DERIVED EQUIVALENCE AND THE PICARD VARIETY

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If X = smooth projective variety over $k = \overline{k}$, denote

$$\mathbf{D}(X) := \mathbf{D}^b(\mathrm{Coh}(X))$$

the bounded derived category of coherent sheaves on X.

Central problem originating in mirror symmetry (now also birational geometry):

Given X and Y two smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ (linear equivalence of triangulated categories), what is the relationship between basic numerical invariants of X and Y, or between geometric properties of X and Y?

Known results (due to **Bondal, Orlov, Kawamata**): if $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then

- $\dim X = \dim Y$.

- $\kappa(X) = \kappa(Y)$ (Kodaira dimension).

- $\nu(X) = \nu(Y)$ (numerical dimension).

- ω_X and ω_Y have the same (possibly ∞) order.

- ω_X is nef $\iff$ ω_Y is nef.

- $R(X) \simeq R(Y)$ as k-algebras, where $R(X) := \oplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$ is the canonical ring. (This implies in particular that $P_m(X) = P_m(Y)$ for all m, so $\kappa(X) = \kappa(Y)$, and *reconstruction*: if ω_X and ω_Y are ample or anti-ample, then $X \simeq Y$.)

On the other hand, there are plenty of examples of X and Y which are **not birational**, but such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$:

• a (non-principally polarized) abelian variety A and its dual $\widehat{A}$ (**Mukai**).

• $S = K3$ surface and M_S a moduli space of sheaves on S (another $K3$) for well-chosen invariants (**Mukai**).

• elliptic surfaces (**Bridgeland, Uehara,...**).

• (strong) Calabi-Yau threefolds (**physicists, Kuznetsov, Borisov-Căldăraru**)

Question: How about other fundamental topological or holomorphic invariants, like Betti numbers, Hodge numbers?

Conjecture (Kontsevich,...): If X and Y are (weak) Calabi-Yau manifolds with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then

$$h^{p,q}(X) = h^{p,q}(Y), \ \forall p, q.$$

(so $b_i(X) = b_i(Y)$ for all i.)

More general question: Is this true for *all* X and Y such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$?

If X and Y are of **general type**, then the answer is yes, by combining two important results:

- $\mathbf{D}(X) \simeq \mathbf{D}(Y) \implies X \sim_K Y$ (**Kawamata**).

- $X \sim_K Y \implies h^{p,q}(X) = h^{p,q}(Y), \forall p, q.$ (**Kontsevich, Batyrev, Denef-Loeser**)

Recall that $X \sim_K Y$ means that there exists a smooth birational model Z dominating X and Y

$$f : Z \to X,\ g : Z \to Y \text{ such that } f^* K_X = g^* K_Y.$$

Aside: The second result implies that Hodge numbers are invariant for birational Calabi-Yau manifolds, which are in fact also conjectured to be derived equivalent. This is known in dimension up to three:

• If X and Y are birational Calabi-Yau threefolds, then $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ (**Bridgeland**).

But, as we saw above, there are non-birational derived equivalent Calabi-Yau's.

Another general invariant: Hochschild (co)homology. (**Kontsevich; Căldăraru, Orlov...**) An invariant with no apparent birational geometry interpretation, but related to the deformation theory of derived categories, and with numerical consequences:

$$\mathbf{D}(X) \cong \mathbf{D}(Y) \implies HH(X) \cong HH(Y),$$

where (j denotes the diagonal embedding of X):

$$HH(X) := \bigoplus_{i,l} \mathrm{Ext}^i_{X \times X}\big(j_* \mathcal{O}_X, j_* \omega_X^{\otimes l}\big)$$

and the induced isomorphism preserves the natural bigrading on HH. This contains the following statements:

- When $i = 0$, we obtain the canonical ring $R(X) = \bigoplus_{l \geq 0} HH_{0,l}(X)$.

- When $l = 0$ we obtain the *Hochschild cohomology*

$$HH^i(X) := \mathrm{Ext}^i_{X \times X}\big(j_*\mathcal{O}_X, j_*\mathcal{O}_X\big) \cong \bigoplus_{p+q=i} H^p(X, \bigwedge^q T_X),$$

where the last isomorphism is the Hochschild-Kostant-Rosenberg isomorphism for cohomology.

- When $l = 1$ we obtain the *Hochschild homology*

$$HH_i(X) := \mathrm{Ext}^i_{X \times X}\big(j_*\mathcal{O}_X, j_*\omega_X\big) \cong \bigoplus_{p+q=i} H^p(X, \bigwedge^q T_X \otimes \omega_X),$$

where the last isomorphism is the Hochschild-Kostant-Rosenberg isomorphism for homology.

Remarks. (1) The isomorphism $HH^1(X) \cong HH^1(Y)$ is equivalent to

$$H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X) \cong H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O}_Y),$$

which in particular gives

$$h^0(X, \Omega^1_X) + h^0(X, T_X) = h^0(Y, \Omega^1_Y) + h^0(Y, T_Y).$$

(2) Via Serre duality, the isomorphism $HH_i(X) \cong HH_i(Y)$ is equivalent to

$$\bigoplus_{p-q=i} H^p(X, \Omega^q_X) \cong \bigoplus_{p-q=i} H^p(Y, \Omega^q_Y).$$

so the sum of the Hodge numbers on the columns in the Hodge diamond is constant, i.e. for all i

$$\sum_{p-q=i} h^{p,q}(X) = \sum_{p-q=i} h^{p,q}(Y).$$

An immediate calculation shows then the following:

Corollary. Assume that $\mathbf{D}(X) \cong \mathbf{D}(Y)$.

(i) If X and Y are surfaces, then $h^{p,q}(X) = h^{p,q}(Y)$, for all p, q.

(ii) If X and Y are threefolds, the same thing holds, except for

$$2h^{1,0}(X) + h^{2,1}(X) = 2h^{1,0}(Y) + h^{2,1}(Y).$$

So the invariance of $h^{1,0}$ would imply the invariance of all Hodge numbers for threefolds. But of course $h^{1,0}(X) = q(X) = \dim \mathrm{Pic}^0(X)$.

Natural question becomes: if $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, what is the relationship between $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$?

Since for an abelian variety A we have $\mathbf{D}(A) \simeq \mathbf{D}(\widehat{A})$, we cannot expect isomorphism. For abelian varieties the situation is in fact completely understood:

Theorem (Orlov). Let A and B be two abelian varieties. Then $\mathbf{D}(A) \cong \mathbf{D}(B)$ if and only if there exists an isometric isomorphism $\Psi : A \times \widehat{A} \cong B \times \widehat{B}$, i.e. with $\tilde{\Psi} = \Psi^{-1}$, where if

$$\Psi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ then } \tilde{\Psi} := \begin{pmatrix} \widehat{\delta} & -\widehat{\beta} \\ -\widehat{\gamma} & \widehat{\alpha} \end{pmatrix}.$$

(In particular A and B are isogenous.)

Given this, the most we can hope for in general is

Conjecture. If $\mathbf{D}(X) \cong \mathbf{D}(Y)$, then $\mathbf{D}(\mathrm{Pic}^0(X)) \cong \mathbf{D}(\mathrm{Pic}^0(Y))$.

Don't know how to prove this conjecture, but the main result is its principal consequence, equally good in applications.

Theorem (—-Schnell). Let X and Y be smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then

(1) $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ are isogenous.

(2) $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(Y)$ unless X and Y are étale locally trivial fibrations over isogenous positive dimensional abelian varieties (hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0$).

(3) In particular

$$h^0(X, \Omega_X^1) = h^0(Y, \Omega_Y^1) \quad \text{and} \quad h^0(X, T_X) = h^0(Y, T_Y).$$

Corollary. Let X and Y be smooth projective threefolds with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then

$$h^{p,q}(X) = h^{p,q}(Y)$$

for all p and q.

Other quick applications:

• Let X and Y be smooth projective fourfolds with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then $h^{2,1}(X) = h^{2,1}(Y)$. If in addition $\mathrm{Aut}^0(X)$ is not affine, then $h^{2,0}(X) = h^{2,0}(Y)$ and $h^{3,1}(X) = h^{3,1}(Y)$.

• Simple example of classification use of the invariance of the irregularity:

If $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, and X is an abelian variety, then so is Y (Huybrechts, Nieper-Wisschirchen).

Proof: By the invariance of Kodaira dimension we get $\kappa(Y) = 0$, and by the above $q(Y) = \dim Y$. A result of Kawamata says that Y is then birational to an abelian variety B. But $\omega_X \simeq \mathcal{O}_X$, so derived invariance implies $\omega_Y \simeq \mathcal{O}_Y$ as well. Hence $Y \simeq B$.

Idea of proof of Theorem: use a result of Rouquier on the invariance of certain types of derived autoequivalences, and the theory of actions of non-affine algebraic groups.

Well-known result of Orlov: if $\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$ is an equivalence, then there exists an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, unique up to isomorphism, such that Φ is the integral functor

$$\Phi = \Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y), \ \Phi_{\mathcal{E}}(\cdot) = \mathbf{R}p_{2*}(p_1^*(\cdot) \overset{\mathbf{L}}{\otimes} \mathcal{E}).$$

Theorem (Rouquier). Let $\Phi = \Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ be an equivalence, induced by $\mathcal{E} \in \mathbf{D}(X \times Y)$. Then Φ induces an isomorphism of algebraic groups

$$F\colon \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \simeq \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$$

defined by:

$$F(\varphi, L) = (\psi, M) \iff \Phi_{\mathcal{E}} \circ \Phi_{(\mathrm{id},\varphi)_* L} \cong \Phi_{(\mathrm{id},\psi)_* M} \circ \Phi_{\mathcal{E}}.$$

(Notation: $(\mathrm{id}, \varphi) : X \to X \times X, \ x \mapsto (x, \varphi(x))$.)

Actions of non-affine algebraic groups. G = connected algebraic group over a field. According to Chevalley's theorem:

$$1 \longrightarrow \mathrm{Aff}(G) \longrightarrow G \longrightarrow \mathrm{Alb}(G) \longrightarrow 1$$

where:

- $\mathrm{Aff}(G)$ = unique maximal connected affine subgroup of G

- $\mathrm{Alb}(G) = G/\mathrm{Aff}(G)$ is an abelian variety, which is the Albanese variety of G. (The map $G \to \mathrm{Alb}(G)$ is the Albanese map of G, i.e. the universal morphism to an abelian variety (Serre); it is locally trivial in the Zariski topology.)

Now let X be a smooth projective variety, and take $G \subset \mathrm{Aut}(X)$. G acts naturally on $\mathrm{Alb}(X)$, inducing a map of abelian varieties

$$\mathrm{Alb}(G) \longrightarrow \mathrm{Alb}(X),$$

with image contained in the Albanese image $\mathrm{alb}_X(X)$.

Theorem 1 (Nishi, Matsumura). The group G acts on $\mathrm{Alb}(X)$ by translations, and the kernel of the induced homomorphism $G \to \mathrm{Alb}(X)$ is affine. Consequently, the induced map $\mathrm{Alb}(G) \to \mathrm{Alb}(X)$ has finite kernel.

Take now: $G := \mathrm{Aut}^0(X)$ the connected component of the identity in $\mathrm{Aut}(X)$.

Note: By Chevalley + Rouquier, if $\mathrm{Aut}^0(X)$ and $\mathrm{Aut}^0(Y)$ are affine, then $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(Y)$. Otherwise general results of Brion imply the condition in part (b) of the Theorem (so the obstruction to isomorphism is the presence of abelian varieties!).

I will sketch a proof of $q(X) = q(Y)$. The isogeny statement uses similar methods, but is more technical.

Recall that by Rouquier's theorem there is an isomorphism of algebraic groups

$$F\colon \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \simeq \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$$

Lemma. $F(\varphi, L) = (\psi, M)$ is equivalent to having an isomorphism

$$p_1^* L \otimes (\varphi \times \mathrm{id})^* \mathcal{E} \simeq p_2^* M \otimes (\mathrm{id} \times \psi)_* \mathcal{E}$$

on the product $X \times Y$.

If this isomorphism sends $\mathrm{Aut}^0(X)$ to $\mathrm{Aut}^0(Y)$ and $\mathrm{Pic}^0(X)$ to $\mathrm{Pic}^0(Y)$, we're done. Otherwise we take advantage of the **mixing** between the two.

Consider the induced map

$$\pi\colon \mathrm{Pic}^0(X) \to \mathrm{Aut}^0(Y), \quad \beta(L) = p_1\big(F(\mathrm{id}, L)\big),$$

and let $A = \mathrm{Im}\ \pi$, an abelian variety which acts on Y.

Fix a point $(x, y) \in \mathrm{Supp}\ \mathcal{E}$, where $\mathcal{E}$ is the kernel of the Fourier-Mukai equivalence. Take the orbit map

$$f : A \longrightarrow Y = \{x\} \times Y,\ a \to (x, a \cdot y).$$

By the Nishi-Matsumura theorem we have that the induced map $A \to \mathrm{Alb}(Y)$ has finite kernel, which gives that the pull-back map

$$f^* : \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(A)$$

is **surjective**.

Define $\mathcal{F} = (\mathrm{id}_x \times f)^* \mathcal{E} \in \mathbf{D}(A)$. The relation in the Lemma above can be translated into the following

$$t_a^* \mathcal{F} \otimes f^* M \cong \mathcal{F}$$

where $a \in A$ is the point corresponding to the automorphism ψ. $(F(\mathrm{id}, L) = (\psi, M).)$ This makes every cohomology sheaf $\mathcal{H}^i(\mathcal{F})$ into a **semihomogeneous vector bundle** on A, which implies by a simple calculation that

$$\dim \mathrm{Ker}\ \pi \leq \dim \mathrm{Ker}\ f^*.$$

By the above this gives

$$q(X) - \dim A = \dim \mathrm{Ker}\ \pi \leq \dim \mathrm{Ker}\ f^* = q(Y) - q(A)$$

i.e. $q(X) \leq q(Y)$.

• More refined properties of semi-homogeneous vector bundles due to Mukai lead to the isogeny statement.

HAPPY BIRTHDAY
ALESSANDRO,
CIRO,
FABRIZIO!