Hochschild homology and cohomology of admissible subcategories

Alexander Kuznetsov

Steklov Math Institute

Moscow, Russia

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- examples.

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Solution: Replace $\mathsf{EndFun}(\mathcal{C})$ by an appropriate category which has necessary structure.

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Theorem (Hochschild–Kostant–Rosenberg):

There are isomorphisms

$$\mathsf{HH}^k(X) = \bigoplus_{q+p=k} H^q(X, \Lambda^p T_X),$$

$$\mathsf{HH}_k(X) = \bigoplus_{q-p=k} H^q(X, \Omega^p_X),$$

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Example: If $E \in \mathcal{T}$ is an ***exceptional object*** ($\mathsf{Hom}(E, E) = \mathsf{k}$, $\mathsf{Ext}^{\neq 0}(E, E) = 0$), then the functor $\alpha : \mathcal{D}^b(\mathsf{k}) \to \mathcal{T}$, $V^\bullet \mapsto V^\bullet \otimes_{\mathsf{k}} E$ is fully faithful and admits adjoint functors $\alpha^* : F \mapsto \mathsf{RHom}(F, E)^*$, $\alpha^! : F \mapsto \mathsf{RHom}(E, F)$,

if $\mathcal{T}$ has finite-dimensional Hom-spaces.

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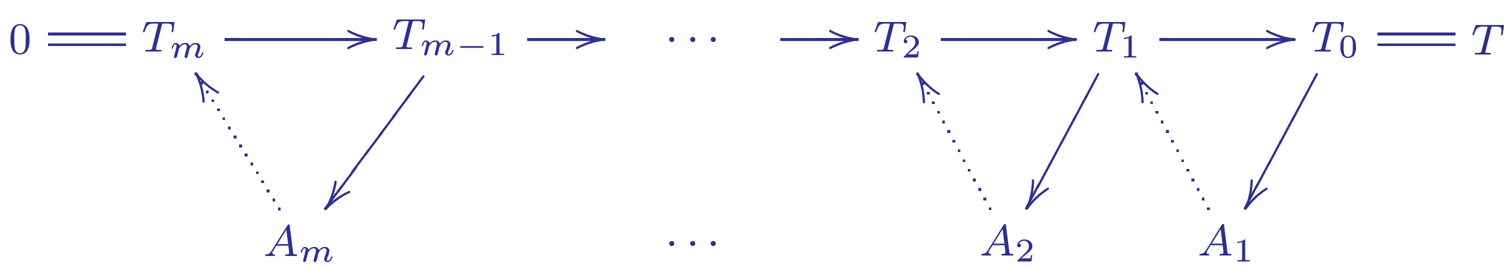

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Semiorthogonal decompositions

Definition: A ***semiorthogonal decomposition*** of $\mathcal{T}$ is a collection of subcategories $\mathcal{A}_1, \dots, \mathcal{A}_m \subset \mathcal{T}$, s.t.

- $\mathsf{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$ (***semiorthogonality***), and
- for any object $T \in \mathcal{T}$ there is a chain of maps $0 = T_m \to T_{m-1} \to \cdots \to T_1 \to T_0 = T$, such that

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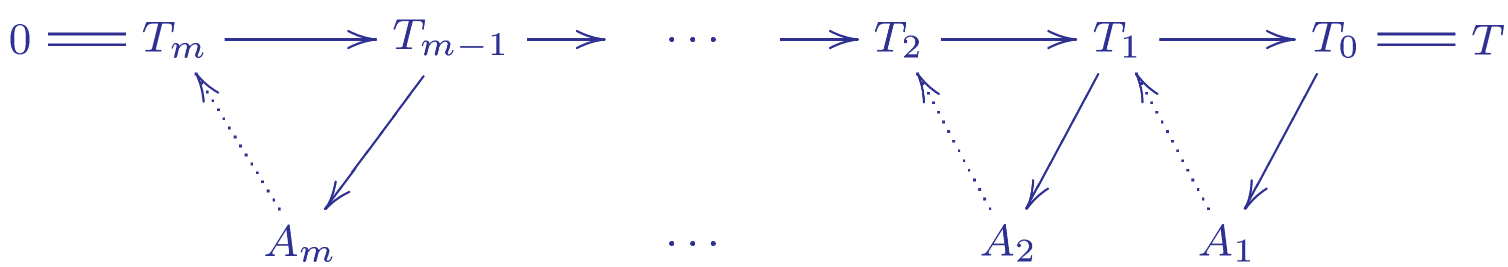

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Notation: $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ — ***s.o.d.***

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Answer: The kernel of the projection functor $\mathcal{D}^b(X) \to \mathcal{A}_i$.

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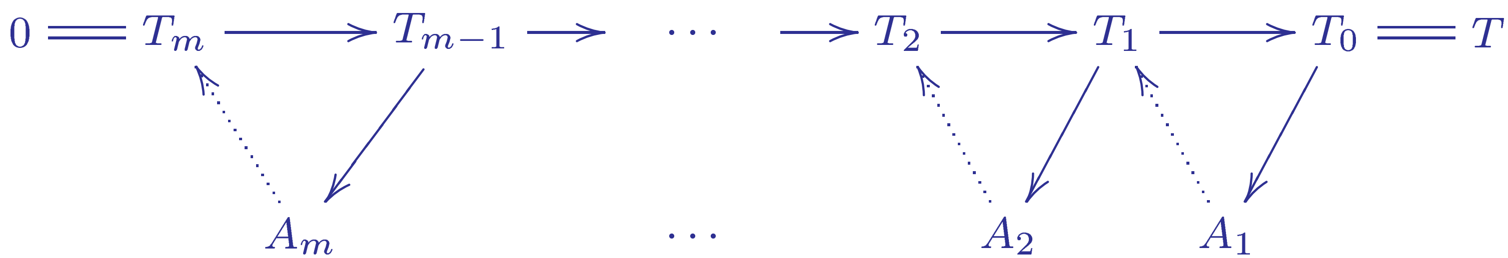

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 & & \nwarrow & & \swarrow & & & & \nwarrow & \swarrow & \nwarrow & \swarrow & & & \\
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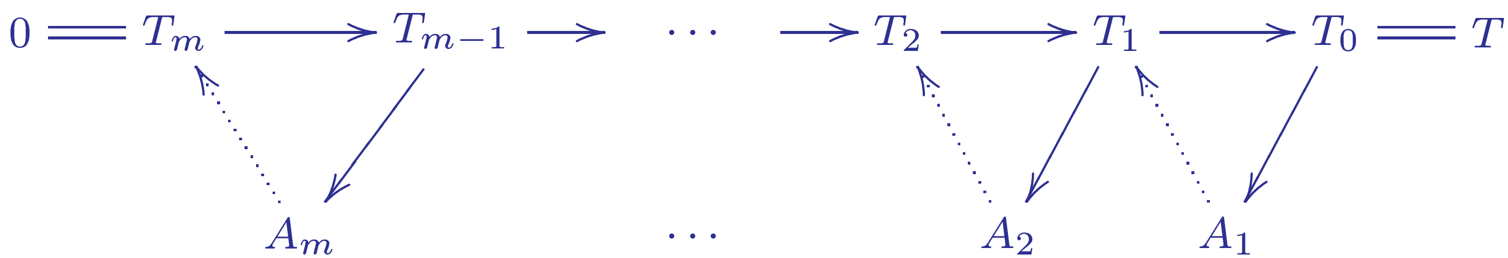

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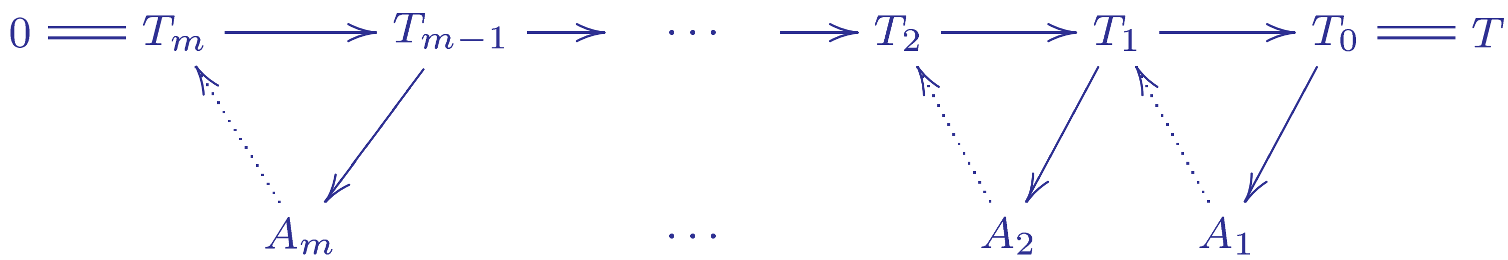

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Theorem: There is an object $P_i \in \mathcal{D}^b(X \times X)$ s.t. $\alpha_i \cong \Phi_{P_i}$ (Fourier–Mukai functor).

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Corollary:

$\mathsf{HH}^\bullet(B_i) = \mathsf{HH}^\bullet(\mathcal{A}_i) \qquad \mathsf{HH}_\bullet(B_i) = \mathsf{HH}_\bullet(\mathcal{A}_i)$.

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Remark: A nonzero element in $\mathsf{HH}_{-d}(\mathcal{A})$
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In this case $\mathsf{HH}_\bullet(\mathcal{A})$ is a free module over $\mathsf{HH}^\bullet(\mathcal{A})$, $\mathsf{HH}^\bullet(\mathcal{A}) = \mathsf{Ext}^\bullet(P, P)$ and $0 \neq \mathsf{id}_P \in \mathsf{Hom}(P, P)$.