Classification & Construction of general singular fibers of proper holomorphic Lagrangian fibrations

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References

1) ArXiv: 0710.2376

2) ArXiv: 0907.4869

3) ArXiv: 1007.2043

Definition

(M, σ) : (holomorphic) symplectic manifold

$\Leftrightarrow \begin{cases} M : \text{complex manifold} \\ \quad (\text{connected, not necessarily compact}) \\ \sigma : d\text{-closed holomorphic 2-form on } M \\ \quad \text{s.t. everywhere non-degenerate} \end{cases}$

$$TM \times TM \longrightarrow \mathcal{O}_M$$
$$(v_1, v_2) \longmapsto \sigma(v_1, v_2)$$

at $\forall x \in M$ non-deg. alternating

$\Rightarrow$ 1) $\dim_{\mathbb{C}} M = 2d$ even

2) $TM \underset{\jmath_\sigma := \iota_\sigma^{-1}}{\overset{\iota_\sigma \atop \cong}{\rightleftarrows}} T^*M \quad v \mapsto \sigma(v, -)$

- $V \subset (M^{2d}, \sigma)$ closed subvariety

V : Lagrangian $\Leftrightarrow \begin{cases} \sigma|V_{reg} \equiv 0 & \text{— (i)} \\ \dim V = d & \text{— (ii)} \end{cases}$

<u>Rem.</u> (i) $\Rightarrow \dim V \leqq d$.

$f : (M^{2d}, \sigma) \twoheadrightarrow B$ (B : smooth)

proper, surjective, holomorphic map with connected fibers

· f : Lagrangian fibration

$\Leftrightarrow$ $\forall$ irred. comp. of f is Lagrangian & Class $\mathcal{C}$ (bimeromorphic to Kähler mfd)

($\Rightarrow$ $\dim \forall$ fiber $= d$ & $\dim B = d$)

Rem. f : smooth at $x \in M$, $b = f(x) \in B$

$$\Rightarrow \quad \begin{array}{ccccccccc} & & T_b^*B & & & & & & \\ & & \wr\wr & & & & & & \\ 0 \to & & N^*_{M/M_b} \text{ at } x & \to & T_x^*M & \to & T_x^*M_b & \to 0 \\ & & & & c_{\sigma,x} \uparrow \downarrow \iota_{\sigma,x} & & & & \\ 0 \to & & T_xM_b & \to & T_xM & \to & N_{M/M_b,x} & \to 0 \\ & & & & & & \wr\wr & & \\ & & & & & & T_bB & & \end{array}$$

$$\underset{\text{Lagrangian}}{\Rightarrow} \begin{cases} T_b^*B \xrightarrow[\iota_{\sigma,x}]{\cong} T_xM_b \\ T_bB \xrightarrow[c_{\sigma,x}]{\cong} T_x^*M_b \end{cases}$$

In particular: For smooth $M_b = f^{-1}(b)$

$(z_1, \cdots, z_d)$ local coord. at $b \in B$

$\Rightarrow \iota_\sigma f^* dz_1, \cdots, \iota_\sigma f^* dz_d \in H^0(M_b, TM_b)$

linearly independent at $\forall x \in M_b$

$\Rightarrow TM_b \cong \mathcal{O}_{M_b}^{\oplus d} \Rightarrow T^*M_b \cong \mathcal{O}_{M_b}^{\oplus d}$

$\Rightarrow M_b \cong$ d-dim. complex torus

classe (Liouville's Thm)

Albanese map

Fact (H.-O.) $D := \{ b \in B \mid f^{-1}(b) : \text{singular} \} \subset B$ is pure of codimension 1 if $D \neq \phi$.

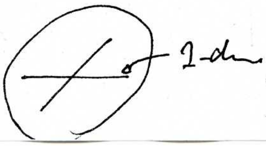
(Rem. <u>In general</u>, this is false even if $f : X \to Y$ is flat & X, Y smooth. (Mumford-Ramanujan))

<u>Question</u> How $f^{-1}(b)$ ($b \in D$: general) looks like?

Local Question on the base B May/Will assume:

$B = \Delta^d_{(z_1, \cdots, z_d)} \supset D = (z_d = 0) \cong \Delta^{d-1}$

$f^*D = \sum a_i H_i$, $n = GCD(a_i)$ n: multiplicity of gen. fiber over D



Ex1 (d = 1, Kodaira)

$f: S = S^2 \to \Delta^1 \ni 0$

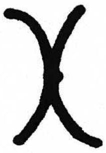
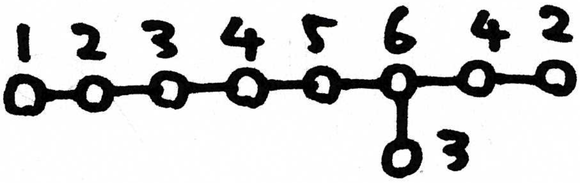
S : symplectic $\Leftrightarrow K_S \cong \mathcal{O}_S$

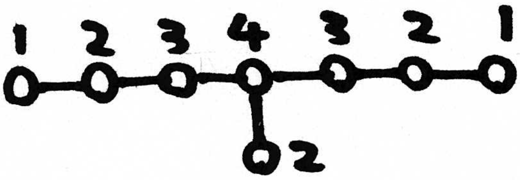
(& f : automatically Lagrangian)

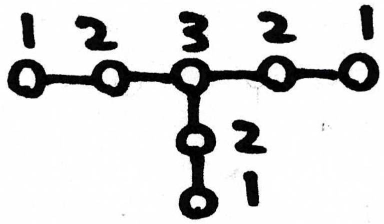
$\Rightarrow$ f : relatively minimal elliptic fibration without multiple fiber

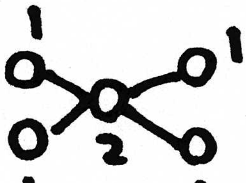
$f^{-1}(0)$ is either:

(I_0 : smooth elliptic curve), I_1  II  (irreducible)

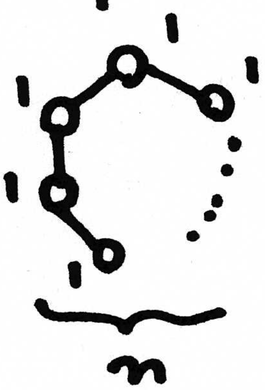
III  IV  II^* 

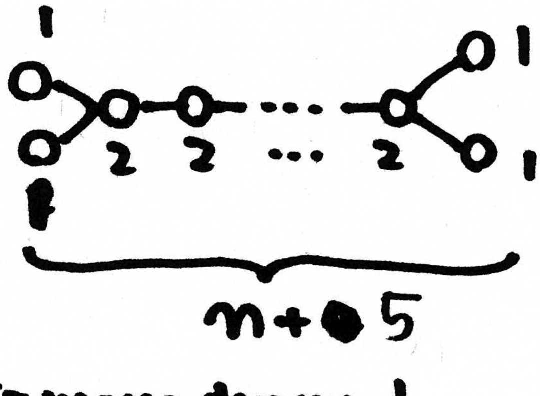
III^* 

IV^* 

I_0^* 

(finite monodromy)

I_n ($n \geqq 2$) 

I_n^* ($n \geqq 1$) 

(infinite monodromy)

Question

For $d \geqq 2$,

How $\left\{\begin{array}{l}\text{close to} \\ \text{different from}\end{array}\right\}$ Kodaira's classif.

are (general) singular fibers?

Aim

Give a fairly complete answer to this Question by finding Kodaira's singular fibers of an ell. surf. in a general singular fiber of Lagrangian fib. & by covering it by a natural $\mathbb{C}^{d-1}$ action (also see some different phenomena).

Rem

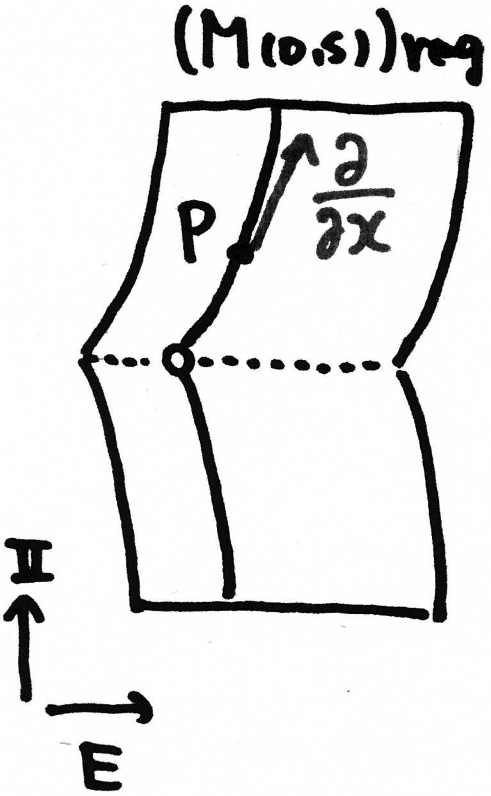
Different from Matsushita's view

Key Idea for the Aim

- characteristic vector field → (Kodaira's sing. fib. (+ one additional case)
- Hamiltonian vect. field → $\mathbb{C}^{d-1}$ action

↑ integration

· Characteristic vector field $v = \iota_\sigma(f^*dt)$ of discriminant hypersurface H <u>on $(M_{(0,s)})_{reg}$</u>:



$$\sigma|_H = dz \wedge ds \qquad (H = (f^*t = 0))$$

$$\Rightarrow \sigma|_H\left(\frac{\partial}{\partial x}, TH_{reg}\right) = 0$$

$\frac{\partial}{\partial s}$

$$\Rightarrow v \parallel \frac{\partial}{\partial x}$$

$\Rightarrow (M_{(0,s)})_{reg} \supset (\mathrm{II})_{reg}$ integral curve of v

&

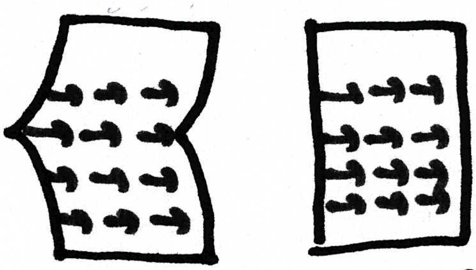
$\mathrm{II} = \left(\begin{array}{l}\text{closure of the integral} \\ \text{curve of } v\end{array}\right)$

· Hamiltonian vector field $u = \iota_\sigma(f^*ds)$ <u>on M</u>:

By obvious + shape of σ

$$\begin{cases} \sigma\left(\iota_\sigma(f^*ds), \frac{\partial}{\partial x}\right) = f^*ds\left(\frac{\partial}{\partial x}\right) = 0 \\ \sigma\left(\iota_\sigma(f^*ds), \frac{\partial}{\partial t}\right) = 0 \\ \sigma\left(\iota_\sigma(f^*ds), \frac{\partial}{\partial z}\right) = 1 \\ \sigma\left(\iota_\sigma(f^*ds), \frac{\partial}{\partial s}\right) = 0 \end{cases}$$

$$\Rightarrow \iota_\sigma(f^*ds) = \frac{\partial}{\partial z}$$



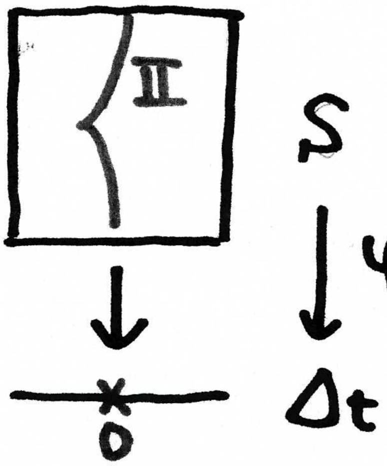
E direction

So: by the integrated action $\mathbb{C} \to \mathrm{Aut}(M/D^2)$

$$M_{(0,s)} = \coprod \mathrm{II}, \quad \mathbb{C} \curvearrowright \{\mathrm{II}\} \text{ transitive}$$

Ex2-1 ($d=2$)

$\varphi: S \to \Delta_t \quad y^2 = x^3 + t$

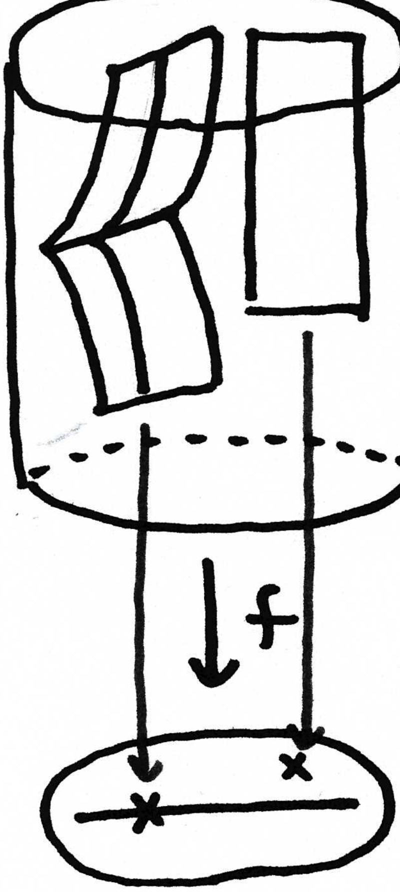


$\sigma_S = \frac{dx}{y} \wedge dt$

$M = S \times \underset{z}{E} \times \Delta_s \xrightarrow{f} \Delta^2_{(t,s)}, \quad ((x,y,t),z,s) \mapsto (t,s)$ (E: ell. curve)

$\sigma_M = \sigma_S + dz \wedge ds = \frac{dx}{y} \wedge dt + dz \wedge ds$

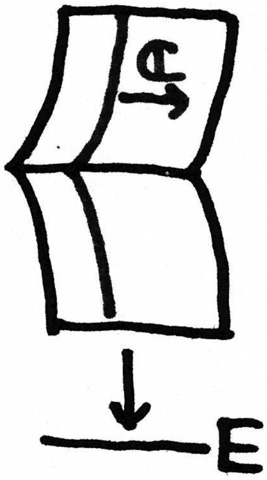
Lagrangian fibration



$t \neq 0: f^{-1}(t,s) = S_t \times E$ (ab. surf.)

$t = 0: f^{-1}(0,s) = \{ \times E$ (where $\{$ = II)

$f^{-1}(0,s) =$



foliated by II under $\mathbb{C}$-action

$\mathbb{C} \twoheadrightarrow E$

What are II $= \{$ & $\mathbb{C}$?

$D = (t=0) \subset \Delta^2_{(t,s)}$

$H = f^*D = (f^*t = 0)$

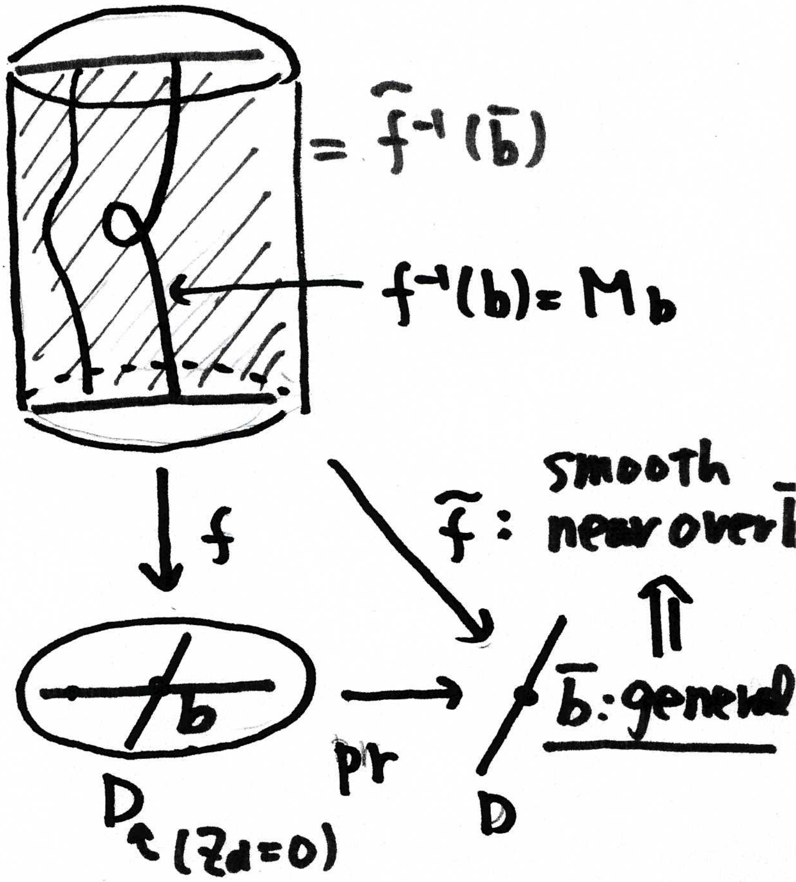
red. discriminant hypersurface

Hamiltonian vector fields & automorphisms

$$M^{2d} \xrightarrow{f} B = \Delta^d \xrightarrow{pr} D = \Delta^{d-1}, \quad \tilde{f}: M^{2d} \to D$$

$(z_1, \cdots, z_d)$ $\quad$ $(z_1, \cdots, z_{d-1})$

f – Lagrangian



$v_i := \iota_\sigma(f^* dz_i)$ $\quad i = 1, 2, \cdots, d-1$

(Hamiltonian vector fields)

⇒ ① linearly independent at each point near over $\bar{b}$ (by smoothness of $\tilde{f}$)

② v_i : tangent to fibers (as so is in gen. fibers)

③ $[v_i, v_j] = 0$ comm. (〃)

$\underset{①\&③}{\Rightarrow}$ $\mathbb{C}^{d-1} \underset{\text{Lie alg.}}{\xrightarrow{\cong}} \mathbb{C}\langle v_1, \cdots, v_{d-1} \rangle \subset H^0(M, TM)$

$\underset{②\&\text{integration}}{\Rightarrow}$ $\mathbb{C}^{d-1} \longrightarrow \mathrm{Aut}(M/B)$ $(\subset \mathrm{Aut}\, M)$

Hamiltonian action of $\mathbb{C}^{d-1}$

By ① : $\forall p \in M_x$, $\mathbb{C}^{d-1} \cdot p$ is $(d-1)$-dim & $\subseteq M_x$ — ④

An Application

- $X \subset (M_b)_{red}$ irreducible component

$\Rightarrow$ ($\dim X = d$ & $\mathbb{C}^{d-1} \curvearrowright X$ Hamiltonian action
orbit $\mathbb{C}^{d-1} \cdot p \subset X$ $(p \in X)$ $(d-1)$-dim (but not necessarily closed))

- $\nu : \hat{X} \longrightarrow X$ normalization

$\Rightarrow$ $\hat{X} \xrightarrow{\nu} X$ equivariant, $\mathbb{C}^{d-1} \circlearrowright \hat{X}$, $\mathbb{C}^{d-1} \circlearrowright X$

So: $\mathbb{C}^{d-1}\hat{p}$ $(\hat{p} \in \hat{X})$ is also $(d-1)$-dim.

$\Rightarrow$

- $\mathrm{Sing}\,\hat{X} = \phi$ ($\because$ $\mathbb{C}^{d-1} \curvearrowright \mathrm{Sing}\,\hat{X}$, $\dim \mathrm{Sing}\,\hat{X} \leqq d-2$)
  i.e. $\hat{X}$ is smooth.

- $\mathrm{Sing}\,X \cup \bigcup_j (X \cap X_j) = \coprod \begin{pmatrix} (d-1)\text{-dim} \\ \text{torus} \end{pmatrix}$ (X_j: other irred. comp. of M_b)

  "every bad loci of X" are codim 1 disjoint union of tori

characteristic vector field (torus)

$f^*D = \sum a_i H_i \qquad H = H_1,\ a = a_1$

$D = (z_d = 0)$

$M = \bigcup U$ s.t. $H \cap U = (h_U = 0)$

$h_U = g_{UV} h_V$ So: $dh_U = g_{UV}\, dh_V$ on $H \cap U \cap V$

- $V_H^{char} := \{ \iota_\sigma (dh_U) \}_U \in H^0(H, TM \otimes \mathcal{O}(H))$ — globally on H

  ↑ $\iota_\sigma(dh_U)$ locally on M: holom. vect. field vanishing at Sing H

[characteristic vector field w.r.t. H] Also:

- $V_H^{Ham} := \left\{ \iota_\sigma \left(\frac{d(f^* z_d)}{h_U^{a-1}} \right) \right\}_U \in H^0(H, TM \otimes \mathcal{O}(-(a-1)H))$ ← globally on H

  $\iota_\sigma\left(\frac{d(f^*z_d)}{h_U^{a-1}}\right)$ locally on M: holom. vect. field vanishing at Sing $H \cup \bigcup (H \cap H_i)$

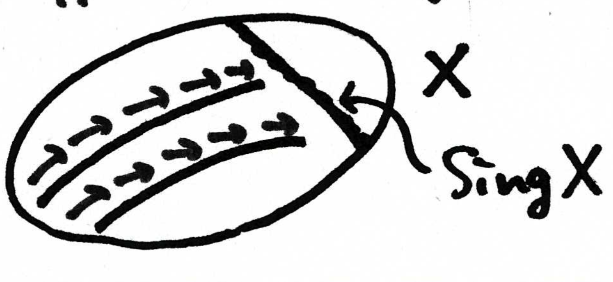
(Rem. $f^* z_d = h_U^a g_U$, $\frac{d(f^* z_d)}{h_U^{a-1}} = a \cdot g_U \cdot dh_U$ on $U \cap H$)

- $V_H^{char} \,/\!/\, V_H^{Ham}$: tangent to fibers

$\Rightarrow X_{reg} = \coprod C_\lambda^0$ $\quad C_\lambda^0$: integral curves of V_H^{char} on X_{reg} <u>smooth</u>

($X \subset M_b$: irred comp.)

$X \supset \overline{C_\lambda^0}$: characteristic curve w.r.t. H (if this is 1-dim)



Application If we would know

$\overline{C_\lambda^\circ} = C \subset X$ $\quad C$: smooth curve

$$\Rightarrow \quad 0 \not\equiv V_H^{char}|_C \in H^0(C, TC \otimes \mathcal{O}_C(H))$$

$$0 \not\equiv V_H^{Ham}|_C \in H^0(C, TC \otimes \mathcal{O}_C(-(a-1)H))$$

$$\Rightarrow \begin{cases} \deg T(C) \otimes \mathcal{O}_C(H) \geqq 0 \\ \deg T(C) \otimes \mathcal{O}_C(-(a-1)H) \geqq 0 \end{cases}$$

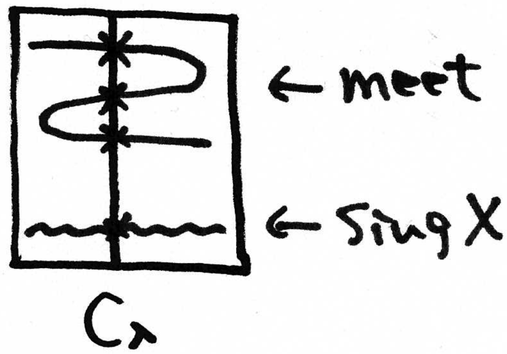
$$\Rightarrow \begin{cases} 2-2g(C) + \deg(H|_C) \geqq 0 \\ 2-2g(C) - \underbrace{\deg}_{\substack{(a-1)\cdot \\ \vee\!\shortparallel \\ 0}} (H|_C) \geqq 0 \end{cases}$$

$\Rightarrow$ • $g(C) = 0$ or 1 i.e. $C \cong \mathbb{P}^1$ or elliptic curve

• Moreover, when $g(C)=1$,

$V_H^{char}|_C$ & $V_H^{Ham}|_C$ has no zero,

$\Rightarrow$ X smooth & $(M_b)_{red} = X$.

Rem $X =$ 

$\Big\}\ V_H^{Ham} = 0$

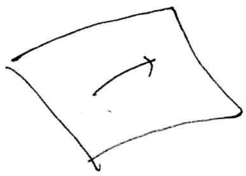
Rem. In general:

$(M^{2d}, \sigma) \supset H$ irred. red. hypersurface

$\Rightarrow \{\iota_\sigma(dh_v)\}_v \in H^0(H, TM \otimes \mathcal{O}_H(H))$

$\quad\quad \| $

$\quad V_H^{char}$ has another meaning



. $x \in H_{reg}$

$F_x = \{\theta_x \in T_x H_{reg} \mid \sigma(\theta_x, v_x) = 0 \ \forall v_x \in T_x H_{reg}\}$

$= \mathbb{C} \cdot \iota_\sigma(dh_v)_x \quad \begin{pmatrix} \sigma(\iota_\sigma(dh_v)_x, v_x) \\ = dh_v(v_x) = v_x(h_v) = 0 \end{pmatrix}$

rk=1 &

$\bigcup_{x \in H_{reg}} F_x$ rank 1 foliation on H_{reg}

characteristic foliation

Hwang closure of leaves are compact curve?

• H : ample & smooth $\Rightarrow$ No (Hwang-Viehweg)

M : in Ex 2-1

M has automorphism:

$$g^*((x,y,t),z,s) = ((\zeta_5^2 x, \zeta_5^3 y, \zeta_5 t), z+p, s)$$

$\zeta_5 = \exp\left(\frac{2\pi\sqrt{-1}}{5}\right)$, $p \in (E)_5$

free, order 5 & $g^*\sigma_M = \sigma_M$

$$\Rightarrow \overline{M} := M/\langle g\rangle \xrightarrow{\overline{f}} \Delta^2_{(u,s)} = \Delta^2_{(t,s)}/\langle t\mapsto \zeta_5 t\rangle \quad (u=t^5)$$

$$\cup$$

$$\overline{D} = (u=0)$$

new Lagrangian fibration with sing. fibers

$$(\overline{M}_{(0,s)})_{red} = \left((y^2=x^3)\times E \Big/ \left\langle \begin{matrix}(x,y,z)\\ \mapsto(\zeta_5^2 x, \zeta_5^3 y, z+p)\end{matrix}\right\rangle \to E/\langle p\rangle \right)$$

Again foliated by $\mathbb{I}$ under $\mathbb{C} \twoheadrightarrow E/\langle p\rangle$

($\underset{\text{locally}}{\cong}$ $\mathbb{I}\times\mathbb{C}$). But, by $\overline{f}^*\overline{D} = 5\overline{H}$,

multiplicity 5 (different from d=1)

Rem By
$$\begin{matrix} M & \longrightarrow & \overline{M} \\ \downarrow & & \downarrow \\ \Delta^2 & \underset{\mathbb{Z}/5}{\longrightarrow} & \Delta^2 \end{matrix},$$

$\mathbb{I}$ = { = closure of integral curve of char. vect. field w.r.t. $\overline{H}$

$\mathbb{C}$: induced by Ham. vect. field

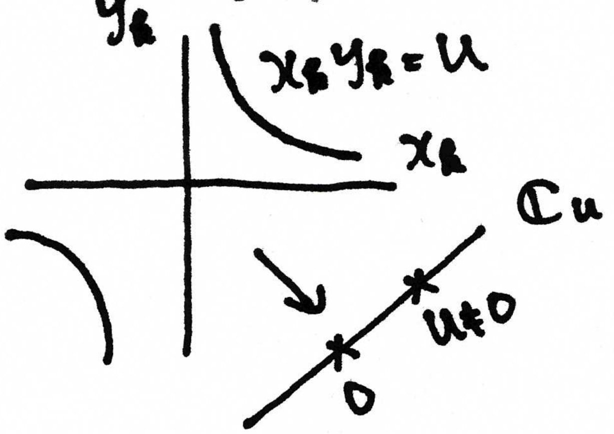
Ex3 ($d=2$ after I. Nakamura)

$$R_k = \operatorname{Spec} \mathbb{C}[u^{k+1}v^{-1}, u^{-k}v] \cong \mathbb{C}^2_k \qquad \begin{cases} x_k = u^{k+1}v^{-1} \\ y_k = u^{-k}v \end{cases}$$

($k \in \mathbb{Z}$)

$$R_k \xrightarrow[\varphi_k]{} \mathbb{C}_u \quad (x_k, y_k) \mapsto x_k y_k = u$$

$$\sigma_k = dx_k \wedge dy_k = du \wedge \frac{dv}{v}$$



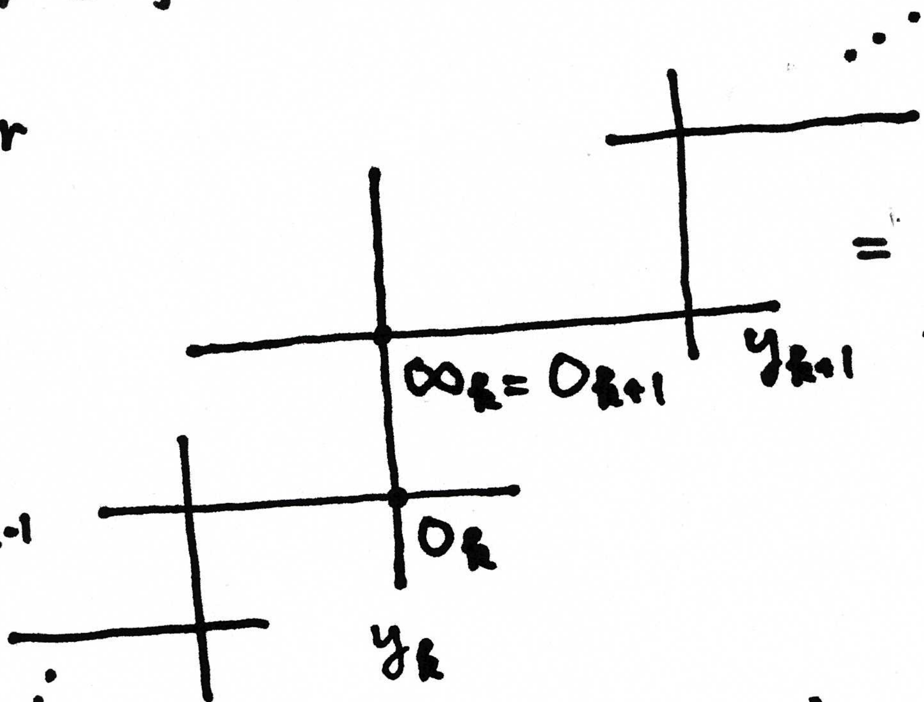
$$(\widehat{R}, \sigma_{\widehat{R}}) = \bigcup_{k \in \mathbb{Z}} (R_k, \sigma_k) \text{ glued by } \begin{cases} y_{k+1} = \dfrac{1}{x_k} \\ x_{k+1} y_{k+1} = x_k y_k \ (= u) \end{cases}$$

$$\varphi := \bigcup_{k \in \mathbb{Z}} \varphi_k : (\widehat{R}, \sigma_{\widehat{R}}) \longrightarrow \mathbb{C}_u$$

non-proper Lagrangian fibration

central fiber

$\varphi^{-1}(0) =$



$= \bigcup_{k \in \mathbb{Z}} \mathbb{P}^1_k$

other fibers $\varphi^{-1}(u) = (x_k y_k = u) = \mathbb{C}^\times$, $u \neq 0$

$$\widehat{M} = \widehat{R} \times \underset{z}{\overset{\text{ell. curve}}{E}} \times \mathbb{C}_s \xrightarrow{\tilde{f}} \mathbb{C}^2_{(u,s)} \quad \text{nat. projection by } \varphi$$

$\sigma_{\widehat{M}} = du \wedge \frac{dv}{v} + dz \wedge ds$ symplectic form

$\tilde{f}$: non-proper Lagrangian fibration

Restrict to $|u| \ll 1$ ($|s| \ll 1$) & fix $p \in E$

$\mathbb{Z} \curvearrowright \widehat{M}$ by:

$$1 : (x_k, y_k, z, s) \mapsto (\underset{x_{k+1}}{\underset{\parallel}{u x_k}}, \underset{y_{k+1}}{\underset{\parallel}{u^{-1} y_k}}, z + p, s)$$

free, proper discontinuous & preserve $\sigma_{\widehat{M}}$

$\Rightarrow M := \widehat{M}/\mathbb{Z} \xrightarrow{f} \Delta^2_{(u,s)} \supset D = (u = 0)$

proper Lagrangian fib. w.r.t. σ_M induced by $\sigma_{\widehat{M}}$

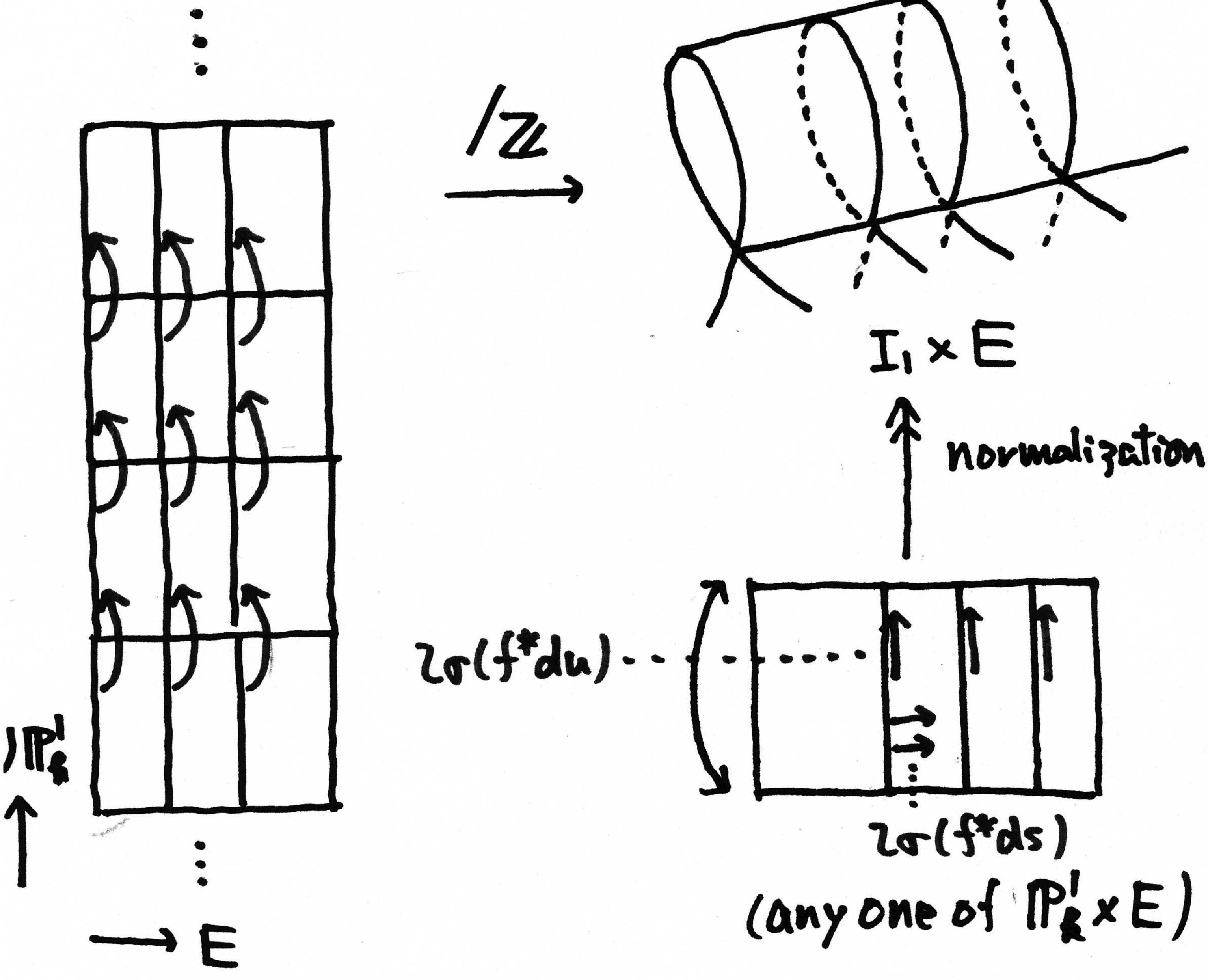
- $f^{-1}((u,s)) = \mathbb{C}^\times_x \times E \big/ \langle (x,z) \mapsto (ux, z+p) \rangle$, $u \neq 0$
  (torus fibration over torus $\mathbb{C}^\times_x / \langle u \rangle$)
  2-dim complex torus

- singular fibers $f^{-1}(0,s) = M_{(0,s)}$?
  Depends on the choice of $p \in E$.

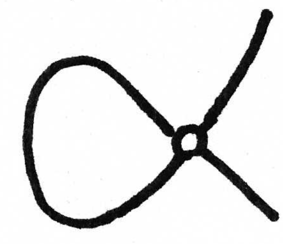
($H = f^* D$: reduced discrim. hyp. surface)

$P = 0 \in E$

$$\tilde{f}^{-1}(0,s) = \tilde{M}_{(0,s)} \xrightarrow{/\mathbb{Z}} M_{(0,s)}$$



So: $M_{(0,s)} = \coprod I_1$

I_1 = closure of  Int. curve of char. v. field $2\sigma(f^*du)$

Char. curve (→ I_1)

&

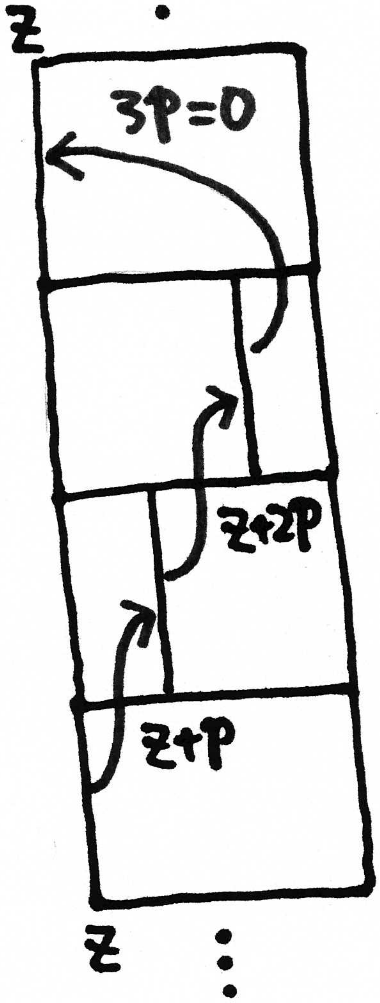
$\mathbb{C} \curvearrowright \{I_1\}$ transitive

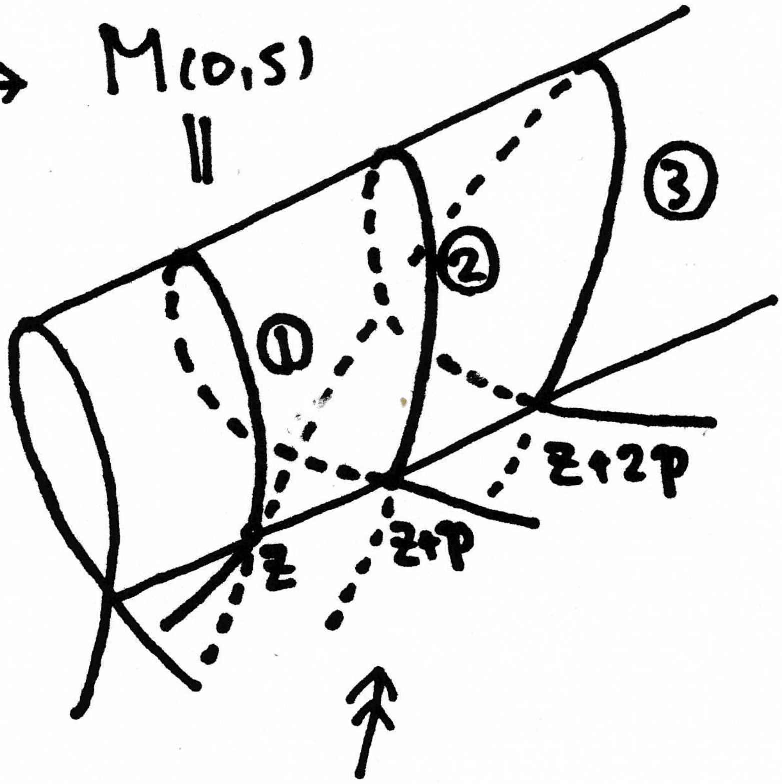
↑ from Ham. v. field $2\sigma(f^*ds)$

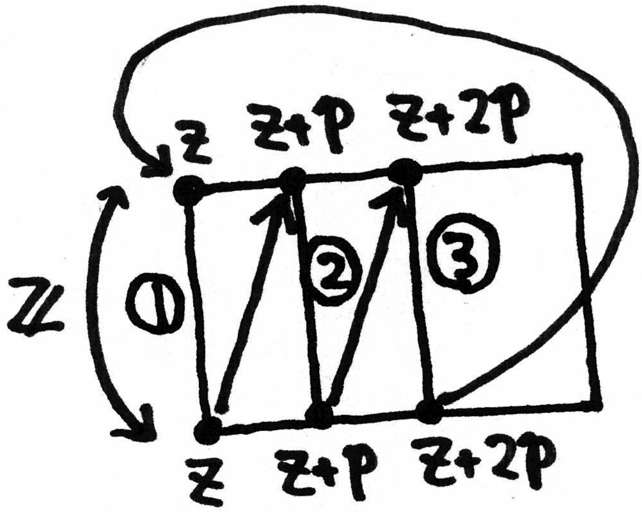
$p = (3\text{-torsion}) \in E$

$\widetilde{M}_{(0,S)} \xrightarrow{\ /\mathbb{Z}\ } M_{(0,S)}$

$\parallel$



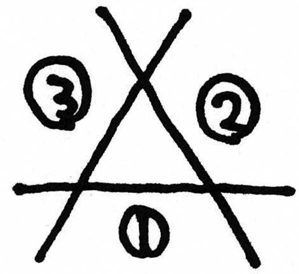




any one of $\mathbb{P}^1 \times E$
(fund. dom. of $\mathbb{Z}$-action)

So:

$M_{(0,S)} = \coprod$



I_3

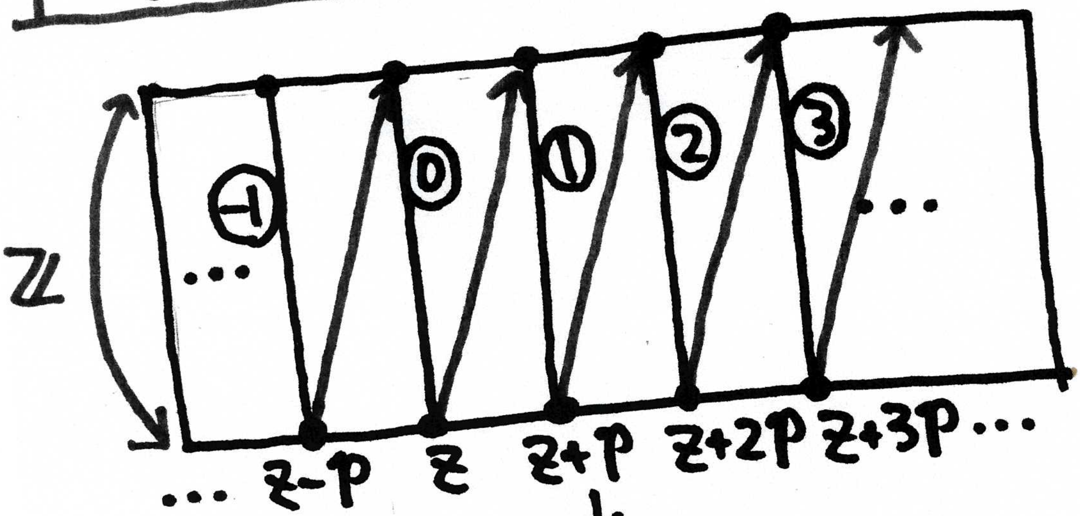
$\mathbb{C} \curvearrowright \{I_3\}$ transitive

Ham. action

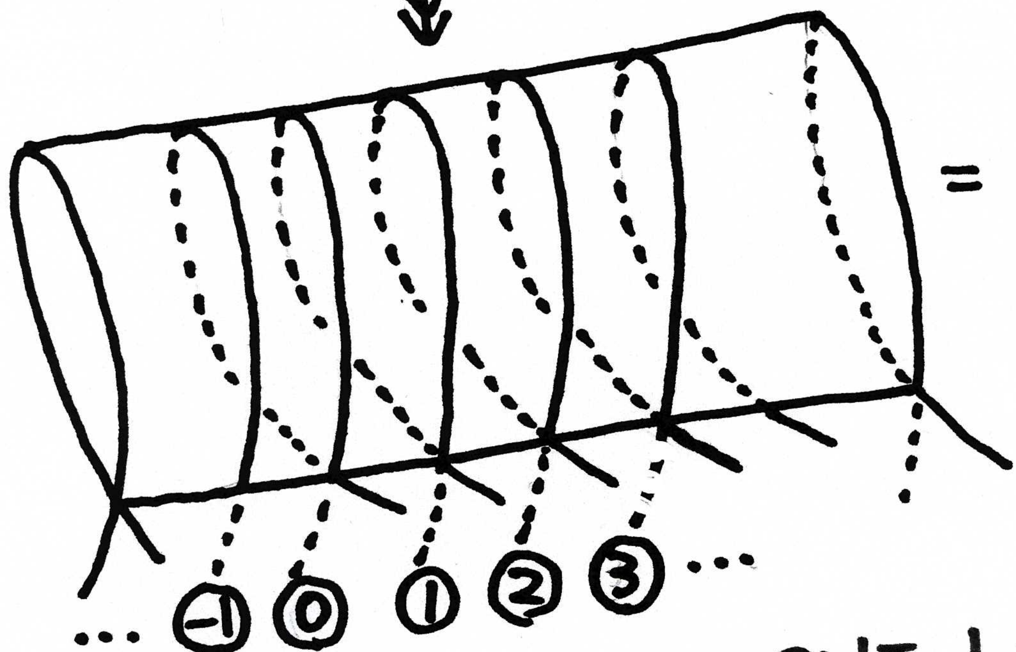
I_3 = conn. comp. of closure of integral curves of char. vert. field

(locally: $M_{(0,S)} \cong I_3 \times \mathbb{C}$ still close to Kodaira's fiber)

<u>$p =$ (non-torsion) $\in E$</u>



$\mathbb{P}^1 \times E$
(fund. domain)



$= M(0,S)$

$\mathbb{C} \curvearrowright \{I_\infty\}$ transitive Ham. action

$M(0,S) = \coprod I_\infty$

$I_\infty =$

∞-chain of $\mathbb{P}^1$

conn. comp. of closure of integral curves of char. v. fi

Very roughly:

$M(0,S) \underset{\text{“locally”}}{\cong} I_\infty \times \mathbb{C}$ new phenomen

Rem.

(1) divide M by $n\mathbb{Z} \Rightarrow M_{(0,s)}$: n irred. comp.

(2) $M_2 = \widetilde{M}/2\mathbb{Z} \xrightarrow{f_2} \Delta^2_{(u,s)}$ (2-torsion of E)

$g: (u, v, z, s) \mapsto (-u, v^{-1}, z+\delta, s)$

free, order 2, $g^* \sigma_{M_2} = \sigma_{M_2}$

$\Rightarrow \overline{M_2} := M_2/\langle g \rangle \xrightarrow{\overline{f_2}} \Delta^2_{(\bar{u},s)} \supset \overline{D} = (\bar{u}=0)$, $\bar{u} = u^2$

Lag. fib.
sing. fiber $= \coprod I_{2n}$ (or $\coprod I_\infty$)
But multiplicity 2

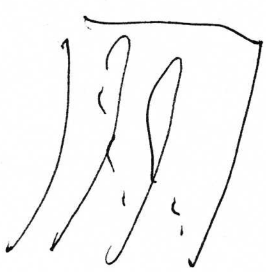
(3) (Similar but slightly more complicated constr.)

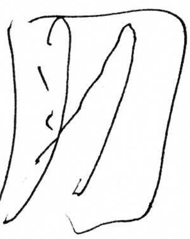
$\exists\, M^4 \xrightarrow{f} \Delta^2_{(u,s)} \supset D = (u=0)$ Lag. fib. s.t.
sing. fibers $M_{(0,s)} = f^{-1}(0,s)$ are:

$$\begin{cases} s \notin \mathbb{Q}(\sqrt{-1}) \Rightarrow M_{(0,s)} = \coprod I_\infty \\ s \in \mathbb{Q}(\sqrt{-1}) \Rightarrow M_{(0,s)} = \coprod I_{k(s)} \end{cases}$$

$0 < k(s) < \infty$

$\{k(s)\}_{s \in \mathbb{Q}(\sqrt{-1})}$
$= \{$almost all pos. integers$\}$





So far, through several examples:

$$\begin{pmatrix}\text{singular fibers}\\ \text{of Lag. fib.}\end{pmatrix} = \coprod (\text{Kodaira fiber})$$

transitive under Ham. action $\mathbb{C}^{d-1}$ $(d=2)$

except: $\begin{cases}\text{I}_\infty \text{ fiber can happen}\\ \text{slight differences in multiplicity}\end{cases}$

<u>Main Result</u>:

This is in fact true for general sing. fibers of $\forall$ Lagrangian fibrations

(general $f^{-1}(b)$: $b \in D$ general pt of D)

<u>Setting</u>:

$f: (M^{2d}, \sigma) \to \Delta^d \supset \underbrace{D = (z_d = 0)}_{\text{discrim. locus}}$ Lag. fib.

$f^*D = \sum_{i=1}^{\ell} a_i H_i = n \sum \overline{a_i} H_i \quad \mathrm{GCD}(\overline{a_i}) = 1$

For reduced 1-cycle $\sum_{i,j} C_{ij}$ $(C_{ij} \subset H_i \setminus \bigcup_{i' \neq i} H_{i'})$

we call:

$\begin{cases}\sum \overline{a_i} C_{ij} : \text{radicated 1-cycle}\\ \quad n \quad : \text{multiplicity}\end{cases}$

<u>Theorem</u> (Jun-Muk Hwang & —)

Under the setting, let $b \in D$ general & fix. Then:

(1) $(M_b)_{red} = \coprod_{\lambda} C_\lambda$

C_λ = conn. comp. of closure of integral curves of char. vect. fields ass. to $^\forall H_i$ $(i=1,\cdots,\ell)$

(reduced characteristic 1-cycle)

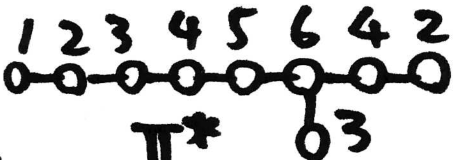
& Hamiltonian action $\mathbb{C}^{d-1} \to Aut(M/\Delta^d)$ acts transitively on $\{C_\lambda\}$ (So $C_\lambda \cong C_{\lambda'}$)

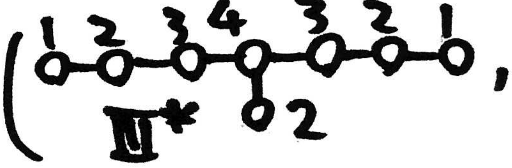
(2) (radicated 1-cycle of C_λ : multiplicity) is one of:

(<u>E: ell. curve</u> ; <u>2, 3, 4, 6</u>) $\left(\begin{array}{l} 3, 6 : \text{only for } E_{\zeta_3} \\ 4 : \text{only for } E_{\zeta_4} \end{array}\right)$

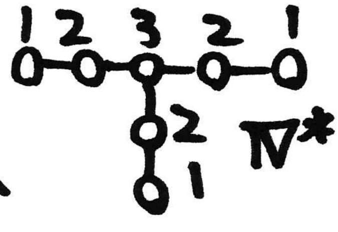
( : 1), ( : 1, 5), ( : 1, 3), ( : 1, 2, 4)

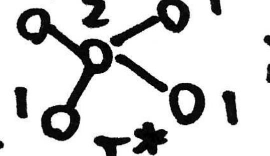
I_1, II, III, IV

(

; 1), (

, 1)

(

; 1, 2), (

; 1, 2, 3)

$(I_{2\ell} ; 1, 2)$ $(\ell \geqq 2)$, $(I_{2\ell+1} ; 1)$, $(I_\ell^* , 1)$ $(\ell \geqq 1)$

or $(I_\infty ; 1, 2)$.

(3) For each $d \geqq 2$, all are realizable.

Outline of Proof

$b \in D$ gen. $f^*D = n \sum \overline{a_i} H_i$ $\quad H = H_1$

$X \subset \overline{M_b} := (M_b)_{red}$ irred. comp. $\quad X \subset H$

Step 1 (Structure of the normalization $\nu : \hat{X} \to X$)

Prop 1 $\hat{X}$ is smooth &

(1) $H^0(\hat{X}, \Omega^1_{\hat{X}}) = \mathbb{C}\langle \varphi_1, \cdots, \varphi_{d-1} \rangle \cong \mathbb{C}^{d-1}$

$\{\varphi_i\}_{i=1}^{d-1}$: pointwisely dual to the lifts

$\{\hat{v}_i\}_{i=1}^{d-1} \subset H^0(T_{\hat{X}})$ of

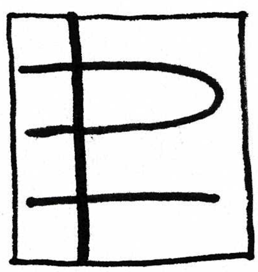
Ham. vect. fields $\{v_i\}_{i=1}^{d-1} \subset H^0(T_M)$.

In part: $\hat{X} \circlearrowleft \mathbb{C}^{d-1}$

$\mathrm{alb}_{\hat{X}} \downarrow \quad \circlearrowleft \mathbb{C}^{d-1}$ nat. translation

$\mathrm{Alb}\,\hat{X}$

(d-1)-dim



} . preimage of $\mathrm{Sing}\, X \cup \bigcup_i (X \cap X_i)$ étale/$\mathrm{Alb}\,\hat{X}$

$\downarrow \mathrm{alb}_{\hat{X}}$

. $\forall$ fibers $\hat{C}$ are isom & either $\mathbb{P}^1$ or ell. curve

(2) $\nu(\hat{C})$ = closure of int. curve of V_H^{char} (always compact & algebraic) char. curve

Cor If $\hat{C}$ = ell. curve $\Rightarrow \hat{X} = X = \overline{M_b}$ & $\overline{M_b}$ is multiple fibers.

In what follows $\hat{C} = \mathbb{P}^1$

Step 2 (local structure of $\overline{M_b}$ at $P \in \mathrm{Sing}\,\overline{M_b}$)

$\begin{cases} \text{(i) } P \in \mathrm{Sing}\,\overline{M_b} : \text{ locally (analytically) irred.} \\ \text{(ii) } P \in \mathrm{Sing}\,\overline{M_b} : \quad \text{"} \qquad \text{"} \quad \text{not irred.} \end{cases}$

Prop 2 In (i):

(1) $X = \overline{M_b}$ globally irred.

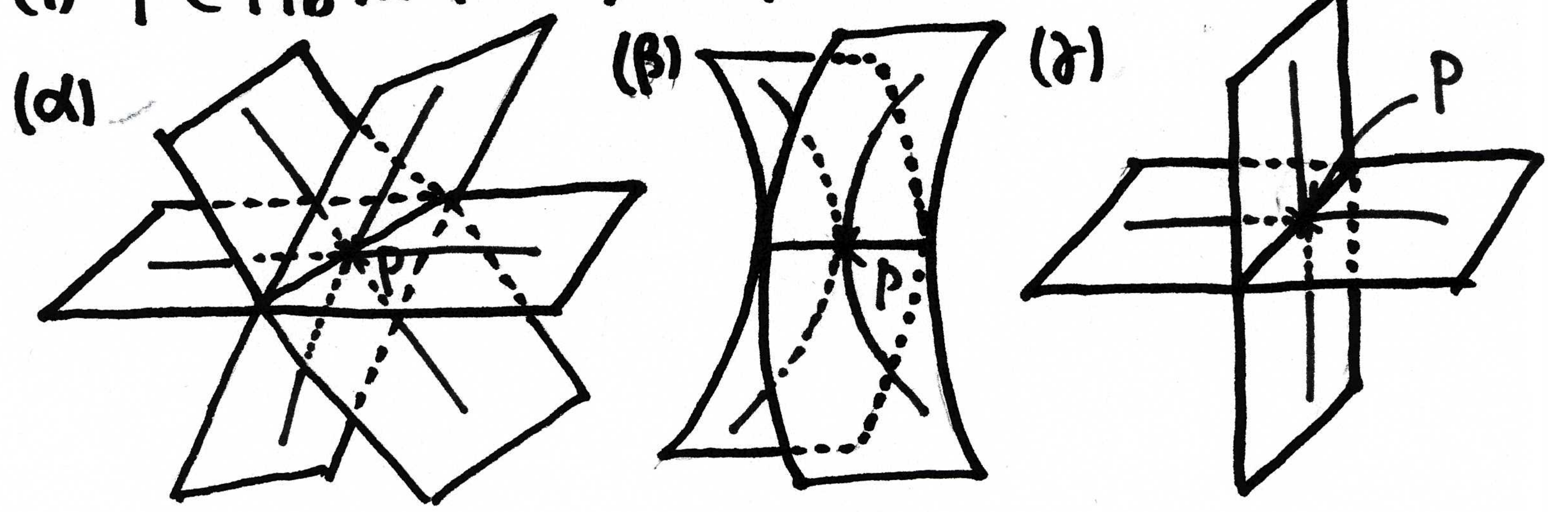
(2) $\overline{M_b} \underset{\text{locally}}{\cong}$  $\times\, \mathbb{C}^{d-1}$
II

Ham. action

(3) $\forall$ char. curves $\cong$ II & $\mathbb{C}^{d-1} \curvearrowright \{\text{II}\}$ transitive

Prop 3 In (ii):

(1) $P \in \overline{M_b}$ is locally analytically isom. to either:



(α) 3 comps

(β) 2 comps cont. order 2

(γ) 2 comps transversal

(2) If $\overline{M_b}$ is globally reducible, then $\nu(\hat{C}) = C \underset{\text{smooth}}{\cong} \mathbb{P}^1$.

Step3 (global str. of radicated char. 1-cycles)

$\Theta = \sum b_i \Theta_i$ radicated char. 1-cycles on M_b

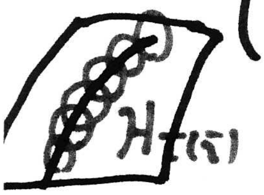
$\begin{cases} b_i = \overline{a}_{\tau(i)} \\ \Theta_i = \text{char. curve} \subset H_{\tau(i)} \end{cases}$

Θ: reducible ($\Rightarrow \forall \Theta_i \cong \mathbb{P}^1$)

$\forall \Theta_i \quad \dfrac{\Theta_i \cdot f^*D}{n} = 0$ (by $f^*D = (f^*z_d = 0)$) principal divisor

$\quad = \sum_j b_j (\Theta_j \cdot \Theta_i)$

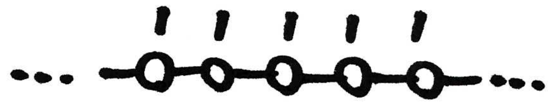
Here: $\begin{cases} \Theta_j \cdot \Theta_i := \#(\Theta_j \cap \Theta_i) \text{ with natural wt. 1 or 2} \quad (j \neq i) \\ \Theta_i \cdot \Theta_i := -2 \end{cases}$



[**Rem** $\underbrace{\Theta_i \cdot \mathcal{H}_{\tau(i)}}_{\text{free}} = K_{\Theta_i}$ free $\Leftarrow \mathbb{C}^{d-1}$ in $\mathcal{H}_{\tau(i)}$]

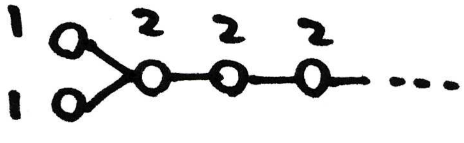
Then: $2b_i = \sum_{j \neq i} b_j (\Theta_j \cdot \Theta_i)$ & $(\Theta_i \cdot \Theta_i) = -2 \quad \forall i$.

$\Rightarrow$ (Kodaira) $\begin{cases} \Theta \text{ is of fin. cycle} \Rightarrow \text{one of Kodaira's type} \\ \Theta \text{ is of infinite cycle} \Rightarrow \end{cases}$



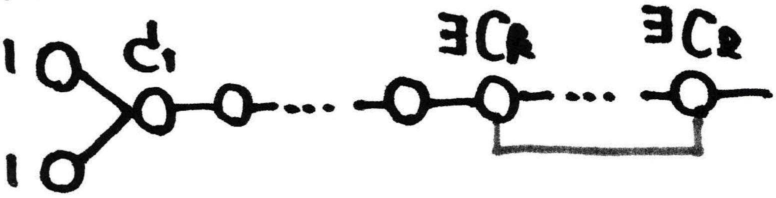
I_∞

or



D_∞

Claim D_∞ is impossible.



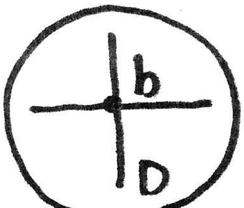
C_k & $C_\ell \subset$ the same H_i

$\Rightarrow$ (inductively) C_1 & $\exists C_j \subset$ the same $H_{i'}$ ($j \geq 2$) ※

Step4 (multiplicity n)

[Consider for each type] Case II

$b \in \Delta \subset \Delta^d$ (gen.)



& restrict to Δ:

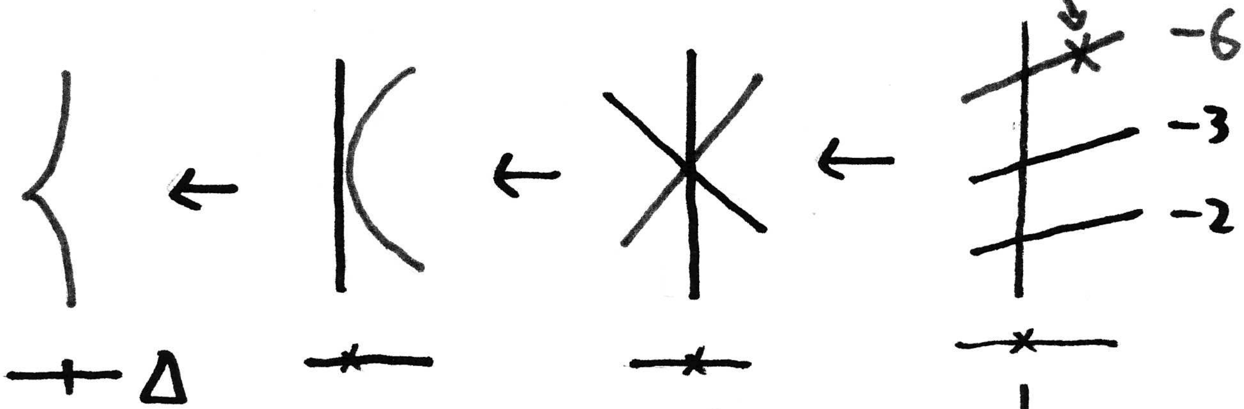
$$\begin{array}{ccccc} M & \supset & M_0 & \supset & nM_b \\ \downarrow & \square\, f_0 & \downarrow & \square & \downarrow \\ \Delta^d & \supset & \Delta & \ni & b \end{array} \qquad (\dim M_0 = d+1)$$

$$\Rightarrow \begin{array}{ccc} M_1 & \longrightarrow & M_0 \\ f_1 \downarrow & \text{norm } \square & \downarrow f_0 \\ \Delta & \xrightarrow{\mathbb{Z}/n} & \Delta \\ \Psi & & \Psi \\ b & & b \end{array}$$

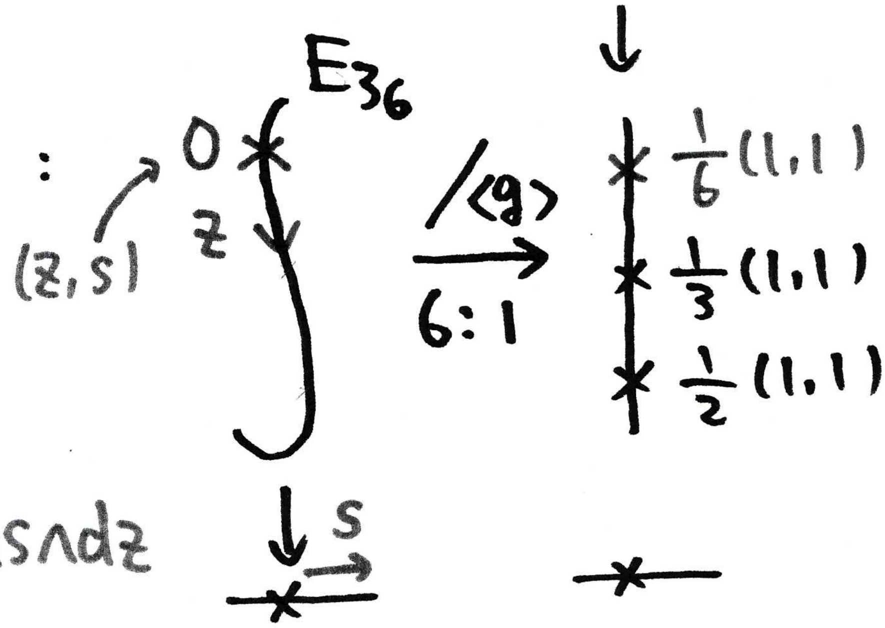
$\left\{ \begin{array}{l} (M_1)_b : \text{mult} = 1 \\ (M_1)_b = \mathbb{C}^{d-1} \end{array} \right\}$ II

$M_1 \to M_0$ étale

Rem.



stable reduction :



- $\begin{cases} g^*s = \zeta_6 s \\ g^*z = \zeta_6 z \end{cases}$
- $du \wedge dy = 6 s^4 ds \wedge dz$

One can perform stable reduction for $M_1 \xrightarrow{f_1} \Delta$ compatibly with Ham. action $\mathbb{C}^{d-1}$:

$$\begin{array}{ccccccc}
F_2 \subset & M_2 & \dashrightarrow & M_1 & \longrightarrow & M_0 \\
\downarrow & \downarrow{\scriptstyle f_2} & & \downarrow & & \downarrow \\
0 \in & \Delta_{s_2} & \xrightarrow[6:1]{/\langle \tau^n \rangle} & \Delta_{s_1} & \xrightarrow[n:1]{/\langle \bar{\tau} \rangle} & \Delta_{s_0}
\end{array}$$

$$\Delta_{s_2} \xrightarrow{/\langle \tau \rangle} \Delta_{s_0}$$

$\langle \tau \rangle = \mathbb{Z}/6n$ $\quad \bar{\tau} = \tau \bmod \langle \tau^n \rangle$

- $\begin{cases} \tau^* s_2 = \zeta_{6n} s_2 \\ \tau^* \omega_{F_2} = \zeta_{6n}^a \omega_{F_2} \end{cases}$ $\quad (0 \le a < 6n)$ $\quad \underline{\zeta_{6n}^a \in \langle \zeta_6 \rangle}$
- $(\tau^n)^* \omega_{F_2} = \zeta_6 \omega_{F_2}$

$\Rightarrow a = 6g+1$

- induced action of $\bar{\tau} := \tau \bmod \langle \tau^n \rangle$ on ω_{M_1}:

$$(\bar{\tau})^* \omega_{M_1} = \zeta_{6n}^4 \cdot \zeta_{6n} \cdot \zeta_{6n}^a \omega_{M_1} = \zeta_{6n}^{6g+6} \omega_{M_1}$$

$\bar{\tau}$ is Gorenstein:

$g+1 \equiv 0 \pmod{n}$ i.e. $g = ng' - 1$

So: $a = 6g+1 = 6ng' - 5$ i.e.

$$\zeta_{6n}^a = \zeta_{6n}^{-5} \in \langle \zeta_6 \rangle \Rightarrow n = 1 \text{ or } 5 \quad //$$