The s. s. K3 surface with Artin inv. 1 in char 2 and the Leech lattice

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$\mathbb{P}^2(\mathbb{F}_4)$ $\quad \dfrac{4^3-1}{4-1} = 21$ pts & lines

$\mathbb{P}^1(\mathbb{F}_4)$ $\quad \dfrac{4^2-1}{4-1} = 5$ pts

$(21)_5$ - conf.

PLAN

§ $(16)_6$-conf. of Kummer surf.

§ s.s. K3

§ Main results

§ Relation with the Leech lattice

§. $(16)_6$-conf.

C : a smooth curve of genus $2/\mathbb{C}$

$J(C)$: the Jacobian $\circlearrowleft -1$

$$\begin{array}{l} J(C) \\ \cup \\ \Theta_a \quad (a \in J(C)_2) \end{array}$$

$$\begin{tikzcd}
\Theta_a \arrow[r, hook] \arrow[dd] & J(C) \arrow[d] & \\
& J(C)/\pm 1 \arrow[d, equal] & X = \mathrm{Km}(C) \arrow[l] \\
\overline{T}_a \subset & \overline{X} \subset \mathbb{P}^3 &
\end{tikzcd}$$

$\overline{T}_a$: a conic

$\overline{X}$: a quartic with 16 nodes

- $Km(C) \supset \{N_a\}, \{T_a\}$
  are two families of 16 disjoint $\mathbb{P}^1$.
- Each member from one family meets 6 members from another family.

- $$Km(C) \overset{\exists}{\hookrightarrow} \mathbb{P}^5_{(2)\cap(2)\cap(2)}$$

  N_a, T_a : lines

- $$\mathrm{Aut}((16)_6\text{-conf.})$$
  $$\wr\!\parallel$$
  $$S_6 \cdot (\mathbb{Z}/2\mathbb{Z})^5$$

  switch

  $$\underbrace{S_6 \cdot (\mathbb{Z}/2\mathbb{Z})^4} \cdot \mathbb{Z}/2\mathbb{Z}$$

§. s.s. K3

X : alg. K3

S_X : Picard lattice

$\rho(X)$: Picard #

$\Rightarrow$ $1 \leqq \rho(X) \leqq 20$ or 22

X is called super singular K3

if $\rho(X) = 22$

(only occur in char $p > 0$)

M. Artin , T. Shioda , Ogus

Rudakov - Shafarevich , ...

$$S_X^* = \mathrm{Hom}(S_X, \mathbb{Z}) \supset S_X$$

$$S_X^*/S_X \cong (\mathbb{Z}/p\mathbb{Z})^{2\sigma}$$

$$1 \leq \sigma \leq 10$$

Artin invariant.

s.s. K3 with Artin inv. σ form a family of $\dim = \sigma - 1$

In the following, assume

char $= 2$.

s.s. K3 with $\sigma = 1$ in char 2 is <u>Unique</u>.

§. Main result

Thm X : s.s. K3 with $\sigma = 1$
(in char 2)

$\Rightarrow \exists\, \mathcal{A}, \mathcal{B} = 21$ disj. $\mathbb{P}^1$

$\mathcal{A}$ = 21 disjoint $\mathbb{P}^1$

s.t.
each member in one set
meets exactly 5 members
in another set.

$(21)_5$-conf.

4 Constructions

(i) $\mathbb{P}^2(\mathbb{F}_4)$ 21 pts
21 lines

$$\begin{tikzcd} \mathbb{P}^2 & \overline{X} \arrow[l, "2:1"', "\text{insep.}"] & X \arrow[l, "\text{mini. resol.}"'] \end{tikzcd}$$

$\cup$

$x_0 x_1 x_2 (x_0^3 + x_1^3 + x_2^3) = 0$

Shimada : generalized this for all s.s. K3 in char = 2.

$$\boxed{\mathbb{P}^2 \xleftarrow[\substack{2:1 \\ \text{insep.}}]{\exists} X}$$

(ii)

$$\overset{\text{Switch}}{S} \circlearrowright \overset{(x_0:x_1:x_2)}{\mathbb{P}^2} \times \overset{(y_0:y_1:y_2)}{\mathbb{P}^2}$$

$$X : \begin{cases} x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2 = 0 \\ x_0^2 y_0 + x_1^2 y_1 + x_2^2 y_2 = 0 \end{cases}$$

$$(a_0 : a_1 : a_2) \in \mathbb{P}^2(\mathbb{F}_4)$$

$$A \xleftrightarrow[S]{} B$$

S. Mukai sent me a Post Card

......

(iii)

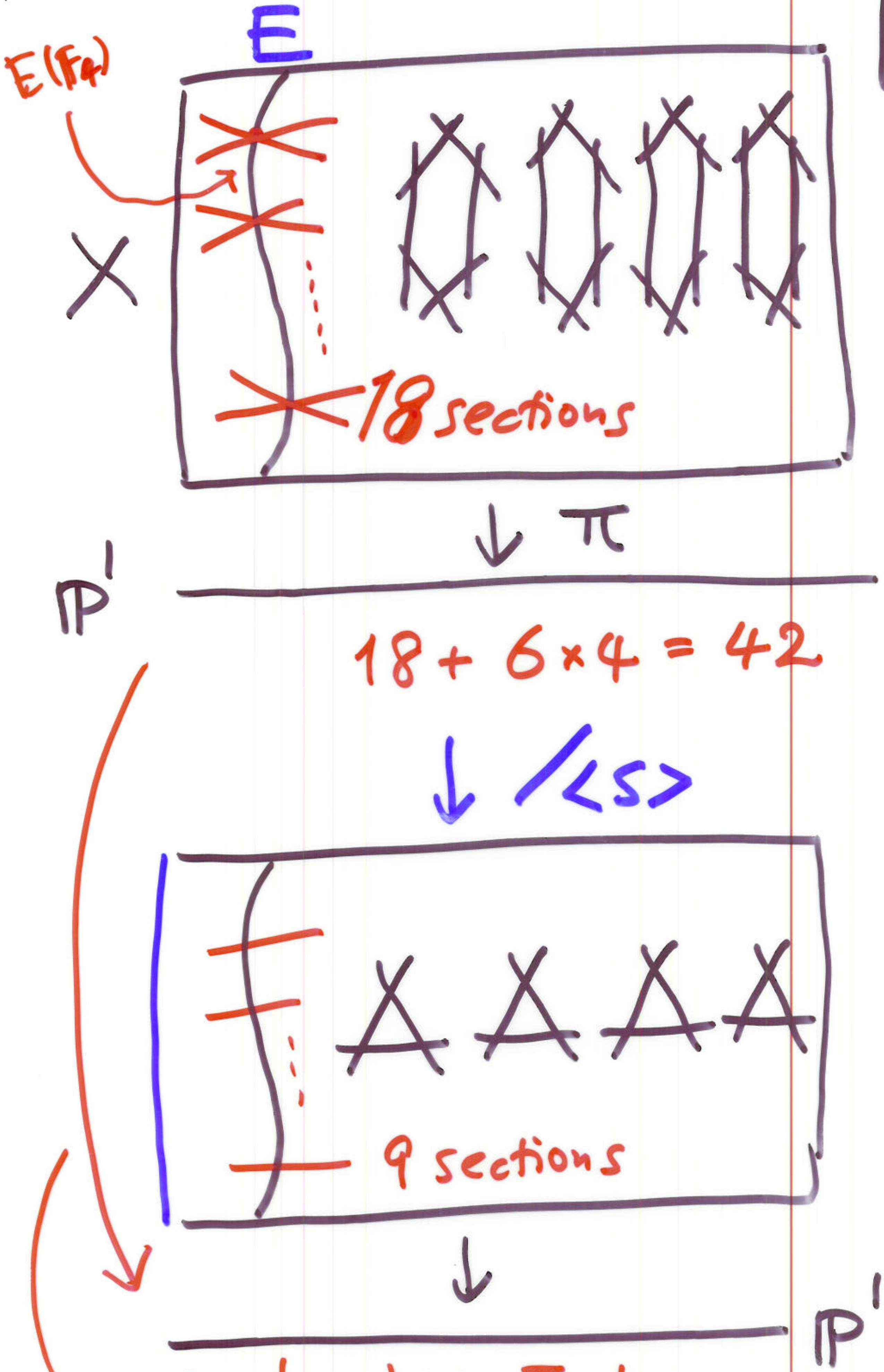
The set of fixed points of the switch S

$$= \{ x_0^3 + x_1^3 + x_2^3 = 0 \}$$

$=: E$ s.s. elli. curve

$|E| : X \xrightarrow{\pi} \mathbb{P}^1$

elli. fibration



We also obtain π by Frobenius base change.

$\boxed{10'}$

X is s.s. K3 with $\sigma = 1$.

- $\rho = 2 + (6-1) \times 4 = 22$

- $|S_X^* / S_X|$

$$= \frac{6^4}{18^2} = 2^2$$

$\underline{\sigma = 1}$

(iv) Generalized Kummer

$$E : y^2 + y = x^3$$

s.s. elli. curve

$$\circlearrowright\ \sigma : (x, y) \longrightarrow (\zeta_3 x, y)$$

$$\text{Fixed pts} = E(\mathbb{F}_2) = \{(0,1,0), (0,0,1), (0,1,1)\}$$

$$\begin{array}{ccc} E \times E & & \\ \downarrow & & \\ E \times E / \langle \sigma \times \sigma^2 \rangle & \longleftarrow & X \end{array}$$

$\exists\ 18\ \mathbb{P}^1$

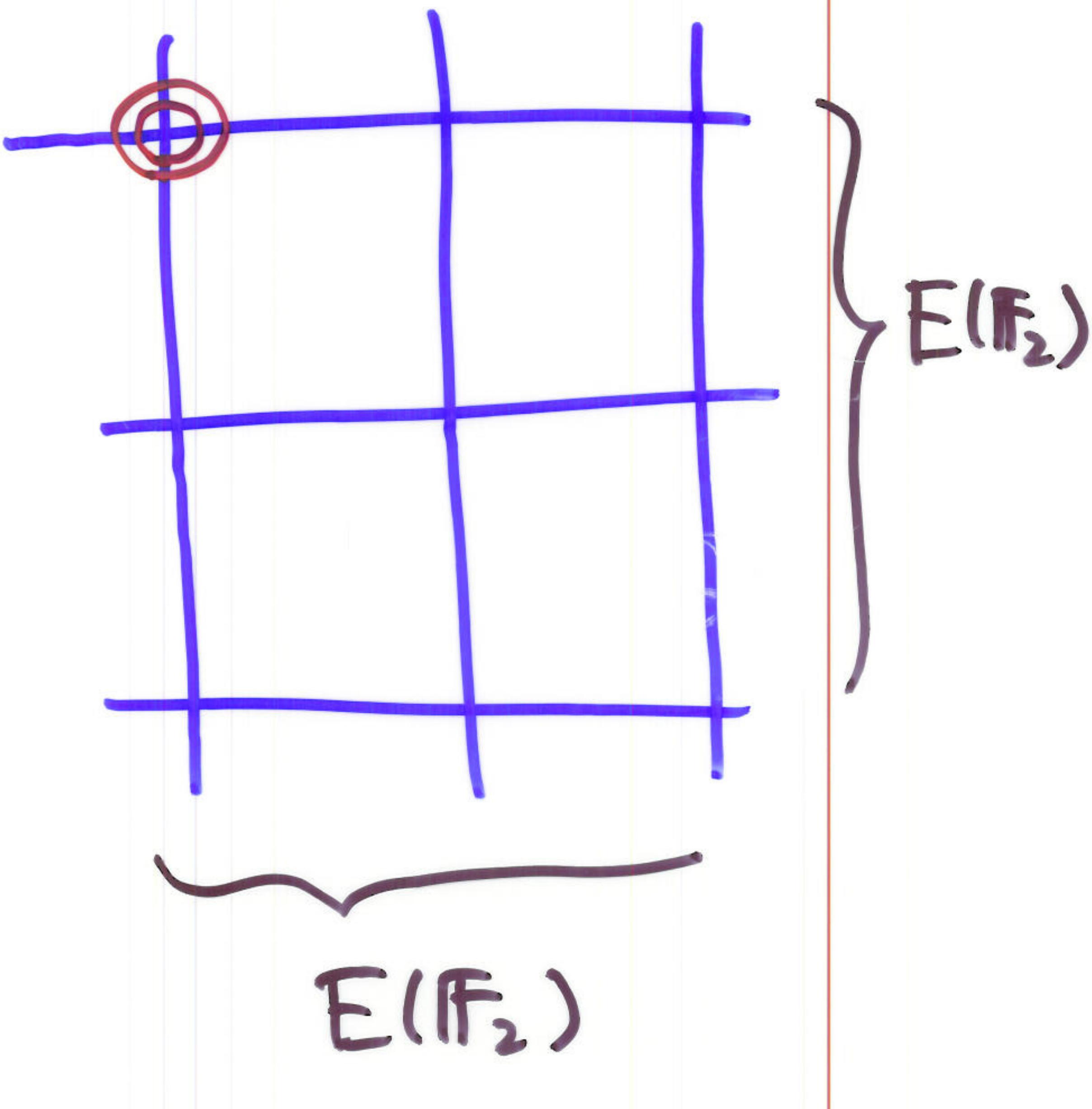
9 rat. double pts of type A_2

Rem.

$E \times E / \pm 1$: NOT K3

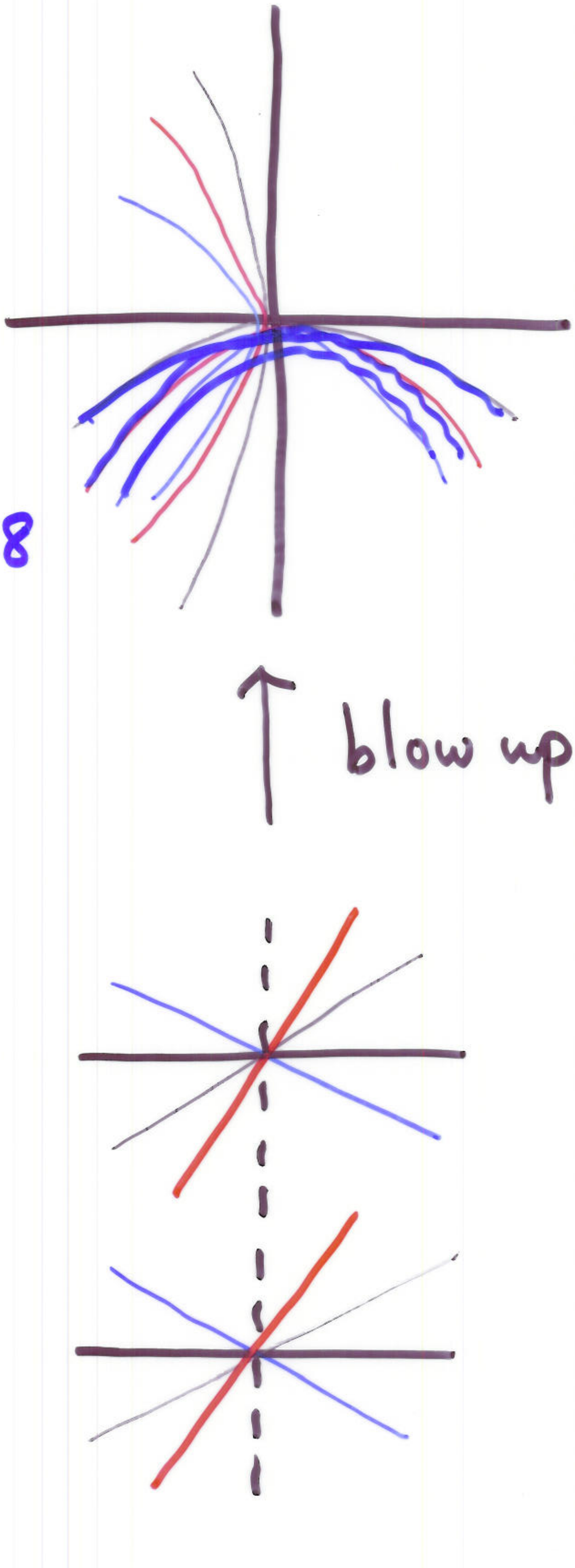
(Shioda)

We need 24 ellip. curves on $E \times E$.

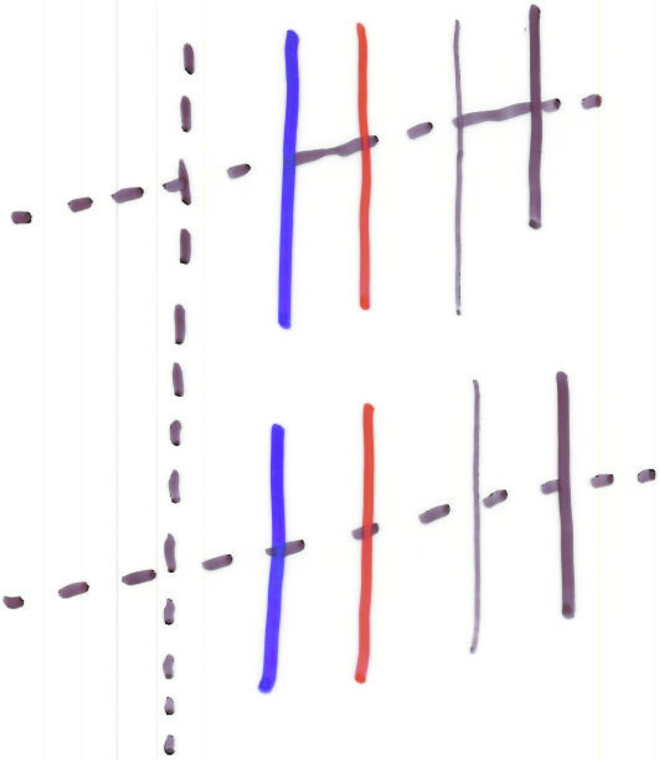


$6 + 18 + \underline{\underline{18}} = 42$

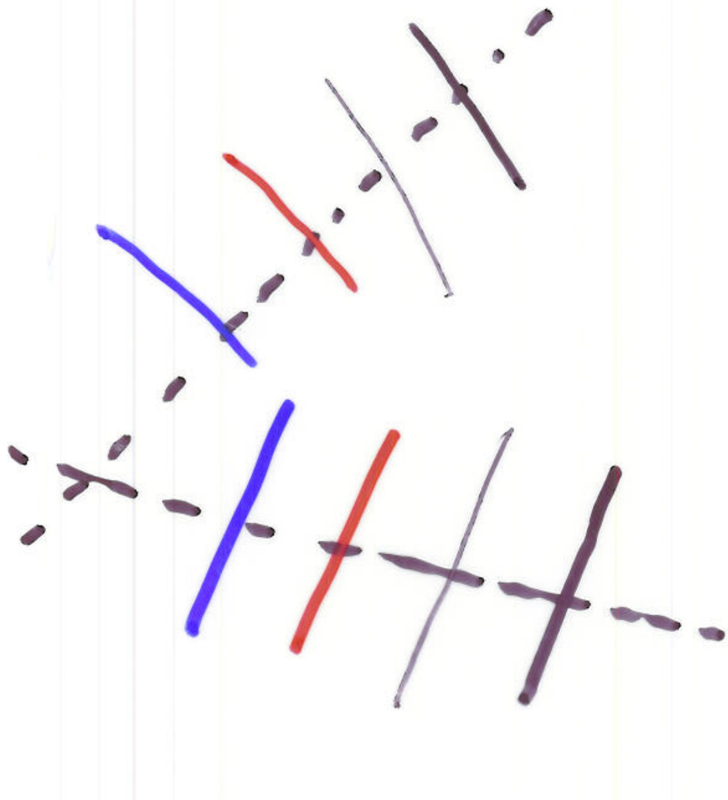
$$\frac{9 \times 6}{3} = 18$$

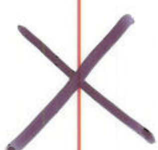


↑ blow ups



$\downarrow$ $/ \langle \sigma \times \sigma^2 \rangle$ contract





§. Lattice theory

① Reflective lattices

S : even lattice with sign $(1, r)$

hyperbolic

Ψ

$r \quad r^2 < 0$

reflection

$s_r : x \longmapsto x - \dfrac{2\langle x, r\rangle}{\langle r, r\rangle} r$

If $\dfrac{2}{\langle r, r\rangle} r \in S^*$, then

$s_r \in O(S)$

$W(S) = \langle \text{reflections} \rangle \lhd O(S)$

S is called reflective

if $[O(S) : W(S)] < +\infty$

Esselman : S : even, hyperbolic reflective

$\Rightarrow$ $\mathrm{rank}(S) \leq 20$ or 22

S_X is only one known example of such lattices of rank 22. (Borcherds)

(up to scales ...)

② Leech lattice

$\mathrm{II}_{1,25}$: even, unimodular
hyperbolic

NOT : reflective

A fund. domain of $W(\mathrm{II}_{1,25})$
can be written in terms
of the Leech lattice Λ.

Λ : even, unimodular
neg. definite
without $\underbrace{(-2)\text{-vectors}}_{\text{roots}}$

$U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ 18

$$\mathrm{II}_{1,25} = U \oplus \Lambda$$

$$\begin{array}{ccc} \mathrm{II}_{1,25} & & \Lambda \\ \Psi & & \Psi \\ x & = & (m, n, \lambda) \end{array}$$

$$x^2 = 2mn + \lambda^2$$

$$\rho = (1, 0, 0)$$

$$\mathrm{II}_{1,25} \ni r = (m, n, \lambda)$$

$$r^2 = -2 \Rightarrow n \neq 0$$

$$(\Lambda \not\ni (-2))$$

r : Leech root

$\Longleftrightarrow \langle r, \rho \rangle = 1$

$(r = (m, 1, \lambda))$

$$\Delta = \{\text{Leech roots}\} \overset{1:1}{\longleftrightarrow} \Lambda$$

$$\begin{array}{ccc} \text{Leech roots} & & \Lambda \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (-1-\frac{\lambda^2}{2}, 1, \lambda) & \longleftarrow\!\shortmid & \lambda \end{array}$$

$$\mathrm{II}_{1,25} \otimes \mathbb{R}$$

$$\cup$$

$\mathcal{P}^+ :=$ a conn. comp. of $\{x^2 > 0\}$

$$C := \left\{ x \in P^+ \;\middle|\; \begin{array}{l} \langle x, r \rangle > 0 \\ \forall r : \text{Leech root} \end{array} \right\}$$

Thm (Conway)

C is a fund. domain of $W(\mathrm{II}_{1,25})$.

$$S_X \overset{\exists}{\hookrightarrow} \mathrm{II}_{1,25}$$

$$S_X \otimes \mathbb{R} \supset \underset{\substack{\cup\!\!\!\!\!\!\cup \\ \text{ample}}}{P^+(S_X)} \text{ pos. cone}$$

$$S_X \otimes \mathbb{R} \subset \mathrm{II}_{1,25} \otimes \mathbb{R}$$

$$D := C \cap P^+(S_X) \qquad \subset \qquad C$$

a finite polyhedron (Borcherds)

◎ D consits of

42 faces defined by 42 (-2)-vectors

and

168 faces def. by 168 (-4)-vectors

- $\mathrm{Aut}(D) \cong PSL(3, \mathbb{F}_4) \cdot D_{12}$
- Projection of ρ = hyperplane section $X \subset \mathbb{P}^2 \times \mathbb{P}^2$
- 168 (-4)-vectors

$\updownarrow$ 1:1

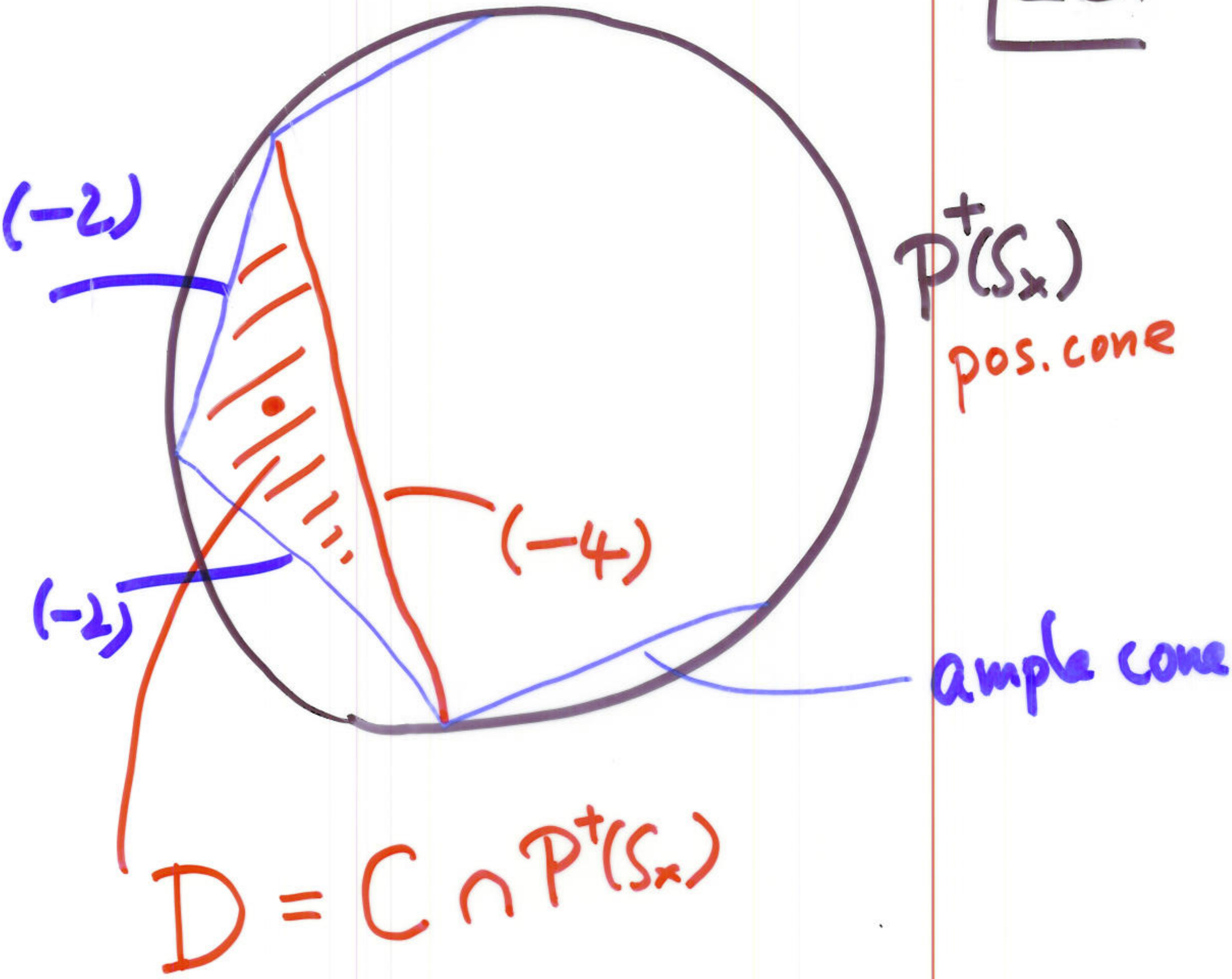
- 6 points on $\mathbb{P}^2(\mathbb{F}_4)$ not collinear

$\updownarrow$ 1:1

- 168 quintic Cremona transf.

$\updownarrow$ 1:1

- 168 involutions of X



Thm

$$\mathrm{Aut}(X) = \left\langle \begin{array}{l} PGL(3, \mathbb{F}_4), \\ \text{switch } S, \\ 168 \text{ involutions} \end{array} \right\rangle$$

Remark

$$S = S_X$$

$S^*(2) \cong$ the Picard lattice of s.s. K3 with $\sigma = 10$.

$$S^*(2) \ni \begin{array}{l} 168\ (-2) \\ 42\ (-4) \end{array}$$

Char = 3

$Y \supset$ 112 lines

$Y : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$

$\circlearrowleft$

$PGU(4, \mathbb{F}_9)$ a herm. form / $\mathbb{F}_9$

Y : s.s. K3 with $\sigma = 1$ in char 3

$$\begin{cases} 112 \ (-2)\text{-vectors} \longleftrightarrow 112 \text{ lines} \\ 5184 \ (-6)\text{-vectors} \longleftrightarrow \text{invol.} \\ 648 \ (-12)\text{-vectors} \longleftrightarrow \text{invol.} \end{cases}$$