Universal Formulas for Counting Nodal Curves on Surfaces

Yu-jong Tzeng

Stanford University
Harvard University

Perspectives on Algebraic Varieties
September 8, 2010

Outline of Topics

1. Introduction: Counting Nodal Curves and Previous Results

Outline of Topics

1. Introduction: Counting Nodal Curves and Previous Results
2. Universal Formulas: Göttsche's Conjecture and Main Theorems

Outline of Topics

Outline of Topics

1. Introduction: Counting Nodal Curves and Previous Results
2. Universal Formulas: Göttsche's Conjecture and Main Theorems
3. Algebraic cobordism
4. Degeneration formula

Introduction

S: a smooth projective surface over $\mathbb{C}$
L: a line bundle on S

Main Question:

How many reduced curves in $|L|$ which

1. have r simple nodes
2. contain no higher singularity
3. pass through $\dim |L| - r$ points in general position?

Notation: this kind of curves are called *r-nodal*.

Previous Results on $\mathbb{P}^2$

To counting nodal curves, we begin with *well-known* surfaces:

On $\mathbb{P}^2$, $\mathcal{O}(d)$: (Ran, Manin, Kontsevich, Harris, Caporaso, Choi, Pandharipande, Fomin, Mikhalkin et. al.)

Previous Results on $\mathbb{P}^2$

To counting nodal curves, we begin with *well-known* surfaces:

On $\mathbb{P}^2$, $\mathcal{O}(d)$: (Ran, Manin, Kontsevich, Harris, Caporaso, Choi, Pandharipande, Fomin, Mikhalkin et. al.)

- Rational curves: [Kontsevich-Manin, 1994] recursive formulas on d

Previous Results on $\mathbb{P}^2$

To counting nodal curves, we begin with *well-known* surfaces:

On $\mathbb{P}^2$, $\mathcal{O}(d)$: (Ran, Manin, Kontsevich, Harris, Caporaso, Choi, Pandharipande, Fomin, Mikhalkin et. al.)

- Rational curves: [Kontsevich-Manin, 1994] recursive formulas on d
- Arbitrary genus: [Caporaso-Harris, 1998] recursive formulas on d, the number of nodes r and tangency conditions

Previous Results on $\mathbb{P}^2$

To counting nodal curves, we begin with *well-known* surfaces:

On $\mathbb{P}^2$, $\mathcal{O}(d)$: (Ran, Manin, Kontsevich, Harris, Caporaso, Choi, Pandharipande, Fomin, Mikhalkin et. al.)

- Rational curves: [Kontsevich-Manin, 1994] recursive formulas on d
- Arbitrary genus: [Caporaso-Harris, 1998] recursive formulas on d, the number of nodes r and tangency conditions

$\Longrightarrow$ We can find the number of r-nodal curves in $|\mathcal{O}(d)|$, for all d and r!

Previous Results on $\mathbb{P}^2$ and Hirzebruch surfaces

On $\mathbb{P}^2$, $\mathcal{O}(d)$:

- Polynomiality: [Fomin-Mikhalkin, 2009] the number of r-nodal curves in $\mathcal{O}(d)$ is a polynomial in d, $\forall d \geq 2r$.

Previous Results on $\mathbb{P}^2$ and Hirzebruch surfaces

On $\mathbb{P}^2$, $\mathcal{O}(d)$:

- Polynomiality: [Fomin-Mikhalkin, 2009] the number of r-nodal curves in $\mathcal{O}(d)$ is a polynomial in d, $\forall d \geq 2r$.

On Hirzebruch surfaces and any line bundle:

- [Vakil, 2000]: Results similar to [Caporaso-Harris] hold.

Previous Results on K3

On K3 surfaces and primitive line bundles L (i.e. Pic $S \cong \mathbb{Z}L$)

Previous Results on K3

On K3 surfaces and primitive line bundles L (i.e. Pic $S \cong \mathbb{Z}L$)

- Rational curves: let $N_r =$ the number of r-nodal rational curves in $|L|$ (with $1 + \frac{1}{2}L^2 = r = g_a$).

Previous Results on K3

On K3 surfaces and primitive line bundles L (i.e. Pic $S \cong \mathbb{Z}L$)

- Rational curves: let N_r = the number of r-nodal rational curves in $|L|$ (with $1 + \frac{1}{2}L^2 = r = g_a$).

The Yau-Zaslow formula

$$\sum_{r=0}^{\infty} N_r q^{r-1} = \frac{1}{\Delta}.$$

where $\Delta = q \prod_{k>0}(1-q^k)^{24} = \eta^{24}$ is a modular form.

Previous Results on K3

On K3 surfaces and primitive line bundle L (i.e. Pic $S \cong \mathbb{Z}L$)

- Arbitrary genus:

Theorem (Bryan-Leung)

$$\sum_{r\geq 0} (\# \text{ of } r\text{-nodal curves in } |L|)\,(DG_2)^r = \frac{(DG_2/q)^{\chi(L)}}{\Delta D^2 G_2/q^2}.$$

G_2: the second Eisenstein series; $D = q\frac{d}{dq}$;
G_2, DG_2 and D^2G_2 are quasi-modular forms.

Counting r-nodal curves on general surfaces

Instead of working on special surfaces, we can work on all surfaces and start with r small.

Counting r-nodal curves on general surfaces

Instead of working on special surfaces, we can work on all surfaces and start with r small.

Price paid: the line bundle is assumed to be sufficiently ample (relative to r).

Counting r-nodal curves on general surfaces

Instead of working on special surfaces, we can work on all surfaces and start with r small.

Price paid: the line bundle is assumed to be sufficiently ample (relative to r).

- $r \leq 3$: standard intersection theory

Counting r-nodal curves on general surfaces

Instead of working on special surfaces, we can work on all surfaces and start with r small.

Price paid: the line bundle is assumed to be sufficiently ample (relative to r).

- $r \leq 3$: standard intersection theory
- $r \leq 6$: [Vainsencher, 1995]

Counting r-nodal curves on general surfaces

Instead of working on special surfaces, we can work on all surfaces and start with r small.

Price paid: the line bundle is assumed to be sufficiently ample (relative to r).

- $r \leq 3$: standard intersection theory
- $r \leq 6$: [Vainsencher, 1995]
- $r \leq 8$: [Kleiman-Piene, 1999]

Counting r-nodal curves on general surfaces

Instead of working on special surfaces, we can work on all surfaces and start with r small.

Price paid: the line bundle is assumed to be sufficiently ample (relative to r).

- $r \leq 3$: standard intersection theory
- $r \leq 6$: [Vainsencher, 1995]
- $r \leq 8$: [Kleiman-Piene, 1999]

They wrote the number of nodal curves explicitly.

Polynomials T_r

Suppose S is a smooth projective surface and L is a sufficiently ample line bundle on S.
Let T_r be the number of r-nodal curves in $|L|$, then Vainsencher proved:

Polynomials T_r

Suppose S is a smooth projective surface and L is a sufficiently ample line bundle on S.
Let T_r be the number of r-nodal curves in $|L|$, then Vainsencher proved:

$$T_1 = 3L^2 + 2LK_S + c_2(S)$$

$$T_2 = \frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2}$$

$$T_3 = (2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40)+(-18c_1(S)^2-117LK+672)L^2$$
$$+(-6c_1(S)^2-25LK-21L^2)c_2(S)-63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6;$$

T_4 = longer

T_5 = longer and longer

T_6 = you really don't want to know...

$$T_1 = 3L^2 + 2LK_S + c_2(S)$$
$$T_2 = \frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2}$$
$$T_3 = (2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40)$$
$$+(-18c_1(S)^2-117LK+672)L^2+(-6c_1(S)^2-25LK-21L^2)c_2(S)$$
$$-63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6;$$
$$T_4 = \dots$$
$$T_5 = \dots$$
$$T_6 = \dots$$

deg

$$
\begin{aligned}
T_1 &= 3L^2 + 2LK_S + c_2(S) \\
T_2 &= \frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2} \\
T_3 &= (2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40) \\
&\quad +(-18c_1(S)^2-117LK+672)L^2+(-6c_1(S)^2-25LK-21L^2)c_2(S) \\
&\quad -63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6; \\
T_4 &= \dots \\
T_5 &= \dots \\
T_6 &= \dots
\end{aligned}
$$

deg

$$T_1 = 3L^2 + 2LK_S + c_2(S) \qquad 1 \qquad L^2, LK, c_2(S)$$

$$T_2 = \frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2}$$

$$T_3 = (2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40)$$
$$+(-18c_1(S)^2-117LK+672)L^2+(-6c_1(S)^2-25LK-21L^2)c_2(S)$$
$$-63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6;$$

$$T_4 = \dots$$
$$T_5 = \dots$$
$$T_6 = \dots$$

		deg	
$T_1 =$	$3L^2 + 2LK_S + c_2(S)$	1	$L^2, LK, c_2(S)$
$T_2 =$	$\frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2}$	2	$L^2, LK, c_1(S)^2, c_2(S)$
$T_3 =$	$(2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40)$ $+(-18c_1(S)^2-117LK+672)L^2+(-6c_1(S)^2-25LK-21L^2)c_2(S)$ $-63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6;$		
$T_4 =$	$\dots$		
$T_5 =$	$\dots$		
$T_6 =$	$\dots$		

		deg	
$T_1 =$	$3L^2 + 2LK_S + c_2(S)$	1	$L^2, LK, c_2(S)$
$T_2 =$	$\frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2}$	2	$L^2, LK, c_1(S)^2, c_2(S)$
$T_3 =$	$(2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40)$ $+(-18c_1(S)^2-117LK+672)L^2+(-6c_1(S)^2-25LK-21L^2)c_2(S)$ $-63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6;$	3	$L^2, LK, c_1(S)^2, c_2(S)$
$T_4 =$	$\dots$		
$T_5 =$	$\dots$		
$T_6 =$	$\dots$		

		deg	
$T_1 =$	$3L^2 + 2LK_S + c_2(S)$	1	$L^2, LK, c_2(S)$
$T_2 =$	$\frac{T_1(-7+T_1)-6c_1(S)^2-25LK-21L^2}{2}$	2	$L^2, LK, c_1(S)^2, c_2(S)$
$T_3 =$	$(2T_2(-14+3L^2+2LK+c_2(S))+T_1(-6c_1(S)^2-25LK-21L^2+40)$ $+(-18c_1(S)^2-117LK+672)L^2+(-6c_1(S)^2-25LK-21L^2)c_2(S)$ $-63(L^2)^2-50(LK)^2+(-12c_1(S)^2+950)LK+292c_1(S)^2)/6;$	3	$L^2, LK, c_1(S)^2, c_2(S)$
$T_4 =$	$\dots$	4	$L^2, LK, c_1(S)^2, c_2(S)$
$T_5 =$	$\dots$	5	$L^2, LK, c_1(S)^2, c_2(S)$
$T_6 =$	$\dots$	6	$L^2, LK, c_1(S)^2, c_2(S)$

Same pattern for T_7 and T_8!

Theorem (Göttsche's conjecture)

For every integer $r \geq 0$, there exists a universal polynomial $T_r(x, y, z, t)$ of degree r such that

$$T_r(L^2, LK, c_1(S)^2, c_2(S)) = \text{ \# of } r\text{-nodal curves in } |L|$$

if L is $(5r-1)$-very ample.

Theorem (Göttsche's conjecture)

For every integer $r \geq 0$, there exists a universal polynomial $T_r(x, y, z, t)$ of degree r such that

$$T_r(L^2, LK, c_1(S)^2, c_2(S)) = \#\text{ of } r\text{-nodal curves in } |L|$$

if L is $(5r-1)$-very ample.

L is called k-very ample if for all zero-dimensional closed scheme $\xi \subset S$ of length $k+1$, $H^0(S, L) \longrightarrow H^0(L|_\xi)$ is surjective.

Remarks:

1. A proof using symplectic geometry was given by A.K. Liu (2000).

Remarks:

1. A proof using symplectic geometry was given by A.K. Liu (2000).
2. T_r is universal because it is independent of S and L.

Remarks:

1. A proof using symplectic geometry was given by A.K. Liu (2000).
2. T_r is universal because it is independent of S and L.
3. T_r can be computed by previous results on $\mathbb{P}^2$ and K3.

Remarks:

1. A proof using symplectic geometry was given by A.K. Liu (2000).
2. T_r is universal because it is independent of S and L.
3. T_r can be computed by previous results on $\mathbb{P}^2$ and K3.

Question Structure of T_r?

Remarks:

1. A proof using symplectic geometry was given by A.K. Liu (2000).
2. T_r is universal because it is independent of S and L.
3. T_r can be computed by previous results on $\mathbb{P}^2$ and K3.

Question Structure of T_r? Yes!

Theorem

There exist power series A_1, A_2, A_3, A_4 in $\mathbb{Q}[[x]]^\times$ such that

$$\sum_{r=0}^{\infty} T_r(L^2, LK, c_1(S)^2, c_2(S))x^r = A_1^{L^2} A_2^{LK_S} A_3^{c_1(S)^2} A_4^{c_2(S)}.$$

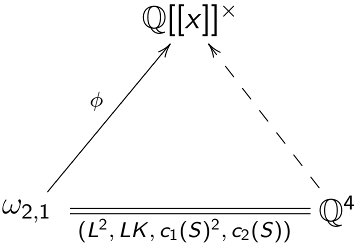
$\longrightarrow$ the generating function is multiplicative.

Theorem (Göttsche-Yau-Zaslow formula)

There exist two power series B_1 and B_2 in q such that

$$\sum_{r\geq 0} T_r(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^r = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Approach: Consider the following diagram



We will prove

1. The bottom is an isomorphism by algebraic cobordism.
2. ϕ is a homomorphism by degeneration formula
3. The theorems will follow from the induced homomorphism $\mathbb{Q}^4$ to $\mathbb{Q}[[x]]^\times$.

History of algebraic cobordism

- The algebraic cobordism theory (of schemes) was first constructed by Levine and Morel (2007).

History of algebraic cobordism

- The algebraic cobordism theory (of schemes) was first constructed by Levine and Morel (2007).
- Levine and Pandharipande constructed another theory using the double point relation (2009).

History of algebraic cobordism

- The algebraic cobordism theory (of schemes) was first constructed by Levine and Morel (2007).
- Levine and Pandharipande constructed another theory using the double point relation (2009).
- Two theories are isomorphic.

History of algebraic cobordism

- The algebraic cobordism theory (of schemes) was first constructed by Levine and Morel (2007).
- Levine and Pandharipande constructed another theory using the double point relation (2009).
- Two theories are isomorphic.
- We use algebraic cobordism of pairs of surfaces and line bundles to count nodal curves.

History of algebraic cobordism

- The algebraic cobordism theory (of schemes) was first constructed by Levine and Morel (2007).
- Levine and Pandharipande constructed another theory using the double point relation (2009).
- Two theories are isomorphic.
- We use algebraic cobordism of pairs of surfaces and line bundles to count nodal curves.
- Lee and Pandharipande generalize it to pairs of schemes and vector bundles of arbitrary dimension and rank.

Double point relation

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

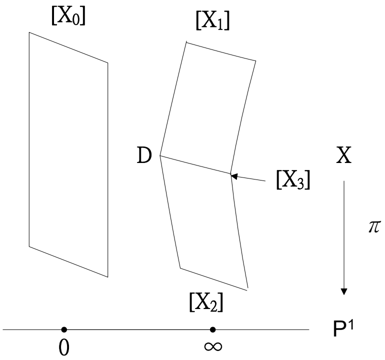
a double point relation if there exist a family X such that

Double point relation

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a double point relation if there exist a family X such that

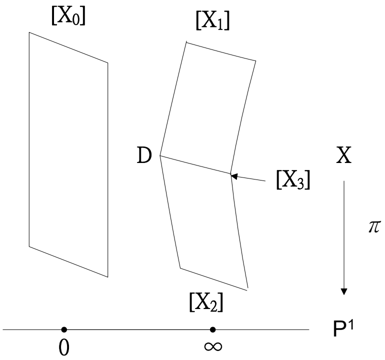


Double point relation

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a double point relation if there exist a family X such that



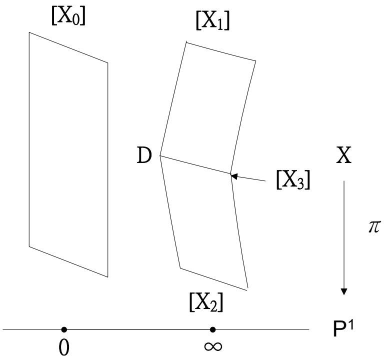
1. X_0 is the smooth fiber over 0.

Double point relation

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a double point relation if there exist a family X such that



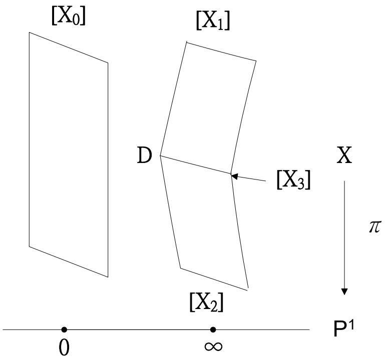
1. X_0 is the smooth fiber over 0.
2. fiber over ∞ has two components X_1 and X_2

Double point relation

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a double point relation if there exist a family X such that



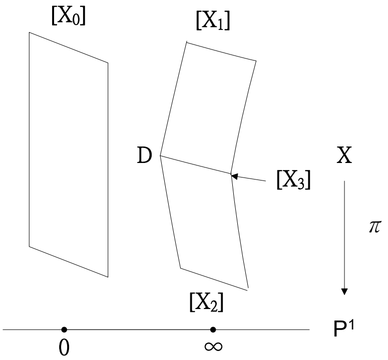
1. X_0 is the smooth fiber over 0.
2. fiber over ∞ has two components X_1 and X_2
3. X_1 and X_2 intersect transversally along a smooth divisor D

Double point relation

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a double point relation if there exist a family X such that



1. X_0 is the smooth fiber over 0.
2. fiber over ∞ has two components X_1 and X_2
3. X_1 and X_2 intersect transversally along a smooth divisor D
4. $X_3 \cong \mathbb{P}_D(N_{X_1/D} \oplus \mathcal{O}_D)$

Definition: Define $\omega_* = \bigoplus_{X \text{ smooth projective}} \mathbb{Q}[X]/$ dp relations

Theorem (Levine and Pandharipande, 2009)

Every smooth projective scheme can be degenerated to the sum of products of projective spaces with $\mathbb{Q}$-coefficients by dp relations, i.e.

$$\omega_* \cong \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_r)} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times \ldots \times \mathbb{P}^{\lambda_r}]$$

Definition: Define $\omega_* = \bigoplus_{X \text{ smooth projective}} \mathbb{Q}[X]/$ dp relations

Theorem (Levine and Pandharipande, 2009)

Every smooth projective scheme can be degenerated to the sum of products of projective spaces with $\mathbb{Q}$-coefficients by dp relations, i.e.

$$\omega_* \cong \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_r)} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times \ldots \times \mathbb{P}^{\lambda_r}]$$

Corollary: For every smooth projective surface S,

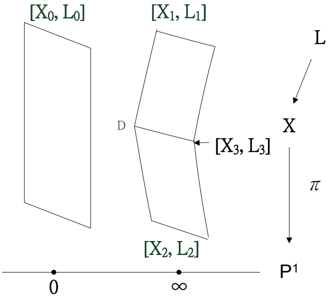
$$[S] = *[\mathbb{P}^2] + *[\mathbb{P}^1 \times \mathbb{P}^1].$$

Extended double point relation

Definition: Let X_i be smooth projective surfaces and L_i be line bundles on X_i. We call

$$[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3]$$

an extended double point relation if there exist a family X and line bundle L on X such that



1. $[X_0] = [X_1] + [X_2] - [X_3]$ is a dp relation
2. $L|_{X_i} = L_i$ for $i = 0, 1, 2$
3. $L_3 =$ the pullback of $L|_D$ to X_3

$[X_i, L_i]$ is called a pair (of a surface and a line bundle).

Definition

Define the algebraic cobordism group $\omega_{2,1}$ to be the $\mathbb{Q}$-vector space spanned by all pairs (of smooth projective surfaces and line bundles) modulo extended double point relations.

Definition

Define the algebraic cobordism group $\omega_{2,1}$ to be the $\mathbb{Q}$-vector space spanned by all pairs (of smooth projective surfaces and line bundles) modulo extended double point relations.

Theorem

$$\omega_{2,1} \xrightarrow{(L^2, LK, c_1(S)^2, c_2(S))} \mathbb{Q}^4$$

is an isomorphism.

Definition

Define the algebraic cobordism group $\omega_{2,1}$ to be the $\mathbb{Q}$-vector space spanned by all pairs (of smooth projective surfaces and line bundles) modulo extended double point relations.

Theorem

$$\omega_{2,1} \xrightarrow{(L^2, LK, c_1(S)^2, c_2(S))} \mathbb{Q}^4$$

is an isomorphism.

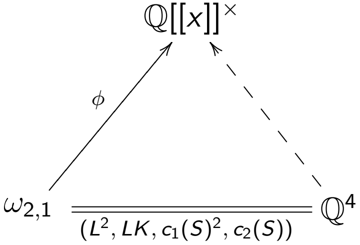
It is also easy to find bases of $\omega_{2,1}$, for example

$$\{[\mathbb{P}^2, \mathcal{O}], [\mathbb{P}^2, \mathcal{O}(1)], [\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}], [\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0)]\} \text{ and}$$

$$\{[\mathbb{P}^2, \mathcal{O}], [\mathbb{P}^2, \mathcal{O}(1)], [\mathrm{K3}, L_1], [\mathrm{K3}, L_2]\}, L_1^2 \neq L_2^2$$

are two bases.

conclusion We have the isomorphism on the bottom of the diagram.



Idea

Goal: Use degeneration of pairs to study the number of nodal curves.

Idea

Goal: Use degeneration of pairs to study the number of nodal curves.

Problem: After degeneration, ample line bundles may not be ample anymore. In that case number of nodal curves does not have good properties.

Idea

Goal: Use degeneration of pairs to study the number of nodal curves.

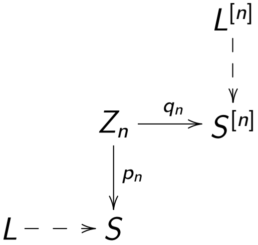
Problem: After degeneration, ample line bundles may not be ample anymore. In that case number of nodal curves does not have good properties.

Solution: Study the enumerative number $d_r(S, L)$ (suggested by Göttsche).

Definition of $d_r(S, L)$

Definition: Suppose L is a line bundle on S. Let

- $S^{[n]}$ be the Hilbert scheme of n points on S.
- $Z_n \subset S \times S^{[n]}$ be the universal closed subscheme.
- Then we define $L^{[n]} := (q_n)_*(p_n)^* L$.



Fact: $L^{[n]}$ is a vector bundle of rank n on $S^{[n]}$.

Definition: Let W^{3r} be the closure of

$$\left\{ \coprod_{i=1}^{r} \mathrm{Spec}(\mathcal{O}_{S,x_i}/m_{S,x_i}^2) \,|\, x_i \text{ are distinct points in } S \right\} \subset S^{[3r]}.$$

Define

$$d_r(S, L) = \int_{W^{3r}} c_{2r}(L_{3r}).$$

Proposition (Göttsche)

$d_r(S, L)$ equals the number of r-nodal curves in $[S, L]$ if L is $(5r-1)$-very ample.

Why is $d_r(S, L) = \int_{W^{3r}} c_{2r}(L_{3r})$ related to the number of r-nodal curves?

- A section $s \in |L| \implies$ a section $s^{[n]} \in |L^{[n]}|$.

Why is $d_r(S, L) = \int_{W^{3r}} c_{2r}(L_{3r})$ related to the number of r-nodal curves?

- A section $s \in |L| \implies$ a section $s^{[n]} \in |L^{[n]}|$.
- If $C := (s = 0)$ contains $\mathrm{Spec}(\mathcal{O}_{S,x}/m_{S,x}^2)$, then C has multiplicity two at x. Generically it is a node.

Why is $d_r(S,L) = \int_{W^{3r}} c_{2r}(L_{3r})$ related to the number of r-nodal curves?

- A section $s \in |L| \Longrightarrow$ a section $s^{[n]} \in |L^{[n]}|$.
- If $C := (s = 0)$ contains $\mathrm{Spec}(\mathcal{O}_{S,x}/m_{S,x}^2)$, then C has multiplicity two at x. Generically it is a node.
- W^{3r} =the closure of

$$\left\{ \coprod_{i=1}^{r} \mathrm{Spec}(\mathcal{O}_{S,x_i}/m_{S,x_i}^2) \,|\, x_i \text{ are distinct points in } S \right\}$$

$\longrightarrow$ the condition of r nodes

The advantage of using $d_r(S, L)$:

The advantage of using $d_r(S, L)$:

- it can be defined for all line bundles

The advantage of using $d_r(S, L)$:

- it can be defined for all line bundles
- behaves well in flat families

The advantage of using $d_r(S, L)$:

- it can be defined for all line bundles
- behaves well in flat families

So we hope to derive a formula about d_r in an extended double relation

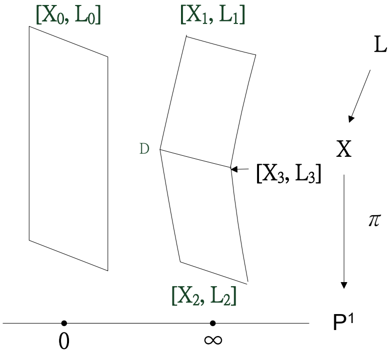
$$[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3].$$

Key tool

J. Li and B. Wu's construction of "the moduli stack of Hilbert schemes":

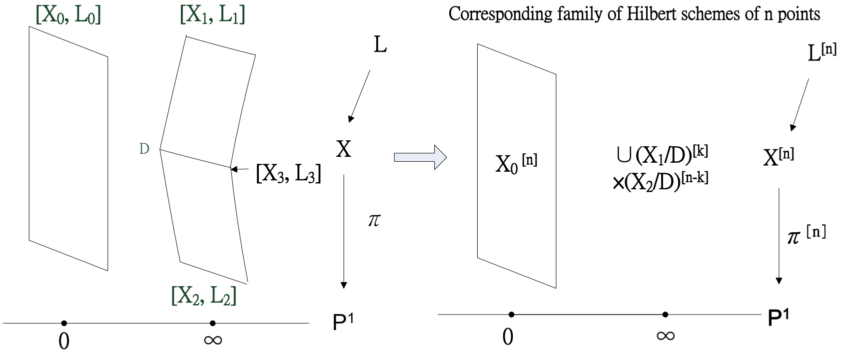
Key tool

J. Li and B. Wu's construction of "the moduli stack of Hilbert schemes":



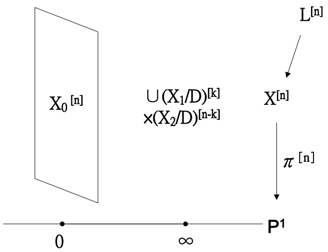
Key tool

J. Li and B. Wu's construction of "the moduli stack of Hilbert schemes":



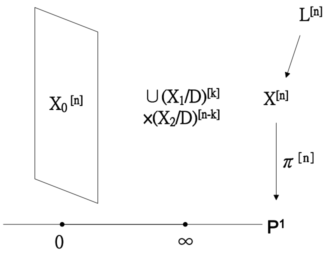
- $d_r(X_t, L_t)$ is a intersection number on $X_t^{[3r]}$
- The limit is $\sum_{k=0}^{n} d_k(X_1/D, L_1)d_{n-k}(X_2/D, L_2)$

Corresponding family of Hilbert schemes of n points



- $d_r(X_t, L_t)$ is a intersection number on $X_t^{[3r]}$
- The limit is $\sum_{k=0}^{n} d_k(X_1/D, L_1) d_{n-k}(X_2/D, L_2)$

Corresponding family of Hilbert schemes of n points



Proposition

Define $\phi(S, L)(x) = \sum_{r=0}^{\infty} d_r(S, L) x^r$, *then*

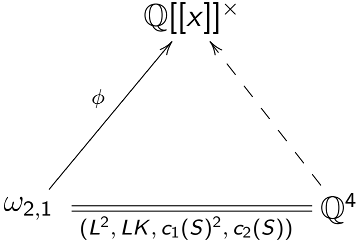
$$\phi(X_0, L_0) = \phi(X_1/D, L_1)\phi(X_2/D, L_2)$$

Theorem

If $[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3]$ *is an extended double point relation, then*

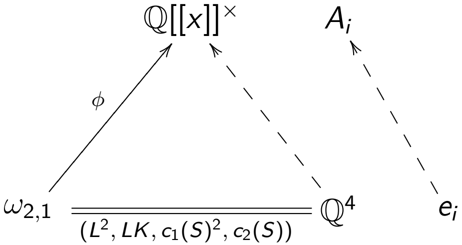
$$\phi(X_0, L_0) = \frac{\phi(X_1, L_1)\phi(X_2, L_2)}{\phi(X_3, L_3)}$$

i.e. ϕ *induced a homomorphism from* $\omega_{2,1}$ *to* $(\mathbb{Q}[[x]]^{\times}, \cdot)$.

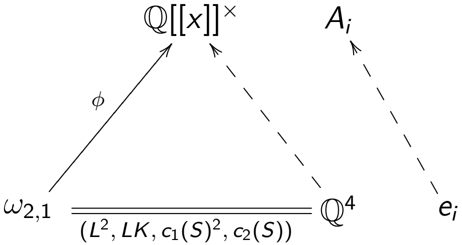


Now we are ready to prove theorems.

Now we are ready to prove theorems.

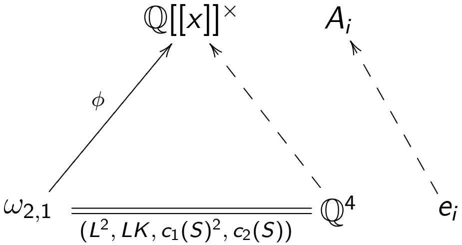


Now we are ready to prove theorems.



Let $\{e_i\}$ be the Standard basis of $\mathbb{Q}^4$ and $\{A_i(x)\}$ be their images.

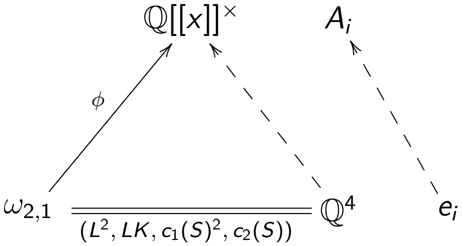
Now we are ready to prove theorems.



Let $\{e_i\}$ be the Standard basis of $\mathbb{Q}^4$ and $\{A_i(x)\}$ be their images. The commutative diagram implies

$$\phi(S,L)(x) = A_1(x)^{L^2} A_2(x)^{LK_S} A_3(x)^{c_1(S)^2} A_4(x)^{c_2(S)}.$$

Now we are ready to prove theorems.



Let $\{e_i\}$ be the Standard basis of $\mathbb{Q}^4$ and $\{A_i(x)\}$ be their images. The commutative diagram implies

$$\phi(S, L)(x) = A_1(x)^{L^2} A_2(x)^{LK_S} A_3(x)^{c_1(S)^2} A_4(x)^{c_2(S)}.$$

$\longrightarrow$ $d_r(S, L)$ is ALWAYS a degree r polynomial in L^2, LK, $c_1(S)^2$ and $c_2(S)$.

- $d_r(S, L)$ is ALWAYS a degree r polynomial in L^2, LK, $c_1(S)^2$ and $c_2(S)$.

- $d_r(S, L)$ is ALWAYS a degree r polynomial in L^2, LK, $c_1(S)^2$ and $c_2(S)$.

+ **Recall**: $d_r(S, L)$ equals the number of r-nodal curves when L is $(5r-1)$-very ample.

- $d_r(S, L)$ is ALWAYS a degree r polynomial in L^2, LK, $c_1(S)^2$ and $c_2(S)$.

+ **Recall**: $d_r(S, L)$ equals the number of r-nodal curves when L is $(5r-1)$-very ample.

$\longrightarrow$ $d_r(S, L)$ is the universal polynomial we are looking for!!!

- $d_r(S, L)$ is ALWAYS a degree r polynomial in L^2, LK, $c_1(S)^2$ and $c_2(S)$.

+ **Recall**: $d_r(S, L)$ equals the number of r-nodal curves when L is $(5r-1)$-very ample.

$\longrightarrow$ $d_r(S, L)$ is the universal polynomial we are looking for!!!

This proves

Theorem (Göttsche's conjecture)

For every integer $r \geq 0$, there exists a universal polynomial $T_r(x, y, z, t)$ of degree r such that

$$T_r(L^2, LK, c_1(S)^2, c_2(S)) = \text{ \# of r-nodal curves in } |L|$$

if L is $(5r-1)$-very ample.

Now we have $T_r(L^2, LK, c_1(S)^2, c_2(S)) = d_r(S, L)$ and recall

$$\phi_(S, L) = A_1^{L^2} A_2^{LK_S} A_3^{c_1(S)^2} A_4^{c_2(S)}.$$

Now we have $T_r(L^2, LK, c_1(S)^2, c_2(S)) = d_r(S, L)$ and recall

$$\phi_(S, L) = A_1^{L^2} A_2^{LK_S} A_3^{c_1(S)^2} A_4^{c_2(S)}.$$

Theorem

There exist power series A_1, A_2, A_3, A_4 in $\mathbb{Q}[[x]]^\times$ such that

$$\sum_{r=0}^{\infty} T_r(L^2, LK, c_1(S)^2, c_2(S))x^r = A_1^{L^2} A_2^{LK_S} A_3^{c_1(S)^2} A_4^{c_2(S)}.$$

(Because both sides are equal to $\sum_{r=0}^{\infty} d_r(S, L)x^r = \phi(S, L)$)

Theorem (Göttsche-Yau-Zaslow formula)

There exist two power series B_1 and B_2 in q such that

$$\sum_{r\geq 0} T_r(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^r = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Proof:

1. LHS$=\phi(S, L)(DG_2)$

Theorem (Göttsche-Yau-Zaslow formula)

There exist two power series B_1 and B_2 in q such that

$$\sum_{r\geq 0} T_r(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^r = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Proof:

1. LHS$=\phi(S, L)(DG_2)$
2. ϕ is a homomorphism from $\omega_{2,1}$ to $\mathbb{Q}[[x]]^{\times}$.

Theorem (Göttsche-Yau-Zaslow formula)

There exist two power series B_1 and B_2 in q such that

$$\sum_{r\geq 0} T_r(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^r = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Proof:

1. LHS$=\phi(S, L)(DG_2)$
2. ϕ is a homomorphism from $\omega_{2,1}$ to $\mathbb{Q}[[x]]^\times$.
3. $\{[\mathbb{P}^2, \mathcal{O}], [\mathbb{P}^2, \mathcal{O}(1)], [K3, L_1], [K3, L_2]\}$ is a basis if $L_1^2 \neq L_2^2$.

Theorem (Göttsche-Yau-Zaslow formula)

There exist two power series B_1 and B_2 in q such that

$$\sum_{r\geq 0} T_r(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^r = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Proof:

1. LHS$=\phi(S, L)(DG_2)$
2. ϕ is a homomorphism from $\omega_{2,1}$ to $\mathbb{Q}[[x]]^\times$.
3. $\{[\mathbb{P}^2, \mathcal{O}], [\mathbb{P}^2, \mathcal{O}(1)], [K3, L_1], [K3, L_2]\}$ is a basis if $L_1^2 \neq L_2^2$.
4. Bryan-Leung found the LHS function on K3 and primitive line bundles.

Thank you!!