Lagrangian Surfaces

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joint work with

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Galois Closure and Lagrangian varieties (Adv. Math. 2010)

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Levico, 6 September 2010

Let X be a smooth complex projective variety of dimension n, $\mathcal{O}_X$ be the structure sheaf and Ω^1_X be the cotangent of X.

Definition

1. X is irregular if $q = h^1(\mathcal{O}_X) = h^0(\Omega^1_X) > 0$.
2. X is *b*-irregular if $\phi : \bigwedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ is not injective.
3. X is *hb*-irregular if $\phi^{2.0} : \bigwedge^2 H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$ is not injective.

$q = \frac{1}{2} b_1$ is the irregularity,
b stands either for badly or **bitterly**, h for holomorphically.

Remarks

1. A curve C of genus $g \geq 2$ is b (and hb)-irregular.
2. if $f : X \to Y$ is a dominant morphism and Y is b-irregular then X is b-irregular.
3. if $f : X \to Y$ is a dominant rational map and Y is hb-irregular then X is hb-irregular.

Lagrangian varieties

Definition

(X, Ω) is a **generalized** Lagrangian variety ($\dim X = n$) if there are $\omega_1, \ldots, \omega_{2n}$ in $H^0(X, \Omega^1_X)$ independent forms:
$W = span < \omega_1, \ldots, \omega_{2n} >$, $\dim W = 2n$

$$\Omega = \omega_1 \wedge \omega_2 + \ldots + \omega_{2n-1} \wedge \omega_{2n} :$$

1. the evaluation map $ev : W \otimes \mathcal{O}_X \to \Omega^1_X$ is generically surjective
2. $\Omega_{|X} = \phi^{2.0}(\Omega) = 0$

Ω is a Lagrangian structure on X,
$sing(\Omega) = \{x \,|\, ev_x : W \otimes \mathcal{O}_{X,x} \to \Omega^1_{X,x}$ not surjective$\}$.

If (X, Ω) is a **generalized** Lagrangian variety, $U \subset X$ is simply connected and $p \in U$, we define
$f: U \longrightarrow \mathbb{C}^{2n}$

$$f(q) = \int_p^q (\omega_1, ..., \omega_{2n})$$

$f(U)$ is Lagrangian with respect to

$$dz_1 \wedge dz_2 + ... + dz_{2n-1} \wedge dz_{2n}$$

$$sing(\Omega) \cap U = \{q \in U : \; df \;\text{ not } \;\text{ surjective}\}.$$

Definition

(X, Ω) is **Lagrangian** if there is a gen. finite map $f : X \longrightarrow A$, A is an abelian variety, $\dim A = 2n$ and $W = f^* H^0(A, \Omega^1_A)$.

Remark

If X is Lagrangian then $f(X)$ is a Lagrangian subvariety of A and $sing(\Omega)$ is the branch of f. If $W = H^0(X, \Omega^1_X)$ and $q = 2n$, then f is the Albanese map (up to isogenies).

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Curves of genus $g > 1$ are gen. Lagrangian, curves in abelian surface are Lagrangian, products of Lagrangian are Lagrangian.

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Algebras

Let $a : X \to Alb(X)$ be the Albanese map. Consider the algebras:

Definition

1. $\mathbb{H}_{\mathbb{Q}} = \oplus H^p(X, \mathbb{Q})$ $(\mathbb{H}_{\mathbb{C}} = \mathbb{H}_{\mathbb{Q}} \otimes \mathbb{C})$
2. $\mathbb{H}_{hol}(X) = \oplus H^{p.0}$
3. $\mathbb{H}' = \oplus \bigwedge^p H^{1.0} \equiv \mathbb{H}_{hol}(Alb(X))$
4. $\mathbb{H}'' = a^*(\mathbb{H}_{hol}(Alb(X))$.

$\mathbb{H}''$ is the subalgebra of $\mathbb{H}_{hol}(X)$ generated by $H^{1.0}$.

FORMALITY

Theorem (Deligne, Griffiths, Morgan, Sullivan)

The rational homotopy groups $\pi_i(X) \otimes \mathbb{Q}$ ($i > 1$) depend only on the algebra $\mathbb{H}_{\mathbb{Q}}$.

This usually requires $\pi_1(X) = 0$.

Nilpotent tower FORMALITY

$G = \pi_1(X)$
$G_0 = G,\ G_1 = [G, G],\ \ldots,\ G_n = [G, G_{n-1}]$
$N_n = \sqrt{G_n} = \{x \in G : x^m \in G_n\}$ (normal)

$$G = N_0 \supset N_1 \supset \ldots \supset N_n \supset \ldots$$

$N_i / N_{i+1} = (G_i / G_{i+1}) / torsion$

$$N = \text{Malcev completion of } \{G/N_i\}$$

Theorem (Chen, Morgan, Hain, Campana, etc...)

N depends only on $\phi : \bigwedge^2 H^1(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$.

In particular if $\ker\phi \neq 0$ then $\pi_1(X)$ is not abelian. To illustrate, we recall the following

Theorem (J. Amòros-I. Bauer)

Let X be a compact algebraic variety whose fundamental group admits a presentation with n generators and s relations; then

$$s - n \geq \dim Im\phi - 2q.$$

Castelnuovo-de Franchis

There is a classical case where $\pi_1(X)$ is not abelian:

Theorem

Let S be an algebraic surface; $0 \neq \omega_1 \wedge \omega_2 \in \ker \phi^{2.0}$ $\iff$ there is a non-constant map $f : S \to C$, C a curve of genus $g \geq 2$.

Corollary

If a surface S has irregularity q and no fibrations over curves of genus > 1, then ($p_g = \dim H^0(\Omega^2_X)$)

$$p_g \geq 2q - 3.$$

Definition

We call the line $p_g = 2q - 3$ the Castelnuovo-de Franchis line (or **trench**).

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Conjecture (M. Mendes Lopes, R. Pardini)

Let X be without fibrations over curves of genus > 1, and $q > 2$. If X is on the trench then

$$q = 3 \quad and \quad X = C_2$$

where C_2 is the 2-symmetric product of C genus 3 curve.

Remark

If there are not (generalized) Lagrangian structures then:

$$p_g \geq 4q - 11.$$

(The trench is far away).

Catanese and BGG

There are many generalizations of the Castelnuovo-de Franchis theorem (one should also mention Ran, Beauville, Siu, *et cetera*):

Definition

We say that X is of Albanese strict type if α is generically finite, but α is not surjective.

Definition

We say that a (rational map) $f : X \to Y$ is an s-Albanese fibration if

1. Y is of Albanese strict type;
2. $\dim X - \dim Y = s$.

Theorem (Fabrizio Catanese (Invent. Math. '91))

There is a one-to-one correspondence between fibrations of Albanese strict type and maximal isotropic subspaces (isotropic Hodge-substructure) of the first cohomology group of X.

Remark

Fabrizio's results gives that the Lagrangian structures are the simplest tensors in the kernel of: $\phi^{2.0} : \bigwedge^2 H^{1.0} \to H^{2.0}$ which are not pull-back of a map $f : X \to Y$.

Using *BGG* correspondence, $\mathbb{H}_{\mathbb{C}} = \oplus H^p(X, \mathbb{C})$ is a module over $\mathbb{H}' = \oplus \bigwedge^p H^{1.0}$. Inequalities of Castelnuovo-de Franchis type have been obtained by Lazarsfeld and Popa. In particular:

Theorem (R. Lazarsfeld-M. Popa)

If X is a surface without fibrations on curves of genus $g \geq 2$ then $h^{1.1} = \dim H^1(X, \Omega^1_X) \geq 3q - (1)$.

Remark

For minimal surfaces of general type Bogomolov-Miyaoka-Yau gives

$$h^{11} \geq 1 + q + p_g.$$

A symplectic topological result

Let (X, Ω) be a Lagrangian structure, set $U = X \setminus sing(\Omega)$.
Lagrangian duality gives the exact sequence :

$$0 \to {\Omega^1_U}^* \to W \otimes \mathcal{O}_U \to \Omega^1_U \to 0.$$

Theorem (M.A. Barja, J.C. Naranjo, G.P.)

If (X, Ω) is a generalized Lagrangian variety such that $codim(sing(\Omega)) \geq 2$ then $c_1(X)^2 - 2c_2(X)$ is a nef class.

If $\dim X = 2$ and $\dim(sing(\Omega)) < 1$, or $sing(\Omega)$ is a normal-crossing **connected divisor**, *then the topological signature $\tau(X)$ of X is not negative, that is:*

$$K_X^2 \geq 8\chi(X).$$

We made the following

Conjecture

If (X, Ω) is a Lagrangian surface then $\tau(X) \geq 0$ (similarly for generalized Lagrangian).

Remark

If the conjecture was true one could divide the irregular varieties:

1. fibred over curves of genus > 1;
2. non fibred:
   - i) no Lagrangian structure $p_g > 4q - 11$,
   - ii) generalized Lagrangian $K^2 \geq 8\chi$.

A good weapon to avoid the trench!

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Trench warfare

Mais, mon colon, celle que j'préfère
C'est la guerr' de quatorz'-dix-huit

G. Brassens

Theorem (M. Mendes Lopes, R. Pardini, G.P.)

No nonfibred surface with $q = 5$ and $p_g = 7$.

The proof use some *hard* analysis of the canonical and bicanonical map (ML-P) and some earlier estimate on h^{11} from Hodge theory (A. Causin-G.P.).

Examples of h-irregular (but not hb) given by

1. Sommese-van de Ven
2. Campana
3. Arapura-Nori
4. ...

Example (Lagrangian surfaces of Bogomolov-Tschinkel)

They use a correspondence on

$$\Gamma \subset K_1 \times K_2$$

K_i are Kummer surfaces of A_i, $i = 1, 2$, Γ is a $K3$, then the pull-back

$$X \subset A_1 \times A_2$$

is a Lagrangian surface.

Geometric Galois Closure

Used many times: *e.g.* M. Teicher's school (Amram-Teicher-Vishne: examples of non fibred surfaces with non finite nilpotent tower).

Let $f : Z \to Y$ be a dominant generically finite map (or rational) of degree $d > 2$. Set

$$V = \{z = (z_1, ..., z_d) \in Z^d : \{z_1, ..., z_d\} = f^{-1}(y), y \in Y\}.$$

Remark

$$V \equiv \{z \in Z^{d-1} : \exists z_d \in Z : \{z_1, ..., z_d\} = f^{-1}(y),\ y \in Y\}.$$

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Definition

Let X be the normalization of the closure of a component V^0 of V; let $g\colon X \to Y$ be defined by

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Z \\ g\searrow & & \swarrow f \\ & Y & \end{array}$$

π projection. Then (X, g) is the Galois closure of f.

Theorem

Assume

1. *the Galois group of f is the full-symmetric group σ_d (i.e. V^0 is irreducible);*
2. $h^{1.0}(Y) = h^{2.0}(Y) = 0$ *(e.g. $Y = \mathbb{P}^n$);*
3. *the Albanese map $Z \to A = Alb(Z)$ is generically finite ($q(Z) \geq n = \dim Z$).*

Then

1. $h^{1.0}(X) \geq (d-1)q(Z)$;
2. $\dim(\ker \phi^{2.0}) \geq \binom{q(Z)}{2}$.

In particular X is hb-irregular and $\pi_1(X)$ is not abelian.

$\phi^{2.0}_X : \bigwedge^2 H^0(\Omega^1_X) \to H^0(X, \Omega^2_X)$

Proof.

Set $q = q(Z)$. The σ_d equivariant map: $j : X \to Z^d$ gives σ_d-representation maps: $j^* : H^{p.0}(Z^d) \to H^{p.0}(X)$:

1. $H^{1.0}(Z^d) = \Gamma^q + \mathbb{C}^q$, ($\Gamma$ standard, $\mathbb{C}$ trivial repr.). One has
$$\Gamma^q \hookrightarrow H^{1.0}(X).$$
2. We have
$$\bigwedge^2 \Gamma^q \hookrightarrow \bigwedge^2 H^{1.0}(Z).$$
Since $H^{2.0}(Y) = 0$ the invariant part is in the kernel: $\left(\bigwedge^2 \Gamma^q\right)^{\sigma_d} \subset \ker \phi^{2.0}$, then (easy computation)
$$\left(\bigwedge^2 \Gamma^q\right)^{\sigma_d} \supset \mathbb{C}^{\binom{q}{2}}.$$

Remark

Easy to satisfy the assumption: embed an abelian variety in $\mathbb{P}^n$ and take generic projections.

Remark

The elements $\ker \phi^{2.0}$ constructed have the type

$$\alpha_1 \wedge \beta_1 + \ldots + \alpha_{d-1} \wedge \beta_{d-1}.$$

They give Lagrangian structures only if $n = d - 1$. We have only one example with $d = n + 1$.

Degree of irrationality

One defines the degree of irrationality of a variety X:

$$d_i(X) = \min\{\deg(g) \mid g\colon X \dashrightarrow \mathbb{P}^n \text{ dominant}\}.$$

Remark

Yoshihara-Tokunaga proved that if S is an abelian surface with polarization (1.2) then $d_i(S) = 3$.
Bastianelli (Trans. 2010) has proved that $d_i(C_2) \geq g - 1$ of the 2-symmetric product C_2 of a curve C of genus g.

Problem

Let $A = J(C)$ be the general principal polarized abelian surface. Is there a $3:1$ rational map $g\colon A \to \mathbb{P}^2$?

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(1.2)-abelian surfaces (Barth)

Let S be an abelian surface with an irreducible polarization of type (1.2) – *i.e.* there is a smooth curve $C \subset S$, $C^2 = 4$, and genus 3 (S is the Prym variety of bielliptic curves of genus 3).

- $|C|$ is a pencil with 4 base points, P_0, P_1, P_2, P_3

- if $P_0 = O_S$ is the origin of S then $2P_i = O_S$.

The construction (Xiao, Yoshihara-Tokunaga)

Blow-up the point P_i then we have $p : S' \longrightarrow \mathbb{P}^1$ and 4 sections E_i. The general curve C of the pencil is not hyperelliptic. Now use the point P_0 to define $C \longrightarrow |K_C - (P_O)| = \mathbb{P}^1$, and glue:

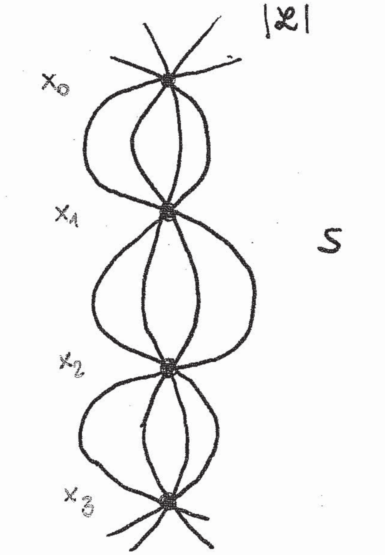
$$p_*(\omega_{S'}(-E_0))$$

to get a $3:1$ rational map $g : S' \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.
Solving the singularity along the smooth 6 hyperelliptic curves $Z \longrightarrow S'$ we get a finite map

$$f : Z \longrightarrow F_3$$

(F_3- Hirzebruch).

F.B.



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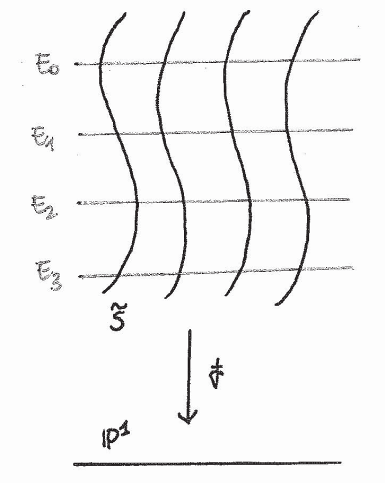

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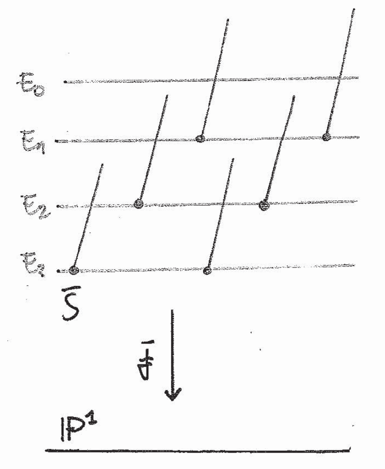

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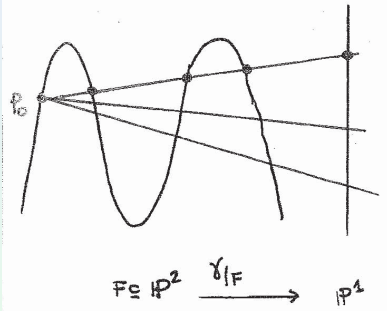

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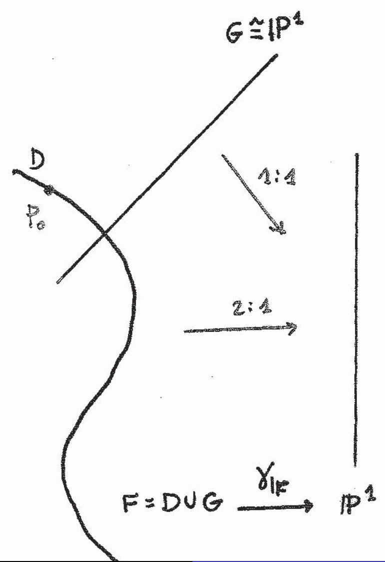

LG Surface

We define the Galois closure $g : X' \longrightarrow Z$ to be the Galois closure of $f : Z \longrightarrow F_3$.

Definition

We call LG (Lagrangian-Galois) surface the minimal desingularization X of X'.

Theorem

The LG surface X is a surface of general type with invariants

$$K_X^2 = 198, \ c_2(X) = 102, \ \chi(\mathcal{O}_X) = 25, \ q = 4, \ p_g = 28.$$

X is Lagrangian with $Alb(X) = S \times S$, and it does not have any fibration over curves of genus ≥ 2.

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Corollary

Let $\tau(X)$ be the signature of X. Then

$$\tau(X) = -2.$$

The conjecture (BNP) is false and the connectedness of $sing(\Omega)$ is important:

Proposition

For a general S the Galois closure $X = X'$ is smooth. If Ω is the Lagrangian structure of X, then $sing(\Omega)$ consists of the 6 smooth rational connected components disjoint (-3)-curves.

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ingredients

1. fibred Galois covering;
2. the Galois covering has degree 2;
3. 2- and 3-torsion points;
4. the special fibers;
5. the group action;
6. moduli of (1.2) surfaces.

proof 1

The maps $f : X \to Z$ and $g : Z \to F_3$ are fiber space map:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ p' \searrow & & \swarrow p \\ & \mathbb{P}^1 & \end{array}$$

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & F_3 \\ p \searrow & & \swarrow k \\ & \mathbb{P}^1 & \end{array}$$

The general fibre D of $p' : X \to \mathbb{P}^1$ are the Galois closure of the fiber C, $p : Z \to \mathbb{P}^1$:

$$f : D \to C.$$

proof 2

-The Galois closure $f : X \to Z$ of $g : Z \to F_3$ is $2:1$.

We need the branch divisor R of f.

-The branch of f is the ramification divisor of g.

The intersection with the general fiber is the C branch of the $3:1$ map $f : C \to \mathbb{P}^1$:

$$R \cdot C = 10.$$

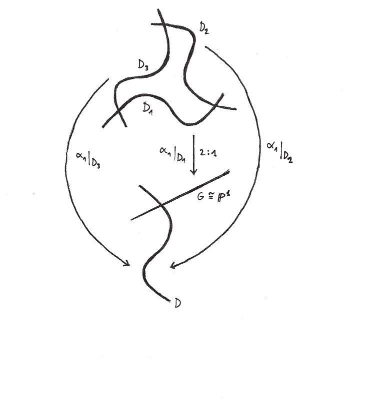
proof 3: torsion points

1. $P_0 = O$ origin of the abelian surface;
2. P_1, P_2, P_3 in the kernel of the polarization;
3. 12 other torsion points (corresponds to the singular points);
4. the 3-torsion points locus S_3 (non trivial)

$$S_3 \setminus \{O\} \subset R.$$

proof 4: special fiber

The smooth hyperelliptic curves gives the picture (F.B.):



proof 5

We get that Néron Severi group of Z general is generated by C, E_i, G_{ij}.

1. The branch divisor R is numerically equivalent to

$$-2E_0 + 4\sum_{k=1}^{3} E_k + 20C - 4\sum_{k=1}^{3}(G_{1k} + G_{2k}).$$

2. The branch divisor R is reduced and has at most simple singularities.

proof 6

From theory of double covering:

$$K_X^2 = 198, \; c_2(X) = 102, \; \chi(\mathcal{O}_X) = 25$$

from Hodge index

$$\tau(X) = -2.$$

proof 7: irregularity

Using representation theory of σ_3 one proves that the irregularity is 4.

Analysing the special fibers and the group action one shows that

$$Alb(X) = S \times S.$$

proof 8: no irrational pencil of genus $g \geq 2$

The tricky part. By contradiction let $h : X \to B$ be the pencil. Then $genus(B) = 2$, and

$$J(B) = S/ < P_0, P_i >$$

for any choice of $i = 1, 2, 3$. Define a natural map :

$$M : Moduli.\ ab.(1.2) - surf \to\ hilb^3(Moduli\ \ PPA - surf).$$

$$M(S) = S/ < P_0, P_1 > + S/ < P_0, P_2 > + S/ < P_0, P_3 >$$

We obtain $Image(F) \subset diagonal$.
The product of an elliptic curve $S = E \times E$ shows this is false.

proof 8: complements

Using the monodromy of the points of order 3 and some analysis on special surfaces (the S obtained as Prym of the Fermat curve $X^4 + Y^4 + Z^4 = 0$) one shows that $X = X'$ for general S.

We can use also the methods of (BNP) to compute $\tau(X)$.

proof 9: complements

Corollary

Let $\phi^{1.1} : H^{1.0}(X) \otimes H^{0.1}(X) \to H^1(X, \Omega^1)$, *then* $\dim \ker(\phi^{1.1}) = 5$ *and* $\dim(\ker \phi) = 7$.

$(\phi : \bigwedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C}))$.

Remark (A. Causin-G.P.)

We proved $\dim(\ker \phi) \geq 7$ if $q = 4$ and no irrational pencil of genus ≥ 2. The bound is attained in the example.

With A. Collino and J.C. Naranjo we discover a surface, connected with the the Fano surface F of line on the the smooth cubic $3-$fold V, and a structure similar to the previous surface: $S \subset F^3$

$$S = \{(\ell_1, \ell_2, \ell_3\} \in F^3 : \ell_1 \cap \ell_2 \cap \ell_3 = \{p\};\ < \ell_1, \ell_2, \ell_3 >= \mathbb{P}^2\}$$

We have a σ_3 action on S, the quotient is the vertex map $v : S \to V$

$$v(\ell_1, \ell_2, \ell_3) = p.$$

1. Compute the nilpotent tower of the Galois closures.
2. Compute the fundamental group.
3. (Campana) Find a Lagrangian variety of a simple abelian variety.

I thank you for the attention and in particular

Alessandro, Ciro and Fabrizio.