Curves on irregular surfaces

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Outline of the talk

S:= smooth (minimal) complex projective surface of general type

$p_g := p_g(S) = h^0(K_S) = h^2(\mathcal{O}_S)$, the *geometric genus*
$q := q(S) = h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$, the *irregularity*.
(Alb(S) and Pic^0(S) are dual abelian varieties of dimension q).

alb: $S \to$ Alb(S), the Albanese map
albdim(S) = dim alb(S), the *Albanese dimension*

An *irrational pencil* of genus $b > 0$ is $f : S \to B$, with:

- B smooth curve of genus b;
- $f : S \to B$ morphism with connected fibers.

Recall:

(a) the existence of an irrational pencil of genus > 1 is a topological property;

(b) S has a finite number of pencils of genus > 1;

(c) if S is minimal the general fiber F of a pencil of genus > 1 has genus $\leq \frac{K^2}{8} + 1$;

(d) none of the above is true for pencils of genus 1 (“elliptic pencils”).

Theorem (Castelnuovo–De Franchis)

S has an irrational pencil of genus > 1 iff there exist $\alpha, \beta \in H^0(\Omega^1_S)$ such that:
(i) α, β are independent
(ii) $\alpha \wedge \beta \equiv 0$.

Examples of irregular surfaces without irrational pencils:

- complete intersections of ample divisors in an abelian variety A: in this case $H^2(\mathcal{O}_A) \hookrightarrow H^2(\mathcal{O}_S)$, so $\wedge^2 H^0(\Omega^1_S) \hookrightarrow H^0(K_S)$.
- the symmetric square of a curve of genus $q \geq 3$ (more on this later).

Remark: in both cases there may be elliptic pencils.

Assume S has no irrational pencil of genus > 1; then:

- there are no simple tensors in the kernel of $w\colon \wedge^2 H^0(\Omega^1_S) \to H^0(K_S)$;
- w induces a map $G(2, q) \to \mathbb{P}(H^0(K_S))$ which is finite onto its image.

Castelnuovo De Franchis inequality

If S has no irrational pencil of genus > 1, then:

$$p_g(S) \geq 2q(S) - 3.$$

Usually surfaces for which a general inequality is an equality can be classified.
This is not the case for surfaces S with $p_g = 2q - 3$ that have no irrational pencil of genus > 1.

- if $q = 3$, then S is the symmetric product of a curve of genus 3 (Hacon, – and, independently, Pirola 2002). This is the only known example with;
- no example for $q = 5$ (Mendes Lopes, Pirola, – 2010);
- if $q \geq 6$, then φ_{K_S} is birational and $K_S^2 \geq 7\chi + 2$ (Mendes Lopes, Pirola, – 2010).

So we look for a different approach.

Instead of concentrating on the numerical invariants, here we look at the simplest class of examples.

Let C be a smooth curve of genus $q \geq 3$. The *symmetric square* of C is defined as:

$$S^2(C) := (C \times C)/ < \iota >, \quad \text{where} \ \ (P, Q) \overset{\iota}{\mapsto} (Q, P).$$

Set $S := S^2(C)$. Then:

(a) S is minimal of general type with $K_S^2 = (q-1)(4q-9)$;

(b) there are canonical identifications:

$$H^0(\Omega^1_S) = H^0(\omega_C), \;\; H^0(\omega_S) = \wedge^2 H^0(\Omega^1_S), \;\; \mathrm{Alb}(S) = J(C)$$

(c) $p_g(S) = q(q-1)/2, \quad q(S) = q, \quad \chi(S) = q(q-3)/2 + 1;$

(d) S has no irrational pencil of genus >1;

(e) S has an elliptic pencil iff C has a map onto an elliptic curve.

There is another surface for which $\wedge^2 H^0(\Omega^1_S) \to H^0(\omega_S)$ is an isomorphism: the Fano surface F of lines in a cubic threefold ($q = 5$, $p_g = 10$, $K^2 = 45$) .

Conjecture (Debarre): the surface F and the symmetric square are the only surfaces that represent a minimal class in a PPAV.

The conjecture is proven, in a weaker form, in several cases:
– $q = 4$ (Barton-Clemens, Ran),
– A Jacobian (Debarre),
– A the intermediate Jacobian of a generic cubic threefold (Debarre, Höring).

Question: does the isomorphism $\wedge^2 H^0(\Omega^1_S) \to H^0(\omega_S)$ characterize these surfaces?

Here we take a different point of view, namely we look at curves of small genus of S with positive self-intersection in order to find a characterization.

Notation: given an irreducible curve (a 1-connected divisor) C of S, we denote:

- $r(C) := \dim < \mathrm{alb}(C) >$;
- $g(C)$, the geometric genus of C;
- $p(C) := p_a(C)$, the arithmetic genus of C.

Remarks:

- of course: $r(C) \leq g(C) \leq p(C)$;
- if $C^2 > 0$, then $r(C) = q$.

Curves on a symmetric square

Let $S = S^2(C)$; for $P \in C$, we let $C_P \subset S$ be the image of $\{P\} \times C \subset C \times C$. Then:

- C_P is smooth, isomorphic to C (so $g(C) = q$);
- $C_P^2 = 1$;
- $r(C) = q$, namely $< \text{alb}(C) >= \text{Alb}(S)$;
- as P varies, the curves C_P form a 1-dimensional algebraic system.

Theorem

Let S be an irregular surface of general type.
If S has a 1-connected effective divisor D with $p_a(D) = q$ and $D^2 > 0$, then S is birationally either:

(a) *the product of two curves of genus ≥ 2; or*

(b) *the symmetric product $S^2(C)$, where C is a smooth curve of genus $q \geq 3$.*

Furthermore, if D is 2-connected, only case (b) occurs.

Remarks:

- we do not assume S minimal, nor $\mathrm{albdim}(S) = 2$, nor D irreducible.
- the existence of an effective divisor with certain numerical properties determines the surface completely.

Outline of the proof:

Step 1 if there is a decomposition $D = A + B$ with $AB = 1$, $p(A), p(B) > 0$, then S is birational to a product of curves;

Step 2 if there is no decomposition as in Step 1, then we may assume that D is 2-connected;

Step 3 if D is 2-connected, then D is smooth irreducible;

Step 4 if D is smooth, there exists a d-dimensional family of curves algebraically equivalent to D, where $d := D^2$;

Step 5 we conclude using the classification of systems of curves $\{D\}$ of dimension equal to D^2 (Catanese-Ciliberto- Mendes Lopes 1998).

Step 4 in more detail:

C smooth of genus q, $d := C^2 > 0$ and $\mathrm{albdim}(S) = 2$. Let $V^1(S) = \{\eta \in \mathrm{Pic}^0(S) | h^1(\eta) \neq 0\}$ and $W(C) = \{\eta \in \mathrm{Pic}^0(C) | h^0((C+\eta)|C) > 0\}$. Then:

- the map $\mathrm{Pic}^0(S) \to \mathrm{Pic}^0(C)$ is an isomorphism;
- let $0 \neq \eta \in \mathrm{Pic}^0(S) = \mathrm{Pic}^0(C)$; there is an exact sequence $0 = H^0(\eta) \to H^0(C+\eta) \to H^0((C+\eta)|_C) \to H^1(\eta)$;
- $W(C)$ is irreducible and generates $J(C) = \mathrm{Alb}(S)$, while $V^1(S)$ is a union of proper abelian subvarieties of $\mathrm{Alb}(S)$, hence $W(C) \not\subset V^1(S)$.
- by the above remarks, there exists a d-dimensional family $\{D\}$ of curves algebraically equivalent to C.

Warning: the family $\{D\}$ may not contain the curve C!

Question: What about surfaces carrying a curve of arithmetic genus $p > q$ with p "small"? (and what is "small"?)

Observations:

- Steps 1–3 are based on a numerical analysis which is hopeless for $p > q$. So we focus on irreducible curves C.
- In Step 4 we need to control $V^1(S)$; for this we will assume that S has no irrational pencils, thus $\dim V^1(S) \leq 1$ and $0 \in V^1(S)$ is an isolated point;
- If $g(C) < 2q - 1$ then $h^0(C) = 1$ (Xiao 1987). So a reasonable meaning of "p small" is "$p < 2q - 1$".

A Brill-Noether type result

S irregular without irrational pencils, $C = \sum C_i$ a reduced connected curve of S such that $\mathrm{Pic}^0(C_i) \hookrightarrow \mathrm{Pic}^0(S)$ for every i (e.g., C irreducible and $C^2 > 0$).
Set $d := C^2$ and $\rho(C) := d + q - p(C) > 0$.

Theorem

Assume that $K_S C \geq C^2 > 0$ and $\rho(C) > 0$. If either $\rho(C) > 1$ or C is not contained in the ramification divisor R of the Albanese map of S, then C moves in an algebraic family of dimension $\geq \rho(C)$.

Idea of proof:

Step 1 Generalize Fulton–Lazarsfeld description of the Brill-Noether loci to the following situation:

- $T \subseteq J(C)$ a subgroup such that $T \to J(C_i)$ has finite kernel for every i;
- $L \in \mathrm{Pic}(C)$ such that $\deg L \leq p - 1$, $W^r_T(L) := \{\eta \in T | h^0(L + \eta) \geq r + 1\}$

Step 2 Take $T = \mathrm{Pic}^0(S)$, $L = \mathcal{O}_C(C)$ and compare $W^r_T(L)$ and $V^1(S)$ as before. (Use the description of $V^1(S)$. If C is not contained in the ramification divisor R of alb, then $0 \in W_T(L)$).

Corollary

C an irreducible curve with $C^2 > 0$ and arithmetic genus p. Then:

1. *if $p < 2q - 1$, then $C^2 \leq p - \frac{q-3}{2}$;*
2. *if C is a fixed component of $|K_S|$, then $C^2 \leq p - q + 1$.*

Remarks:

1. This is because by Xiao's result no curve numerically equivalent to C can move in a linear system;
2. If the inequality fails, then there exists $\eta \in \mathrm{Pic}^0(S)$ such that $h^1(\eta) = 0$, $h^0(C + \eta) > 0$. Then
$h^0(K_S + \eta) \geq h^0(K_S - C) = p_g(S) > \chi(S)$: contradiction!

Working harder, one obtains:

Theorem

If C is an irreducible curve with $C^2 > 0$ and arithmetic genus $p < 2q - 1$ such that $C \not\subset R$, then:

$$C^2 \leq 3\frac{p-q}{2} + 2.$$

Remarks:

- to prove this we use a rigidity result for curves with an involution on a surface and "Clifford+";
- the inequality is very good when $p - q$ is small; for $p = q + 1$ we get $C^2 \leq 3$. It should be possible to get down to $C^2 \leq 2$ with ad hoc arguments.