Representations out of polydifferentials and the KZ-system

On the occasion of the sixtieth birthyear of Alessandro, Ciro and Fabrizio

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Outline

Input data for the KZ system

- $\mathfrak{g}$: a simple finite dimensional complex Lie algebra.
- $C \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$: represents a nondegenerate $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$; denote by $q^C : \mathfrak{g}^* \to \mathbb{C}$ the associated quadratic form $q^C(u) := \frac{1}{2}C(u \otimes u)$.
- $V_1, \dots, V_n$ finite dimensional irreps of $\mathfrak{g}$.

Construction of the KZ system

$\mathbf{V} := V_1 \otimes \cdots \otimes V_n$ (a representation of $\mathfrak{g}$).
$C^{(\nu,\mu)}$ $(1 \le \nu < \mu \le n)$: endomorphism of $\mathbf{V}$ by letting C act through the tensor factors V_ν and V_μ; commutes with the $\mathfrak{g}$-action, so preserves $\mathbf{V}^{\mathfrak{g}}$.
Next consider

$$\begin{aligned} U_n :=& \{(p_1, \dots, p_n) \in \mathbb{C}^n \mid p_i\text{'s distinct}\}/(\text{transl grp } \mathbb{C}) \\ =& \operatorname{Inj}(\underline{n}, \mathbb{C})/(\text{transl}), \end{aligned}$$

where $\underline{n} := \{1, \dots, n\}$.
Note that $(z_\nu - z_\mu)^{-1}$ is a regular function on U_n.

Basic theorem about the KZ system

KZ connection ∇^C_{KZ}: on the trivial bundle over U_n with fiber $\mathbf{V}^{\mathfrak{g}}$, given by the End($\mathbf{V}^{\mathfrak{g}}$)-valued 1-form

$$\sum_{1 \leq \nu < \mu \leq n} \frac{d(z_\nu - z_\mu)}{z_\nu - z_\mu} \otimes (C^{(\nu,\mu)}|\mathbf{V}^{\mathfrak{g}}),$$

Theorem (well-known)

∇^C_{KZ} *is flat (and has logarithmic singularities at infinity).*

Yields local system $\mathbb{KZ}^C(V_1, \ldots, V_n) \subset \mathcal{O}_{U_n} \otimes \mathbf{V}^{\mathfrak{g}}$, called the *KZ system*.

Basic question about the KZ system

Question

Is there a topological interpretation of the KZ system?

Only of interest when $\mathbf{V}^{\mathfrak{g}} \neq 0$. Assume this.
Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system of simple roots $\alpha_1, \ldots, \alpha_r$ in $\mathfrak{h}^*$.
If $\lambda_\nu \in \mathfrak{h}^*$ highest weight of V_ν, then assumption implies:

$$\lambda_1 + \cdots + \lambda_n = m_1\alpha_1 + \cdots + m_r\alpha_r$$

for certain $m_k \in \mathbb{Z}_{\geq 0}$. Let $m := m_1 + \cdots + m_r$.

A moduli space

Choose a finite set M of m elements and put

$$U_{n,M} := \mathrm{Inj}(\underline{n} \sqcup M, \mathbb{C})/(\mathrm{transl}).$$

Modular interpretation for U_n as space of triples

$$(C, z : \underline{n}_\infty \hookrightarrow C, \omega)$$

with $C \cong \mathbb{P}^1$, $\underline{n}_\infty := \{1, \dots, n, \infty\}$ and ω a nowhere zero differential on $C - \{z(\infty)\}$. Similarly, $U_{n,M}$ is the moduli space of triples

$$(C, z \sqcup t : \underline{n}_\infty \sqcup M \hookrightarrow C, \omega)$$

Partial completion of the moduli space

Forgetting t defines $p : U_{n,M} \to U_n$.
Factors through a partial Deligne-Mumford compactification:

$$p : U_{n,M} \subset \hat{U}_{n,M} \xrightarrow{\hat{p}} U_n$$

with $\hat{p}$ proper; this allows for stable pointed genus zero curves $(C, z \sqcup t : \underline{n}_\infty \sqcup M \hookrightarrow C)$ provided that after forgetting t, (C, z) can be contracted to define an element of U_n.

Boundary divisors

$\hat{U}_{n,M}$ is smooth and $\Delta_{n,M} := \hat{U}_{n,M} - U_{n,M}$ is a simple normal crossing divisor. Its irr components come in 3 types, each indexed by nonempty subsets X of M, telling us how $\underline{n}_\infty \sqcup M$ is split into two parts by a stable curve with two irreducible components:

$$\Delta(X) : X \Big| (M - X) \sqcup \underline{n}_\infty, \ \#(X) \geq 2,$$

$$\Delta_\infty(X) : X \sqcup \{\infty\} \Big| (M - X) \sqcup \underline{n},$$

$$\Delta_\nu(X) : X \sqcup \{\nu\} \Big| (M - X) \sqcup \underline{n}_\infty - \{\nu\}.$$

A remarkable differential

Now let $\pi : M \to \{1, \dots, r\}$, $i \mapsto \bar{i}$, be a map such that the fiber over k, M_k, has m_k elements. Consider the following multivalued function on $U_{n,M}$:

$$F^C := \prod_{\substack{i \in M \\ \nu=1,\dots,n}} (t_i - z_\nu)^{C(\alpha_i, \lambda_\nu)} \prod_{\substack{i,j \in M \\ i \neq j}} (t_i - t_j)^{-C(\alpha_i, \alpha_j)/2}.$$

Its logarithmic differential is:

$$\eta^C := d \log F^C = \sum_{\substack{i \in M \\ \nu=1,\dots,n}} C(\alpha_{\bar{i}}, \lambda_\nu) \frac{d(t_i - z^\nu)}{t_i - z^\nu} - \frac{1}{2} \sum_{\substack{i,j \in M \\ i \neq j}} C(\alpha_{\bar{i}}, \alpha_{\bar{j}}) \frac{d(t_i - t_j)}{t_i - t_j}.$$

A flat connection

Put $d^C := d - \eta^C = F^C d(F^C)^{-1}$ and regard this as a connection on $\mathcal{O}_{U_{n,M}}$. Its locally flat sections define local system $\mathbb{L}^C \subset \mathcal{O}_{U_{n,M}}$ (determinations of F^C yield local generators). It is unitary if C defined over $\mathbb{R}$ (then determinations of F^C have norm 1). Monodromy of $\mathbb{L}^C$ around $\Delta_{n,M}$ given by residues:

Lemma

Let $\rho \in \mathfrak{h}^*$ be the half sum of the positive roots; for $X \subset M$ write $\alpha_X := \sum_{x \in X} \alpha_{\bar{x}}$. Then

- $\mathrm{Res}_{\Delta(X)}\, \eta^C = q^C(\rho) - q^C(\rho - \alpha_X)$,
- $\mathrm{Res}_{\Delta_\infty(X)}\, \eta^C = q^C(\rho) - q^C(\rho + \alpha_X)$,
- $\mathrm{Res}_\nu\, \Delta(X)\eta^C = q^C(\rho + \lambda_\nu) - q^C(\rho + \lambda_\nu - \alpha_X)$

A local system

Now let

$$U_{n,M} \overset{j}{\hookrightarrow} U \hookrightarrow \hat{U}_{n,M}$$

be obtained by removing some irreducible components of $\Delta_{n,M}$ from $\hat{U}_{n,M}$. Namely: remove (resp. do *not* remove) a component if η^C there a residue in $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Z}_{<0}$) and if the residue is not in $\mathbb{Z}$ do as you like.
A topologically defined local system on U_n is given by

$$\mathbb{H}^C(\lambda_1, \dots, \lambda_n) := R^m(\hat{p}|U)_* j_! \mathbb{L}^C.$$

Note it comes with an action of $\mathfrak{S}(M)_\pi = \mathfrak{S}(M_1) \times \cdots \times \mathfrak{S}(M_r)$.
Stalk at $z \in U_n$ is $H^m(U(z), U(z) \cap \Delta_{n,M}; \mathbb{L}^C)$.

Topological interpretation of the KZ system

Theorem

The KZ local system $\mathbb{KZ}^C(V_1, \ldots, V_n)$ is naturally embedded as a subsystem of

$$\mathrm{Hom}_{\mathfrak{S}(M)_\pi}\left(\wedge^m(\mathbb{C}^M), \mathbb{H}^C(\lambda_1, \ldots, \lambda_n)\right).$$

Remarks:
1. There is a precise description for this as a subsystem.
2. Schechtman-Varchenko constructed a map from the KZ system to a similarly defined local system, but in which U is replaced by $U_{n,M}$. It factors through ours, but is probably not always injective.

The algebraic case

If C defined over $\mathbb{Q}$, then we can be more concrete: F^C is algebraic in the sense that it becomes univalued on an finite covering $U^\natural_{n,M} \to U_{n,M}$. This covering is abelian: if $\mathbb{L}^C$ has monodromy group $\mu_s \subset \mathbb{C}^\times$, then it is a μ_s-covering and $\mathbb{L}^C$ sits in the direct image of $\mathbb{C}_{U^\natural_{n,M}}$ on $U_{n,M}$ as the eigenspace for the tautological character $\chi : \mu_s \subset \mathbb{C}^\times$. Let

$$U^\natural_{n,M} \subset U^\natural \overset{j^\natural}{\hookrightarrow} \hat{U}^\natural_{n,M}$$

have the obvious meaning and let $p^\natural : U^\natural \to U_n$ be the projection. Then

$$\mathbb{H}^C(\lambda_1, \dots, \lambda_n) = (R^m p^\natural_* j^\natural_! \mathbb{C})^\chi.$$

The WZW subsystem

This occurs for the WZW systems (these involve a particular choice of C).
Let $\theta \in \mathfrak{h}^*$ be the highest root, $\check{\theta} \in \mathfrak{h}$ the corresponding coroot.
Fix (a *level*) $\ell \in \{1, 2, \dots\}$ and take $C = C_\ell$ with $C_\ell \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ characterized by $q_{C_\ell}(\theta) = (\rho(\check{\theta}) + 1 + \ell)^{-1}$.
Choose $e_\theta \in \mathfrak{g}_\theta$ a generator of the corresponding root space and let $\mathcal{E}_\theta \in \mathcal{O}_{U_n} \otimes_{\mathbb{C}} \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ be defined by

$$\mathcal{E}_\theta(\mathbf{z}) := \sum_{\nu=1}^{n} 1 \otimes \cdots \otimes z_\nu X_\theta \otimes \cdots \otimes 1.$$

WZW-subsystem

Proposition (Beilinson-Feigin)

Then the subsheaf of $\mathcal{O}_{U_n} \otimes \mathbf{V}^{\mathfrak{g}}$ *defined by*

$$\mathcal{W}_\ell(V_1, \ldots, V_n) := \ker(\mathcal{E}_\theta^{1+\ell} | \mathcal{O}_{U_n} \otimes \mathbf{V}^{\mathfrak{g}})$$

is flat for $\nabla_{KZ}^{c_\ell}$ *and hence defines a local subsystem* $\mathbb{W}_\ell(V_1, \ldots, V_n) \subset \mathbb{KZ}^{c_\ell}(V_1, \ldots, V_n)$.

$\mathbb{W}_\ell(V_1, \ldots, V_n)$ is called the *Wess-Zumino-Witten system of level* ℓ.

A conjecture

Conjecture (1)

$\mathbb{W}_\ell(V_1, \dots, V_n)$ *is a unitary system*

This would in fact follow from the truth of

Conjecture (2)

$\mathbb{W}_\ell(V_1, \dots, V_n)$ *maps under the embedding described above to the direct image of* $\hat{p}^\natural_* \omega_{\hat{U}^\natural_{n,M}/U_n}$.

For the flatness of $\mathbb{W}_\ell(V_1, \dots, V_n)$ would then make it a summand of a polarized *rigid local system*, purely of Hodge type $(m, 0)$.
This last conjecture was proved by Ramadas in case $\mathfrak{g} = \mathfrak{sl}(2)$.

Serre presentation of $\mathfrak{g}$

Let $(c_{k,l})_{k,l=1}^r$ be the Cartan matrix of $\mathfrak{g}$:
$\mathfrak{g}$ has generators $e_1, \dots, e_r, f_1, \dots, f_r$ subject to the relations $[e_k, f_l] = 0$ for $k \neq l$ and if we put $\check{\alpha}_k := [e_k, f_k]$, then

$$[\check{\alpha}_k, e_l] = c_{k,l} e_l, \quad [\check{\alpha}_k, f_l] = -c_{k,l} f_l, \quad [\check{\alpha}_k, \check{\alpha}_l] = 0.$$

and also imposing the *Serre relations*

$$\mathrm{ad}(f_k)^{1-c_{k,l}} f_l = 0, \quad \mathrm{ad}(e_k)^{1-c_{k,l}} e_l = 0 \quad (k \neq l)$$

$\tilde{\mathfrak{g}}$: the Lie algebra that we get if we suppress only the last set of Serre relations.
$\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra spanned by the $\check{\alpha}_k$'s (also a subalgebra of $\tilde{\mathfrak{g}}$).

Polydifferentials on a product of $\mathbb{P}^1$'s

Fix a set $\mathcal{M}$ (soon to be countably inifinite).
Let $\mathcal{B}$ be the graded $\mathbb{C}$-vector space of the relative rational polydifferentials on $(\mathbb{P}^1)^{\mathcal{M}}_{\mathbb{C}} : (\mathbb{P}^1)^{\mathcal{M}} \times \mathbb{C} \to \mathbb{C}$ which is $\mathbb{C}$-spanned by the forms

$$\zeta_I(z) := \frac{dt_{i_N} dt_{i_{N-1}} \cdots dt_{i_1}}{(t_{i_N} - t_{i_{N-1}}) \cdots (t_{i_2} - t_{i_1})(t_{i_1} - z)}.$$

where $I = (i_N, i_{N-1}, \dots, i_1)$ runs over the finite sequences in $\mathcal{M}$ (we stipulate $\zeta_\emptyset = 1$).
Notice that we get zero unless the sequence I is without repetition.

A shuffle algebra of polydifferentials

$\hat{\mathcal{B}}^d$: the space of (possibly infinite) sums of these relative polydifferentials of degree N, $\hat{\mathcal{B}} := \oplus_{d=0}^{\infty} \hat{\mathcal{B}}^d$.

Lemma (Shuffle rule)

The graded vector space $\mathcal{B}$ is closed under product (it is a shuffle algebra): for finite sequences in I and J in $\mathcal{M}$,

$$\zeta_I \zeta_J = \sum_{K \text{ a shuffle of } I \text{ and } J} \zeta_K$$

Algebra of invariants in $\hat{\mathcal{B}}$

Now assume $\mathcal{M}$ equipped with a map $\Pi : \mathcal{M} \to \{1, \ldots, r\}$, $i \mapsto \bar{i}$ such that every fiber $\mathcal{M}_k$ is countably infinite.
The group $\mathfrak{S}_\Pi = \mathfrak{S}(\mathcal{M}_1) \times \cdots \times \mathfrak{S}(\mathcal{M}_r)$ acts in $\hat{\mathcal{V}}$. Additive generators for $\hat{\mathcal{V}}^{\mathfrak{S}_\Pi}$ are indexed by finite sequences S in $\{1, \ldots, r\}$:

$$\zeta(S) := \sum_{\Pi(I)=S} \zeta_I.$$

We will produce the irreducible highest weight representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ inside $\hat{\mathcal{B}}^{\mathfrak{S}_\Pi}$ with $1 \in \hat{\mathcal{B}}^{\mathfrak{S}_\Pi}$ as highest weight vector.

The operators $\tilde{f}_k$

Let $\lambda \in \mathfrak{h}^*$. For $k \in \{1, \dots, r\}$ define an operator $\tilde{f}_k$ in the space of rational polydifferentials on $(\mathbb{P}^1)^{\mathcal{M}}_{\mathbb{C}}$ by

$$\tilde{f}_k := \sum_{i \in \mathcal{M}_k} \Big(\frac{\lambda(\check{\alpha}_k) dt_i}{t_i - z} - \sum_{j \in \mathcal{M} - \{i\}} c_{k,\bar{j}} \frac{dt_i dt_j}{t_i - t_j} \iota_{\partial/\partial t_j} \Big),$$

Here dt_i is the multiplication operator in the space of these polydifferentials and by $\iota_{\partial/\partial t_i}$ its adjoint (which acts in the ith tensor factor by sending dt_i to 1 and 1 to 0). So for a finite subset $X \subset \mathcal{M}$, we have

$$\tilde{f}_k \Big(\prod_{x \in X} dt_x \Big) = \sum_{i \in \mathcal{M}_k \setminus X} \Big(\frac{\lambda(\check{\alpha}_k)}{t_i - z} - \sum_{x \in X} \frac{c_{k,\bar{x}}}{t_i - t_x} \Big) dt_i \prod_{x \in X} dt_x .$$

The operators $\tilde{e}_k$

One checks with the help of the shuffle rule that

$$\tilde{f}_k(\zeta(S)) = \sum_{S=S''S'} (\lambda(\check{\alpha}_k) - c_{k,S'})\zeta(S''kS').$$

so $\tilde{f}_k$ preserves $\hat{\mathcal{B}}^{\mathfrak{S}_\Pi}$.
Define $\tilde{e}_k : \hat{\mathcal{B}}^{\mathfrak{S}_\Pi} \to \hat{\mathcal{B}}^{\mathfrak{S}_\Pi}$ by

$$\tilde{e}_k(\zeta(S)) := \begin{cases} \zeta(S') & \text{if } S = kS' \\ 0 & \text{otherwise} \end{cases}$$

Has an interpretation as a residue taken at $t_i = \infty$, $i \in \mathcal{M}_k$.

Polydifferential realization of $\mathcal{V}(\lambda)$

Theorem

The operators $\tilde{e}_k, \tilde{f}_k$, $k = 1, \ldots, r$, define a representation of $\tilde{\mathfrak{g}}$ on $\hat{\mathcal{B}}^{\mathfrak{S}_n}$ and $\tilde{\mathfrak{g}}$ acts on the $\tilde{\mathfrak{g}}$-submodule $\mathcal{V}(\lambda)$ generated by 1 through the irrep of $\mathfrak{g}$ with highest weight λ and highest weight vector 1.

Towards a tensor product of of irreps 1

Let $\lambda_1, \dots, \lambda_n$ be dominant weights as before. Work now on

$$(\mathbb{P}^1)^{\mathcal{M}}_{\mathbb{C}^n} : (\mathbb{P}^1)^{\mathcal{M}} \times \mathbb{C}^n \to \mathbb{C}^n.$$

For n sequences $I_1, \dots, I_n$ in $\mathcal{M}$ we have the relative polydifferential

$$\omega_{I_1}(z_1)\omega_{I_2}(z_2)\cdots\omega_{I_n}(z_n).$$

on $(\mathbb{P}^1)^{\mathcal{M}}_{\mathbb{C}^n}$ ($z_1, \dots, z_n$ are coordinates of $\mathbb{C}^n$). It is zero unless the concatenated sequence $I_1 \cdots I_n$ is without repetition.

Towards a tensor product of of irreps 2

$\mathcal{B}_n$: the graded vector space spanned by these polydifferentials.
$\hat{\mathcal{B}}_n = \oplus_d \hat{\mathcal{B}}_n^d$ with $\hat{\mathcal{B}}_n^d$ the completion of $\mathcal{B}_n^d$ which allows for infinite sums.
Given n sequences $S^\bullet = (S_1, \ldots, S_n)$ in $\{1, \ldots, r\}$, we observe that

$$\prod_{\nu=1}^{n} \zeta(S_\nu)(z_\nu) = \sum_{\substack{\overline{I_\nu}=S_\nu \\ \nu=1,\ldots,n}} \zeta_{I_1}(z_1) \cdots \zeta_{I_n}(z_n) \in \hat{\mathcal{B}}_n^{\mathfrak{S}_\Pi}.$$

These elements form a $\mathbb{C}$-basis of $\hat{\mathcal{B}}_n^{\mathfrak{S}_\Pi}$.

Towards a tensor product of irreps 3

So the above factorization defines an isomorphism

$$\hat{\mathcal{B}}_n^{\mathfrak{S}_\sqcap} \cong \hat{\mathcal{B}}^{\mathfrak{S}_\sqcap} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \hat{\mathcal{B}}^{\mathfrak{S}_\sqcap}.$$

It is clear that $\mathcal{V}(\lambda_\bullet) = \mathcal{V}(\lambda_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{V}(\lambda_n)$ is the smallest subspace of $\hat{\mathcal{B}}_n^{\mathfrak{S}_\sqcap}$ that contains 1 and is invariant under the operators $\tilde{f}_k^{(\nu)}$ and $\tilde{e}_k^{(\nu)}$. It is the tensor product of n highest weight representations.
It follows from the residue interpretation of $\tilde{e}_k$ that:

Polydifferential realization of a tensor product of irreps

Theorem

The space of $\mathfrak{g}$-invariants $\mathcal{V}(\lambda_\bullet)^{\mathfrak{g}}$ is the space of degree m polydifferentials in $\mathcal{V}(\lambda_\bullet)_m$ that are regular along every hyperplane at infinity $(t_i = \infty)$, $i \in \mathcal{M}$.

If we now choose $M \subset \mathcal{M}$ such that $\pi := \Pi|M$, then we see that $\mathcal{V}(\lambda_\bullet)_m$ can be realized as a space of polydifferentials on $\mathbb{P}^M_{\mathbb{C}^n}$. This leads to the interpretation of the KZ system in terms of $U_{n,M}/U_n$ as given above.