 1/n_0$, here is part of the description:

n_0	7	8	9	10	11	12
$v(n_0)$	1/420	1/450	1/630	1/825	1/1089	1/1404
n_0	13	14	15	16	17	18
$v(n_0)$	1/1728	1/2152.5	1/2640	–	–	–

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- $\mathcal{N}(\frac{1}{2}) = v(\frac{1}{2}) = \frac{1}{12}$. (optimal)

Conclusions

- Fletcher-Reid examples with $g = 1/2$ and $K^3 = 1/12$:

$$X_{22} \subset \mathbb{P}(1,2,3,4,11)$$
$$X_{6,18} \subset \mathbb{P}(2,2,3,3,4,9)$$
$$X_{10,14} \subset \mathbb{P}(2,2,3,4,5,7)$$

Conclusions

- Fletcher-Reid examples with $g = 1/2$ and $K^3 = 1/12$:

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Theorem

Let X be a minimal projective 3-fold of general type. Then

(1) $K_X^3 \geq \frac{17}{30030} > \frac{1}{1767}$. Furthermore, $K_X^3 = \frac{17}{30030}$ if and only if $\mathbb{B}(X) = \{B_{3a}, 0, 3\}$.

(2) (announcement) φ_m is birational for $m \geq 65$.

The method

- We study the m_0-canonical map of X:

$$\varphi_{m_0} : X \dashrightarrow \mathbb{P}^{P_{m_0}-1}.$$

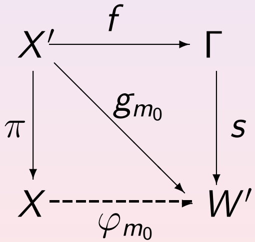
By Hironaka's big theorem, we can take successive blow-ups $\pi : X' \to X$ such that:

(i) X' is smooth;

(ii) the movable part of $|m_0 K_{X'}|$ is base point free;

(iii) the support of the union of $\pi^*(K_{m_0})$ and the exceptional divisors is of simple normal crossings.

- Set $g_{m_0} := \varphi_{m_0} \circ \pi$. Then g_{m_0} is a morphism by assumption. Let $X' \xrightarrow{f} \Gamma \xrightarrow{s} W'$ be the Stein factorization of g_{m_0} with W' the image of X' through g_{m_0}.



• Denote by M_{m_0} the movable part of $|m_0K_{X'}|$. One has

$$m_0\pi^*(K_X) = M_{m_0} + E'_{m_0}$$

for an effective $\mathbb{Q}$-divisor E'_{m_0}. In total, since

$$h^0(X', \llcorner m_0\pi^*(K_X)\lrcorner) = h^0(X', \ulcorner m_0\pi^*(K_X)\urcorner) = P_{m_0}(X') = P_{m_0}(X),$$

one has:

$$m_0K_{X'} = M_{m_0} + Z_{m_0}$$

where Z_{m_0} is the fixed part of $|m_0K_{X'}|$.

- If $\dim(\Gamma) \geq 2$, a general member S of $|M_{m_0}|$ is a nonsingular projective surface of general type. Set $p = 1$.

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- If $\dim(\Gamma) = 1$, a general fiber S of f is an irreducible smooth projective surface of general type. We may write

$$M_{m_0} = \sum_{i=1}^{a_{m_0}} S_i \equiv a_{m_0} S$$

where S_i are smooth fibers of f for all i and $a_{m_0} \geq \min\{2P_{m_0} - 2, P_{m_0} + g(\Gamma) - 1\}$. Set $p = a_{m_0}$.

The method

- Let S be a generic irreducible element of $|m_0K_{X'}|$. Let $|G|$ be a base point free linear system on S. Let C be a generic irreducible element of $|G|$. Kodaira Lemma $\Rightarrow \exists \beta > 0$ such that $\pi^*(K_X)|_S \geq \beta C$.

The method

- Let S be a generic irreducible element of $|m_0K_{X'}|$. Let $|G|$ be a base point free linear system on S. Let C be a generic irreducible element of $|G|$. Kodaira Lemma $\Rightarrow \exists\beta > 0$ such that $\pi^*(K_X)|_S \geq \beta C$.
- Inequality (1):

$$K_X^3 \geq \frac{p\beta}{m_0}\xi \tag{1}$$

where $\xi = \pi^*(K_X) \cdot C$.

- Inequality (2):

$$\xi \geq \frac{\deg(K_C)}{1+\frac{m_0}{p}+\frac{1}{\beta}}. \tag{2}$$

The method

- Inequality (2):

$$\xi \geq \frac{\deg(K_C)}{1+\frac{m_0}{p}+\frac{1}{\beta}}. \quad (2)$$

- Inequality (3): For any positive integer m such that $\alpha_m := (m-1-\frac{m_0}{p}-\frac{1}{\beta})\xi > 1$, one has

$$\xi \geq \frac{\deg(K_C)+\ulcorner \alpha_m \urcorner}{m}. \quad (3)$$

Technical applications

- When $\dim \Gamma > 1$, take $|G| := |S|_S|$. Thus $\beta = \frac{1}{m_0}$.

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- When $\dim \Gamma = 1$, take $G = q\sigma^*(K_{S_0})$ for $q \geq 1$ where $\sigma : S \to S_0$ is the contraction onto the minima model. Here is a key inequality:

$$\pi^*(K_X)|_S \geq \frac{p}{m_0 + p}\sigma^*(K_{S_0}).$$

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- Here is the complete list for 3-folds with small invariants:

No.	$(P_3, \cdots, P_{11})$	P_{18}	P_{24}	μ_1	χ	$B^{(12)} = (n_{1,2}, n_{5,11}, \cdots, n_{1,5})$ or B_{min}	K^3
1	(0, 0, 0, 0, 0, 0, 0, 1, 0)	4	8	14	2	(5, 0, 0, 1, 0, 3, 0, 0, 3, 0, 0, 1, 0, 0, 0)	$\frac{3}{770}$
2	(0, 0, 0, 0, 0, 1, 0, 0, 0)	3	7	15	2	(4, 0, 1, 0, 0, 2, 1, 0, 3, 0, 0, 0, 2, 0, 0)	$\frac{1}{360}$
2*a*		2	3	18		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{1}{1170}$
3	(0, 0, 0, 0, 0, 1, 0, 1, 0)	3	7	15	3	(6, 1, 0, 0, 0, 4, 1, 0, 4, 0, 1, 0, 2, 0, 0)	$\frac{23}{9240}$
3*a*		2	3	18		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{17}{30030}$
4	(0, 0, 0, 0, 0, 1, 0, 1, 0)	4	9	14	3	(7, 0, 1, 0, 0, 4, 0, 1, 3, 0, 1, 0, 2, 0, 0)	$\frac{13}{3465}$
4.5		1	2	14		$\{(4,11),(1,3),*\} \succ \{(5,14),*\}$	$\frac{1}{630}$
5	(0, 0, 0, 0, 0, 1, 0, 1, 0)	5	10	14	3	(7, 0, 1, 0, 0, 4, 1, 0, 4, 0, 0, 1, 1, 0, 0)	$\frac{17}{3960}$
5*a*		4	3	15		$\{(8,20),(3,8),*\} \succ \{(11,28),*\}$	$\frac{1}{1386}$
5*b*		3	3	15		$\{(5,13),(4,15),*\}$	$\frac{1}{1170}$
6	(0, 0, 0, 1, 0, 0, 0, 1, 0)	3	6	14	3	(9, 0, 0, 2, 0, 1, 0, 1, 4, 0, 2, 0, 0, 0, 1)	$\frac{1}{462}$
7	(0, 0, 0, 1, 0, 0, 1, 0, 0)	3	5	14	2	(5, 0, 1, 1, 0, 0, 0, 0, 5, 0, 1, 0, 0, 0, 1)	$\frac{1}{630}$
7*a*		2	3	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$
8	(0, 0, 0, 1, 0, 0, 1, 1, 0)	3	5	14	3	(7, 1, 0, 1, 0, 2, 0, 0, 6, 0, 2, 0, 0, 0, 1)	$\frac{1}{770}$
10	(0, 0, 0, 1, 0, 1, 0, 0, 0)	3	6	14	3	(8, 0, 1, 1, 0, 0, 2, 0, 5, 0, 1, 0, 1, 0, 1)	$\frac{1}{630}$
10*a*		2	4	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$
11	(0, 0, 0, 1, 0, 1, 0, 1, 0)	2	4	14	3	(9, 0, 0, 2, 0, 0, 1, 1, 3, 1, 0, 0, 1, 0, 1)	$\frac{3}{3080}$
12	(0, 0, 0, 1, 0, 1, 0, 1, 0)	5	11	14	3	(9, 0, 1, 0, 0, 1, 2, 0, 4, 0, 2, 0, 0, 0, 1)	$\frac{1}{252}$
12*a*		4	6	14		$\{(2,5),(6,16),*\} \succ \{(8,21),*\}$	$\frac{1}{630}$
13	(0, 0, 0, 1, 0, 1, 0, 1, 0)	3	4	14	4	(12, 0, 0, 2, 0, 2, 0, 2, 4, 0, 2, 0, 0, 1, 0)	$\frac{4}{3465}$
14	(0, 0, 0, 1, 0, 1, 0, 1, 0)	3	6	14	4	(10, 1, 0, 1, 0, 2, 2, 0, 6, 0, 2, 0, 1, 0, 1)	$\frac{1}{770}$
15	(0, 0, 0, 1, 0, 1, 0, 1, 0)	4	8	14	4	(11, 0, 1, 1, 0, 2, 1, 1, 5, 0, 2, 0, 1, 0, 1)	$\frac{71}{27720}$
15*b*		3	4	14		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{23}{36036}$
15*c*		3	5	14		$\{(7,16),(7,19),*\}$	$\frac{31}{31920}$
16	(0, 0, 0, 1, 0, 1, 0, 1, 0)	5	9	14	4	(11, 0, 1, 1, 0, 2, 2, 0, 6, 0, 1, 1, 0, 0, 1)	$\frac{43}{13860}$
16.5		4	3	14		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{85}{72072}$

16b		4	4	14		$\{(2,5),(6,16),*\} \succ \{(8,21),*\}$	$\frac{1}{1386}$
16.6		3	3	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{13}{6160}$
17	$(0,0,0,1,0,1,0,1,1)$	3	6	14	3	$(9,0,0,2,0,0,0,2,3,0,1,0,1,0,1)$	$\frac{3}{1540}$
18	$(0,0,0,1,0,1,0,1,1)$	4	7	14	3	$(9,0,0,2,0,0,1,1,4,0,0,1,0,0,1)$	$\frac{23}{9240}$
18b		4	6	14		$\{(3,8),(4,11),*\} \succ \{(7,19),*\}$	$\frac{83}{43890}$
19	$(0,0,0,1,0,1,1,0,0)$	3	3	14	3	$(8,0,1,1,0,1,0,1,5,0,1,0,0,1,0)$	$\frac{2}{3465}$
20	$(0,0,0,1,0,1,1,0,0)$	4	7	14	3	$(7,0,2,0,0,1,1,0,6,0,1,0,1,0,1)$	$\frac{1}{504}$
21	$(0,0,0,1,0,1,1,1,0)$	4	8	14	2	$(6,0,1,0,0,0,1,0,3,1,0,0,0,0,1)$	$\frac{1}{360}$
23	$(0,0,0,1,0,1,1,1,0)$	3	5	14	3	$(8,0,1,1,0,1,0,1,4,1,0,0,1,0,1)$	$\frac{19}{13860}$
25	$(0,0,0,1,0,1,1,1,0)$	4	7	14	4	$(9,1,1,0,0,3,1,0,7,0,2,0,1,0,1)$	$\frac{47}{27720}$
25a		4	6	14		$\{(5,11),(4,9),*\} \succ \{(9,20),*\}$	$\frac{1}{840}$
26	$(0,0,0,1,0,1,1,1,0,)$	5	9	14	4	$(10,0,2,0,0,3,0,1,6,0,2,0,1,0,1)$	$\frac{41}{13860}$
26a		3	5	14		$\{(4,11),(1,3),*\} \succ \{(5,14),*\}$	$\frac{1}{1260}$
27	$(0,0,0,1,0,1,1,1,0)$	6	10	14	4	$(10,0,2,0,0,3,1,0,7,0,1,1,0,0,1)$	$\frac{97}{27720}$
27.5		5	3	14		$\{(4,10),(3,8),*\} \succ \{(7,18),*\}$	$\frac{1}{1386}$
27b		5	5	14		$\{(5,13),(5,18),*\}$	$\frac{1}{1170}$
28	$(0,0,0,1,0,1,1,1,1)$	4	8	14	2	$(5,1,0,0,0,0,1,0,4,0,1,0,0,0,1)$	$\frac{23}{9240}$
29	$(0,0,0,1,0,1,1,1,1)$	5	10	14	2	$(6,0,1,0,0,0,0,1,3,0,1,0,0,0,1)$	$\frac{13}{3465}$
29.5		3	4	14		$\{(4,11),(1,3),*\} \succ \{(5,14),*\}$	$\frac{1}{630}$
30	$(0,0,0,1,0,1,1,1,1)$	3	5	14	3	$(7,1,0,1,0,1,0,1,5,0,1,0,1,0,1)$	$\frac{1}{924}$
31	$(0,0,0,1,0,1,1,1,1)$	4	6	14	3	$(7,1,0,1,0,1,1,0,6,0,0,1,0,0,1)$	$\frac{1}{616}$
32	$(0,0,0,1,0,1,1,1,1)$	5	8	14	3	$(8,0,1,1,0,1,0,1,5,0,0,1,0,0,1)$	$\frac{2}{693}$
32a		4	6	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{528}$
32b		2	2	14		$\{(4,11),(1,3),*\} \succ \{(5,14),*\}$	$\frac{1}{1386}$
33	$(0,0,0,1,1,0,0,1,0)$	2	4	14	2	$(5,0,0,2,0,0,1,0,1,1,1,0,0,0,0)$	$\frac{1}{840}$
34	$(0,0,0,1,1,0,0,1,0)$	4	8	14	3	$(7,0,1,1,0,2,1,0,3,0,3,0,0,0,0)$	$\frac{1}{360}$
34a		3	6	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{560}$
34b		3	4	14		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{1}{1170}$

No.	$(P_3, \cdots, P_{11})$	P_{18}	P_{24}	μ_1	χ	$(n_{1,2}, n_{4,9}, \cdots, n_{1,5})$ or B_{min}	K^3
35	(0, 0, 0, 1, 1, 0, 0, 1, 1)	3	6	14	2	(5, 0, 0, 2, 0, 0, 0, 1, 1, 0, 2, 0, 0, 0, 0)	$\frac{1}{462}$
36	(0, 0, 0, 1, 1, 0, 1, 1, 0)	3	5	14	2	(4, 0, 1, 1, 0, 1, 0, 0, 2, 1, 1, 0, 0, 0, 0)	$\frac{1}{630}$
36*a*		2	3	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$
36*b*		2	4	14		$\{(3,10),(2,7),*\} \succ \{(5,17),*\}$	$\frac{4}{5355}$
37	(0, 0, 0, 1, 1, 0, 1, 1, 0)	5	9	14	3	(6, 0, 2, 0, 0, 3, 0, 0, 4, 0, 3, 0, 0, 0, 0)	$\frac{1}{315}$
38	(0, 0, 0, 1, 1, 0, 1, 1, 1)	3	5	14	2	(3, 1, 0, 1, 0, 1, 0, 0, 3, 0, 2, 0, 0, 0, 0)	$\frac{1}{770}$
39	(0, 0, 0, 1, 1, 1, 0, 1, 0)	3	6	14	3	(7, 0, 1, 1, 0, 1, 2, 0, 2, 1, 1, 0, 1, 0, 0)	$\frac{1}{630}$
39*a*		2	4	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$
39*b*		2	5	14		$\{(3,10),(2,7),*\} \succ \{(5,17),*\}$	$\frac{4}{5355}$
40	(0, 0, 0, 1, 1, 1, 0, 1, 0)	5	10	14	4	(9, 0, 2, 0, 0, 3, 2, 0, 4, 0, 3, 0, 1, 0, 0)	$\frac{1}{315}$
40.5		4	4	14		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$> \frac{1}{780}$
40*b*		4	5	14		$\{(2,5),(6,16),*\} \succ \{(8,21),*\}$	$\frac{1}{1260}$
41	(0, 0, 0, 1, 1, 1, 0, 1, 1)	5	11	13	2	(5, 0, 1, 0, 0, 0, 2, 0, 1, 0, 2, 0, 0, 0, 0)	$\frac{1}{252}$
42	(0, 0, 0, 1, 1, 1, 0, 1, 1)	3	6	14	3	(6, 1, 0, 1, 0, 1, 2, 0, 3, 0, 2, 0, 1, 0, 0)	$\frac{1}{770}$
43	(0, 0, 0, 1, 1, 1, 0, 1, 1)	4	8	14	3	(7, 0, 1, 1, 0, 1, 1, 1, 2, 0, 2, 0, 1, 0, 0)	$\frac{71}{27720}$
43*b*		3	4	14		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{23}{36036}$
43*c*		3	5	14		$\{(7,16),(7,19),*\}$	$\frac{31}{31920}$
44	(0, 0, 0, 1, 1, 1, 0, 1, 1)	5	9	14	3	(7, 0, 1, 1, 0, 1, 2, 0, 3, 0, 1, 1, 0, 0, 0)	$\frac{43}{13860}$
44*a*		4	4	14		$\{(2,5),(6,16),*\} \succ \{(8,21),*\}$	$\frac{1}{1386}$
44*c*		4	6	14		$\{(7,16),(5,18),*\}$	$\frac{1}{720}$
44.5		4	4	14		$\{(5,13),*\}$	$> \frac{1}{848}$
45	(0, 0, 0, 1, 1, 1, 1, 0, 1)	4	7	14	2	(3, 0, 2, 0, 0, 0, 1, 0, 3, 0, 1, 0, 1, 0, 0)	$\frac{1}{504}$
46	(0, 0, 0, 1, 1, 1, 1, 1, 0)	4	7	14	3	(6, 0, 2, 0, 0, 2, 1, 0, 3, 1, 1, 0, 1, 0, 0)	$\frac{1}{504}$
46*b*		3	6	14		$\{(3,10),(2,7),*\} \succ \{(5,17),*\}$	$\frac{7}{6120}$
48	0, 0, 0, 1, 1, 1, 1, 1, 1)	3	5	14	2	(4, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0)	$\frac{19}{13860}$
49	(0, 0, 0, 1, 1, 1, 1, 1, 1)	4	7	14	3	(5, 1, 1, 0, 0, 2, 1, 0, 4, 0, 2, 0, 1, 0, 0)	$\frac{47}{27720}$

49*a*		4	6	14		$\{(5,11),(4,9),*\} \succ \{(9,20),*\}$	$\frac{1}{840}$
50	(0, 0, 0, 1, 1, 1, 1, 1, 1)	5	9	14	3	(6, 0, 2, 0, 0, 2, 0, 1, 3, 0, 2, 0, 1, 0, 0)	$\frac{41}{13860}$
50*a*		3	5	14		$\{(4,11),(1,3),*\} \succ \{(5,14),*\}$	$\frac{1}{1260}$
51	(0, 0, 0, 1, 1, 1, 1, 1, 1)	6	10	14	3	(6, 0, 2, 0, 0, 2, 1, 0, 4, 0, 1, 1, 0, 0, 0)	$\frac{97}{27720}$
51*a*		5	4	14		$\{(4,10),(3,8),*\} \succ \{(7,18),*\}$	$\frac{1}{1386}$
51*b*		5	5	14		$\{(5,13),(5,18),*\}$	$\frac{1}{1170}$
52	(0, 0, 1, 0, 0, 1, 0, 1, 0)	3	7	14	2	(4, 0, 0, 1, 0, 2, 2, 0, 2, 0, 0, 0, 0, 0, 1)	$\frac{1}{420}$
53	(0, 0, 1, 0, 0, 1, 1, 1, 0)	4	8	14	2	(3, 0, 1, 0, 0, 3, 1, 0, 3, 0, 0, 0, 0, 0, 1)	$\frac{1}{360}$
53*a*		3	4	15		$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{1}{1170}$
54	(0, 0, 1, 0, 1, 0, 0, 1, 0)	2	4	14	2	(2, 0, 0, 2, 0, 3, 1, 0, 1, 0, 1, 0, 0, 0, 0)	$\frac{1}{840}$
56	(0, 0, 1, 0, 1, 0, 1, 1, 0)	3	5	14	2	(1, 0, 1, 1, 0, 4, 0, 0, 2, 0, 1, 0, 0, 0, 0)	$\frac{1}{630}$
56*a*		2	3	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$
57	(0, 0, 1, 0, 1, 0, 1, 1, 0)	3	3	14	3	(3, 0, 1, 2, 0, 5, 0, 0, 4, 0, 0, 1, 0, 0, 0)	$\frac{1}{1386}$
58	(0, 0, 1, 0, 1, 1, 0, 1, 0)	3	6	14	3	(4, 0, 1, 1, 0, 4, 2, 0, 2, 0, 1, 0, 1, 0, 0)	$\frac{1}{630}$
58*a*		2	4	14		$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$
59	(0, 0, 1, 0, 1, 1, 0, 1, 1)	2	4	14	2	(2, 0, 0, 2, 0, 2, 1, 1, 0, 0, 0, 0, 1, 0, 0)	$\frac{3}{3080}$
60	(0, 0, 1, 0, 1, 1, 1, 1, 0)	4	7	14	3	(3, 0, 2, 0, 0, 5, 1, 0, 3, 0, 1, 0, 1, 0, 0)	$\frac{1}{504}$
62	(0, 0, 1, 0, 1, 1, 1, 1, 1)	3	5	14	2	(1, 0, 1, 1, 0, 3, 0, 1, 1, 0, 0, 0, 1, 0, 0)	$\frac{19}{13860}$

• Let X be a nonsingular projective 3-fold of general type. When the geometric genus $p_g \geq 2$, the canonical map $\varphi_1 := \Phi_{|K_X|}$ is usually a key tool for birational classification.

Canonically fibred 3-folds

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- If φ_1 is of fiber type (i.e. $\dim \overline{\varphi_1(X)} < 3$), it is interesting to see if the birational invariants of the generic irreducible component in the general fiber of φ_1 is bounded from above.

- Let X be a nonsingular projective 3-fold of general type. When the geometric genus $p_g \geq 2$, the canonical map $\varphi_1 := \Phi_{|K_X|}$ is usually a key tool for birational classification.
- If φ_1 is of fiber type (i.e. $\dim \overline{\varphi_1(X)} < 3$), it is interesting to see if the birational invariants of the generic irreducible component in the general fiber of φ_1 is bounded from above.
- Chen-Hacon $\Rrightarrow$ When X is Gorenstein minimal and φ_1 is of fiber type, then X is canonically fibred by surfaces or curves with bounded invariants.

- Chen-Cui, 2010 $\Rrightarrow$

Theorem

Let X be a Gorenstein minimal projective 3-fold of general type. Assume that X is canonically of fiber type. Let F be a smooth model of the generic irreducible component in the general fiber of φ_1. Then

(i) $g(F) \leq 91$ when F is a curve and $p_g(X) \geq 183$;
(ii) $p_g(F) \leq 37$ when F is a surface and $p_g(X) \gg 0$, say $p_g(X) \geq 3890$.

Standard construction. Let S be a minimalsurface of g eneral type with $p_g(S) = 0$. Assume there exists a divisor H on S such that $|K_S + H|$ is composed with a pencil of curves and that $2H$ is linearly equivalent to a smooth divisor R. Let $\hat{C}$ be a generic irreducible element of the movable part of $|K_S + H|$. Assume $\hat{C}$ is smooth. Set $d := \hat{C}.H$ and $D := \hat{C} \bigcap H$. Let C_0 be a fixed smooth projective curve of genus 2. Let θ be a 2-torsion divisor on C_0. Set $Y := S \times C_0$. Take $\delta := p_1^*(H) + p_2^*(\theta)$ and pick a smooth divisor $\Delta \sim p_1^*(2H)$. Then the pair (δ, Δ) determines a smooth double covering $\pi : X \to Y$ and $K_X = \pi^*(K_Y + \delta)$.

Since $K_Y + \delta = p_1^*(K_S + H) + p_2^*(K_{C_0} + \theta)$, $p_g(Y) = 0$ and $h^0(K_{C_0} + \theta) = 1$, one sees that $|K_X| = \pi^*|K_Y + \delta|$ and that $\Phi_{|K_X|}$ factors through π, p_1 and $\Phi_{|K_S+H|}$. Since $|K_S + H|$ is composed with a pencil of curves $\hat{C}$, X is canonically fibred by surfaces F and F is a double covering over $T := \hat{C} \times C_0$ corresponding to the data $(q_1^*(D) + q_2^*(\theta), q_1^*(2D))$ where q_1 and q_2 are projections. Denote by $\sigma : F \to T$ the double covering. Then $K_F = \sigma^*(K_T + q_1^*(D) + q_2^*(\theta))$. By calculation, one has $p_g(F) = 3g(\hat{C})$ when $d = 0$ and $p_g(F) = 3g(\hat{C}) + d - 1$ whenever $d > 0$.

Lemma

Let S be any smooth minimal projective surface of general type with $p_g(S) = 0$. Assume $\mu : S \to \mathbb{P}^1$ is a genus 2 fibration. Let H be a general fiber of μ. Then $|K_S + H|$ is composed with a pencil of curves $\hat{C}$ of genus $g(\hat{C})$ and $\hat{C}.H = 2$.

Lemma

Let S be any smooth minimal projective surface of general type with $p_g(S) = 0$. Assume $\mu : S \to \mathbb{P}^1$ is a genus 2 fibration. Let H be a general fiber of μ. Then $|K_S + H|$ is composed with a pencil of curves $\hat{C}$ of genus $g(\hat{C})$ and $\hat{C}.H = 2$.

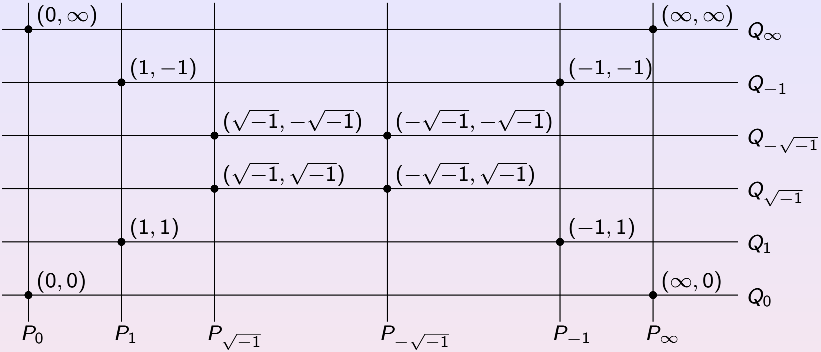
- We take a pair (S, H) which was found by Xiao, where S is a numerical Compedelli surface with $K_S^2 = 2$, $p_g(S) = q(S) = 0$ and $\mathrm{Tor}(S) = (\mathbb{Z}_2)^3$.

- Let $P = \mathbb{P}^1 \times \mathbb{P}^1$. Take four curves C_1, C_2, C_3 and C_4 defined by the following equations, respectively:

$$C_1: \quad x = y;$$
$$C_2: \quad x = -y;$$
$$C_3: \quad xy = 1;$$
$$C_4: \quad xy = -1.$$

These four curves intersect mutually at 12 ordinary double points:

$$(0,0),\ (\infty,\infty),\ (0,\infty),\ (\infty,0)$$
$$(\pm 1, \pm 1),\ (\pm\sqrt{-1}, \pm\sqrt{-1}).$$


(0, ∞)
(∞, ∞)
Q∞
(1, −1)
(−1, −1)
Q−1
(√−1, −√−1)
(−√−1, −√−1)
Q−√−1
(√−1, √−1)
(−√−1, √−1)
Q√−1
(1, 1)
(−1, 1)
Q1
(0, 0)
(∞, 0)
Q0
P0
P1
P√−1
P−√−1
P−1
P∞

Xiao $\Rrightarrow$ There exists a divisor R_1 of bidegree $(14, 6)$ which has exactly 12 simple singularities of multiplicity 4. Then the data (δ_1, R_1) determines a singular double covering onto P.

$$\begin{array}{ccccc} S & \xleftarrow{\sigma} & \tilde{S} & \xrightarrow{\theta} & \tilde{P} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tau} \\ \mathbb{P}^1 & = & \mathbb{P}^1 & \xleftarrow[\varphi]{} & P \end{array}$$

$K_S^2 = 2$ and $p_g(S) = q(S) = 0$.

- Let H be a general fiber of f. Calculations $\Rightarrow$ $|K_S + H|$ has exactly 6 base points, but no fixed parts. Clearly a general member $\hat{C} \in |K_S + H|$ is a smooth curve of genus 6.

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- Now we take the triple $(S, H, \hat{C})$ and run standard construction. What we get is the 3-fold $X_{S,19}$ which is canonically fibred by surfaces F with $p_g(F) = 19$.

$X_{S,19}$

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- Now we take the triple $(S, H, \hat{C})$ and run standard construction. What we get is the 3-fold $X_{S,19}$ which is canonically fibred by surfaces F with $p_g(F) = 19$.
- Thanks very much!