Perspectives on moduli of curves in low genus

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Introduction

Among the favourite, at least to me, moduli spaces related to curves of genus g let me mention the following ones (besides $\mathcal{M}_g$):

- *$Pic_{d,g}$: the universal Picard variety over $\mathcal{M}_g$ i.e. the moduli of pairs (C, N) such that $N \in Pic^d(C)$;*
- *$\mathcal{R}_g$: the Prym moduli space i.e. the moduli of pairs (C, N) such that N is a non trivial square root of $\mathcal{O}_C$;*
- *$\mathcal{S}_g^+ \cup \mathcal{S}_g^-$: the moduli spaces of even or odd spin curves i.e. the moduli of pairs (C, N) such that $N^{\otimes 2} \cong \omega_C$;*
- *$\mathcal{R}_g^p$: the moduli space of pairs (C, N) such that N is a non trivial p-root of $\mathcal{O}_C$.*

$\star$ **In the following C is always a smooth, irreducible, complex projective curve of genus $g \geq 2$.**

The first part of the talk reviews some new or recent results on these spaces, focusing on the following type of questions

- *What about the Kodaira dimension*
- *Uniruledness/rational connectedness/unirationality*
- *Rationality problem.*

Then we will turn to various and diverse geometric descriptions of some of these spaces in very low genus.

Perspectives are changing according to the times. The unirationality of any moduli space $\mathcal{M}$ was maybe believable in the ancient times.
In the recent times the drastic division of the Kodaira dimension between $-\infty$ and $3g-3$ in the known cases, made perhaps believable that either $kod(\mathcal{M}) = 3g-3$ or $\mathcal{M}$ is unirational for a moduli space in my list.
Nowadays too the attention on the various questions is varying and some changes of perspective are occurring, as we will see. In this spirit, let me address some related questions, still open in very low genus:

1. *How much the rationality is extended in the set of rationally connected moduli spaces in the previous list?*

2. *What about uniruled but not rationally connected examples in the previous families of moduli spaces?*

3. *For which values of g the Kodaira dimension is intermediate, i.e. not $-\infty$ nor $3g-3$?*

The universal Picard variety $\mathrm{Pic_{d,g}}$

Rationality

Known in few cases, mostly open.

In the second part $Pic_{0,g}, g = 3, 4, 5$, is discussed.

Unirationality

The unirationality of $Pic_{d,g}$ is known for $g \leq 9$ [—]

Kodaira dimension

The Kodaira dimension of $Pic_{d,g}$ is

- $-\infty$ *for* $g \leq 9$,
- 0 *for* $g = 10$,
- 19 *for* $g = 11$,
- $3g - 3$ *for* $g \geq 12$.

- The intermediate Kodaira dimension appears: [Farkas- –] for $d = g$, extended by [Bini, Fontanari, Viviani] to $(d + g - 1, 2g - 2) = 1$.

The Prym moduli space $\mathcal{R}_g$

Rationality

Known for $g \leq 4$.

Due to Dolgachev for $g \leq 3$ and to Catanese for $g \leq 4$.

Unirationality

The unirationality is known for $g \leq 7$

[Izadi-Lo Giudice-Sankaran] $g = 5$, [Donagi] $g = 6$, [—] $g \leq 7$.

Kodaira dimension

$\mathcal{R}_g$ *is of general type for* $g \geq 12$.

Due to [Farkas-Ludwig]

- What about rationality for $g = 5, 6$?
- What about the (intermediate) Kodaira dimension of $\mathcal{R}_g$, $8 \leq g \leq 11$?

Quick unirationality for $\mathcal{R}_g, g \leq 6$

Let me sketch a quick, different proof for $g \leq 6$:
Let $(C, \eta) \in \mathcal{R}_g, g \leq 6$ be general . Then C is birational to $C' \subset \mathbf{P}^2$, C' a δ-nodal sextic with $\delta := 10 - g$. Let

$$\eta' \in (\nu^*)^{-1}(\eta) \subset Pic^0(C').$$

It turns out that, for a general (C, η), one can apply to (C', η') Beauville's results on determinantal hypersurfaces. This gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-4)^{\oplus 3} \xrightarrow{A} \mathcal{O}_{\mathbf{P}^2}(-2)^{\oplus 3} \longrightarrow \eta' \longrightarrow 0$$

where $A := (a_{ij})$ is a symmetric matrix of quadratic forms on $\mathbf{P}^2$. Fixing coordinates (x, y) we obtain a hypersurface of type $(2, 2)$

$$T := \{\sum a_{ij} y_i y_j = 0\} \subset \mathbf{P}^2 \times \mathbf{P}^2$$

and a conic bundle $p_1 : T \to \mathbf{P}^2$ with discriminant (C', η') (*)

(*) T is nodal: $p_1 : Sing\ T \to Sing\ C'$ is bijective. Moreover $\pi' : \tilde{C}' \to C'$ is not allowable for Pryms.

Let $\mathcal{T}_\delta \subset \mid \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2) \mid$ be the family of δ-nodal threefolds T, $\delta \leq 7$. It follows that there exists a dominant map

$$\phi_\delta : \mathcal{T}_\delta \to \mathcal{R}_{10-\delta}$$

sending $(T, p_1) \longrightarrow (C', \eta') \longrightarrow (C, \eta)$. $\mathcal{T}_\delta$ is rational for $\delta \leq 7$. Hence $\mathcal{R}_g$ is unirational for $3 \leq g \leq 6$. $\square$

- Let $g = 6$ then the nodes of T can be fixed, so that $\mathcal{R}_6$ is dominated by a fixed $\mathbf{P}^{15}$ of 4-nodal threefolds T.
  Is it possible to use this to study the rationality problem?

(Compare with Shepherd-Barron's proof of the rationality of $\mathcal{M}_6$).

Moduli $\mathcal{S}_g^+$ of even spin curves

Rationality

The rationality of $\mathcal{S}_g^+$ is known for $g \leq 4$.

$g \leq 3$: classical. $g = 4$: [Takagi-Zucconi].

Uniruledness

The uniruledness of $\mathcal{S}_g^+$ is known for $g \leq 7$.

Kodaira dimension

- $kod(\mathcal{S}_g^+) = -\infty$ *for* $g \leq 7$,
- $kod(\mathcal{S}_8^+) = 0$,
- $kod(\mathcal{S}_g^+) = 3g - 3$ *for* $g \geq 9$.

$g \leq 8$: [Farkas- —]. $g \geq 9$: [Farkas].

- *What about the (uni)rationality of* $\mathcal{S}_g^+$ *for* $g \leq 7$*?*

Moduli $\mathcal{S}_g^-$ of odd spin curves

Rationality

The rationality of $\mathcal{S}_g^-$ is known for $g \leq 3$.

Unirationality

The unirationality is known for $g \leq 8$.

Uniruledness

The uniruledness is known for $g \leq 11$.

Kodaira dimension

- $kod(\mathcal{S}_g^-) = -\infty$ *for* $g \leq 11$,
- $kod(\mathcal{S}_g^-) = 3g - 3$ *for* $g \geq 12$.

[Farkas- —].

- *What about rational connectedness for* $g = 9, 10, 11$?
- *Experimentally the uniruledness is related to K3 surfaces.*

K3 surfaces and uniruledness: examples

Example (Odd spin moduli space of genus $g \leq 11$)

(C, d) general odd spin curve, $g \leq 11$, $g \neq 10$. Then:

$C \subset S \subset \mathbf{P}^g$, S a smooth K3 surface s.t. $\mathcal{O}_S(1) \cong \mathcal{O}_S(C)$.

$P_d := \mid \mathcal{I}_{d/S}(C) \mid$ defines a pencil of odd spin curves (D, d), $D \in P_d$. The image of the natural map $m : P_d \to \mathcal{S}_g^-$ is a rational curve through a general point. Hence $\mathcal{S}_g^-$ is uniruled. □

(C, d) general spin curve $g = 10$. It is known that a general 1-nodal quotient $C' = C / < x = y >$ embeds as follows:

$C' \subset S \subset \mathbf{P}^{11}$, S a smooth K3 surface s.t. $\mathcal{O}_S(1) \cong \mathcal{O}_S(C')$,

[KFPS]. One shows that x, y can be choosen in d. Let $d = x + y + z$, $o = Sing\ C'$. Then the pencil $P_d := \mid C' - 2o - z \mid$ defines a rational curve through a general point of $\mathcal{S}_{10}^-$. Hence $\mathcal{S}_{10}^-$ is uniruled. □

Example (Prym moduli space of genus 7)

(C, η) general Prym curve, $g = 7$. $\omega_C \otimes \eta$ embeds C as

$C \subset S \subset \mathbf{P}^5$, where S is a smooth c.i. of three quadrics.

Moreover $h^0(\mathcal{I}_{C/\mathbf{P}^5}(2)) = 3$. Surprisingly S is a very special K3, namely a Nikulin surface. The standard exact sequence

$$0 \to \mathcal{O}_S(C - 2H) \to \mathcal{O}_S(2(C - H)) \to \eta^{\otimes 2} \to 0$$

implies that $2(C - H) \sim L$ with L effective, (H hyperplane section). Actually L is the sum of 8 disjoint lines. Let $E := C - H$.

Since $L > 0$, $(D, \mathcal{O}_D(E))$ is a Prym curve for any smooth $D \in \mid C \mid$.

Conclusion: the construction defines a non constant moduli map $m : \mid C \mid \to \mathcal{R}_7$ targeting a general point. Hence $\mathcal{R}_7$ is uniruled. □
The unirationality follows by a different method, [—].

Example (Even spin moduli spaces of genus $g \leq 7$)

(C, θ) a general even spin curve of genus 7. Fix $\eta \in Pic^0_2(C)$ so that $\eta(d) \cong \theta$, d odd theta characteristic. Then consider as previously the embedding defined by $\omega_C \otimes \eta$:

$$C \subset S \subset \mathbf{P}^5.$$

The pencil $P_d := | \mathcal{I}_{d/S}(C) |$ defines a family of even spin curves: indeed S is a Nikulin surface such that $\eta_D := \mathcal{O}_D(E) \in Pic^0_2(D)$, for any $D \in | C |$. (With the previous notations). Then

$$\{(D, \eta_D(d), \ D \in P_d\}$$

is a rational family of even spin curves targeting a general point of $\mathcal{S}^+_7$, which is therefore uniruled. □

Similarily for $g \leq 5$ via Nikulin surfaces. $g = 6$ needs to be treated differently.

$Pic_{0,g}$ in low genus

The second part of this talk deals with some concrete geometry of $Pic_{0,g}$ in very low genus: $g = 3, 4, 5$. Some conventions:

- $\circ$ $(C, N) :=$ a pair such that $N \in Pic^0(C)$, N non trivial.
- $\circ$ $N := \mathcal{O}_C(e)$ with $e \in Div^0(C)$,
- $\circ$ $H_e := H^0(\omega_C(e))$.

To study $Pic_{0,g}$ it will be useful the multiplication map

$$\mu : H_e \otimes H_{-e} \to H^0(\omega_C^2).$$

Any good property of a general (C, N) is assumed. In particular:

Lemma

For a general pair (C, N) μ has maximal rank:
it is surjective for $g \geq 4$ and injective for $g \leq 3$.

Let us consider the Segre embedding

$$\mathbf{P}_e \times \mathbf{P}_{-e} \subset \mathbf{P}^{g^2-2g} := \mathbf{P}(H_e^* \otimes H_{-e}^*),$$

where $\mathbf{P}_e := \mathbf{P}H_e^*$. Then μ defines a morphism

$$f : C \to C' \subset \mathbf{P}_e \times \mathbf{P}_{-e} \subset \mathbf{P}^{g^2-2g}$$

which is birational onto its image C'. For $g \geq 4$ f is an embedding. So we put $C := C'$, reserving the notation C' to the genus 3 case.

- *At least in very low genus one can construct from f a special surface S. S is an invariant of (C, N): its moduli are useful to describe $Pic_{0,g}$.*

- *Step by step we will do this for $g = 3, 4, 5$.*

On the rationality of $Pic_{0,3}$

From (C, N) we have at first a 6-nodal curve of bidegree $(4, 4)$

$$C' \subset \mathbf{P}^1 \times \mathbf{P}^1.$$

Blowing $Sing\ C'$ up, we obtain a degree two Del Pezzo surface

$$\sigma : S \to \mathbf{P}^1 \times \mathbf{P}^1.$$

The strict transform of C' is a copy of C, still denoted by C.
S is endowed with the polarizations

$$\mathcal{O}_S(H) := \sigma^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1) \text{ and } \mathcal{O}_S(H_1) := \sigma^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0),$$

where $N \cong \mathcal{O}_C(H_1)$. $H_2 := H - H_1$. Moreover notice that

$$C \in \mid -2K_S \mid .$$

The construction $(C, N) \longrightarrow (S, H, H_1)$ is equivariant.

For $g = 3$ we see a phenomenon also occuring for $g = 5$:

- *the condition* $(*) : \mathcal{O}_S(H) \cong \omega_C^{\otimes 2}$ *makes the surface* S *special.*

Let $E := E_1 + \cdots + E_6$ be the exceptional divisor of σ. Since $C \sim 4H - 2E \sim -2K_S$, we have the standard exact sequence

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S(3H - 2E) \to \mathcal{O}_C(C - H) \to 0.$$

Moreover $\mathcal{O}_C(C) \cong \omega_C^{\otimes 2}$. Then:

- *(*) holds iff* $\omega_C^{\otimes 2}(-H) \cong \mathcal{O}_C$ *iff* $h^0(\mathcal{O}_S(3H - 2E)) = 1$.

The latter condition is equivalent to say that in $\mathbf{P}^1 \times \mathbf{P}^1$ there exists a curve of type (3,3) with six nodes at *Sing* C'.

Let (S, H, H_1) be any Del Pezzo of degree 2, polarized by (H, H_1):

Proposition

The following conditions are equivalent

- *a smooth* $C \in | -2K_S |$ *satisfies* $\omega_C^{\otimes 2} \cong \mathcal{O}_C(H)$,
- $\sigma_H : S \to \mathbf{P}^1 \times \mathbf{P}^1$ *is the blowing up of Sing* l, *where* $l = l_1 + l_2 + l_3$ *and* $l_1, l_2, l_3 \in | \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1) |$.
- *each smooth* $C \in | -2K_S |$ *satisfies* $\omega_C^{\otimes 2} \cong \mathcal{O}_C(H)$.

[Geometrically: C' is the projection of the bicanonical model $C \subset \mathbf{P}^5$ from $s = Sing\ Q$, Q a rank 4 quadric through C. The six nodes arise from six bisecant lines to C intersecting s. But these are chords of the Veronese surface $V \supset C$. Hence they intersect s in pairs: the projection of V from s is a quartic Steiner surface R and $Sing\ C' \subset Sing\ R$.]

A triple (S, H, H_1) defined by a general *Sing I* as above has trivial automorphisms group, as a triple.

Let $\mathcal{L}$ be the family of the elements *Sing I*. Then we have:

1. *Rational moduli* $\mathcal{M} := \mathcal{L}/PGL(2) \times PGL(2)$ *for* (S, H, H_1).
2. *A universal surface* $p : \mathcal{S} \to U \subset \mathcal{M}$.
3. $\mathbf{P}_{\mathcal{M}} := \mathbf{P}(p_* \omega_{\mathcal{S}/U}^{\otimes 2})$ *with fibre* $\mid -2K_S \mid$ *at* (S, H, H_1).

The assignement $(C, N) \to (S, H, H_1)$ induces a birational map

$$\phi : Pic_{0,3} \to \mathbf{P}_{\mathcal{M}}.$$

Lemma

Given (C,N) let (S,H,H_1) be its associated triple. Then there exists a unique $D \in \mid -2K_S \mid$ such that D is biregular to C and $\mathcal{O}_D(H_1) \cong \omega_C \otimes N$.

Proof.

Let $\sigma_* D = D' \subset \mathbf{P}^1 \times \mathbf{P}^1$, then $Sing\ D' = Sing\ C'$ and there exists $\alpha \in PGL(2) \times PGL(2)$ s. t. $\alpha(D') = C'$. Hence $\alpha(Sing C') = Sing\ C'$. Since $Aut(S,H,H_1) = 1$, then $\alpha = id$ and $D' = C'$. □

Definition

Let $x \in Pic_{0,3}$ be the moduli point of (C,N), y the moduli point of (S,H,H_1). By definition $\phi(x) := (y,D)$.

ϕ admits an obvious inverse sending (y,D) to the moduli point of $(D, \mathcal{O}_D(H_1))$. Hence $Pic_{0,3}$ is rational. □

On the rationality of $Kum_{0,4}$

Starting from (C, N) we have now

$$C \subset \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8.$$

Projecting C in the two factors we obtain the 6-nodal sextics

$$C_i := p_{i*}C \subset \mathbf{P}^2, \ i = 1, 2.$$

Since C is bicanonically embedded it easily follows that:

- *Sing* C_1 *and Sing* C_2 *are related by a double six configuration.*

In other words:
let $\sigma_i : S_i \to \mathbf{P}^2$ be the blowing up of *Sing* C_i and E_i the exceptional divisor. Then:

- S_1 *and* S_2 *are the same cubic surface* S*,*
- E_1, E_2 *is a double six configuration of lines on* S*.*

Essentially, one proceeds as in genus three:

- *Aut* $S = 1$ *for a general* S*,*
- C *embeds in* S *as the strict transform of* C_i *by* σ_i*,*
- σ_1/C *is the map defined by* $\omega_C \otimes N$*.*

The assignement $(C, N) \longrightarrow (S, \sigma_1)$ defines a birational map

$$\phi : Pic_{0,4} \to \mathbf{P} p_*(\omega_{\mathcal{S}/\mathcal{P}}^{\otimes 2}),$$

$\mathcal{P}$ being the moduli space of (S, σ_1) and $\omega_{\mathcal{S}/\mathcal{P}}$ the relative cotangent sheaf of the universal cubic $p : \mathcal{S} \to \mathcal{P}$. Let c be the moduli point of (C, N), s the moduli point of (S, σ_1): $\phi(c) := (s, C)$. Then

$$Pic_{0,4} \cong \mathcal{P} \times \mathbf{P}^9.$$

The rationality of $\mathcal{P}$ is unknown. Let $I : \mathcal{P} \to \mathcal{P}$ be the involution $(S, \sigma_1) \leftrightarrow (S, \sigma_2)$: by Coble $\mathcal{P}/I$ is rational. It's easy to see that

$$Kum_{0,4} := (Pic_{0,4}/ < -1 >) \cong (\mathcal{P}/ < I >) \times \mathbf{P}^9.$$

Hence $Kum_{0,4}$ is rational. □

On the rationality of $Pic_{0,5}$

In this case (C, N) defines a K3 surface S. We have

$$C \subset \mathbf{P}^3 \times \mathbf{P}^3 \cap \mathbf{P}(Ker\ \mu)^\perp \subset \mathbf{P}^{15}.$$

$\mathbf{P}(Ker\ \mu)^\perp$ is transversal of codimension 4 to $\mathbf{P}^3 \times \mathbf{P}^3$, hence

$$S := \mathbf{P}(Ker\ \mu)^\perp \cap \mathbf{P}^3 \times \mathbf{P}^3$$

is a smooth K3 surface. S is endowed with the polarizations:

$$\mid H_1 \mid := \mid \mathcal{O}_S(1,0) \mid, \ , \mid H_2 := \mathcal{O}_S(0,1) \mid, \mid H \mid := \mid H_1 + H_2 \mid, \mid C \mid .$$

Moreover

$$H_i^2 = 4\ ,\ H_1 H_2 = 6.$$

Let p_1, p_2 be the two projections in $\mathbf{P}^3$, it is easy to see that

$$S_i = p_{i*}S,\ \ i = 1, 2$$

is a determinantal quartic surface.

It is not enough for Pic S: once more consider

$$0 \to \mathcal{O}_S(C - H) \to \mathcal{O}_S(2C - H) \to \mathcal{O}_C(2C - H) \to 0.$$

1. It holds $h^1(\mathcal{O}_S(C - H)) = 0$ and $(C - H)^2 = -12$.
2. $\mathcal{O}_C(H)$ bicanonical $\Rightarrow \mathcal{O}_C(2C - H) \cong \mathcal{O}_C \Rightarrow 2C - H > 0$.
3. $2C - H \sim B_1 + \cdots + B_6$, where $B_1 \dots B_6$ are conics.
4. $B_i B_j = -2\delta_{ij}, \; H_1 B_i = H_2 B_i = 1, \; CB_i = 0$.

(S, H_1) is an invariant of (C, N), S is a very special K3 surface.

The lattice of a general S is:

$$\mathbb{L} := \mathbf{Z}c \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}b_1 \oplus \ldots \mathbf{Z}b_6$$

with intersection matrix:

$$c^2 = 8\ ,\ h_1^2 = 4\ ,\ ch_1 = 8\ ,\ cb_i = 0\ , h_1 b_i = 1\ ,\ b_i b_j = -2\delta_{ij}.$$

By definition $\mathcal{K}$ is the moduli of pairs (S, j) such that:

$\circ$ $j : \mathbb{L} \to Pic\ S$ *is a primitive embedding,*
$\circ$ $(S, \mathcal{O}_S(C))$ *is a polarized K3 surface,*
$\circ$ $j(c) = \mathcal{O}_S(C)$.

One can show that $Aut(S, j) = 1$ for a general (S, j). Let $\mathcal{C}$ be the universal line bundle on the universal surface $\mathcal{S} \to U \subset \mathcal{K}$. The assignement $(C, N) \longrightarrow (S, j; C)$ induces a birational map

$$\alpha : Pic_{0,5} \to \mathbf{P}_{\mathcal{K}} := \mathbf{P}\pi_* \mathcal{C}.$$

The rationality of $Pic_{0,5}$ then follows from the rationality of $\mathcal{K}$.

Projective models of S

1. $S \subset \mathbf{P}^3 \times \mathbf{P}^3$, $H_1 + H_2$ Smooth linear section of $\mathbf{P}^3 \times \mathbf{P}^3$.
2. $S_i := p_{1*}S \subset \mathbf{P}^3$, H_i, $i = 1, 2$ Smooth quartic models of S.
3. $S_d := \mathbf{P}(Ker\mu) \cap D$, $D =$ dual of $\mathbf{P}^3 \times \mathbf{P}^3$. Singular quartic.
4. $S_n \subset \mathbf{P}^5$, C. Singular complete int. of 3 quadrics in $\mathbf{P}^5$.

S_n is the most interesting model: $B_1, \ldots, B_6$ are contracted to six nodes $o_1 \ldots o_6$. Let Π be the net of quadrics through S_n, then:

1. $Sing\ S_n = \{o_1 \ldots o_6\}$,
2. each $Q \in \Pi$ has rank ≥ 5,
3. the discriminant $D \subset \Pi$ is an integral six-nodal sextic,
4. $Sing\ D \subset B \subset \Pi$, where B is a smooth conic.

The above properties characterize nets Π: building on them one can deduce the rationality of the moduli space of these nets.

Main remark: Fix a smooth conic $B \subset \mathbf{P}^2$ and consider

$$w = (w_1, \dots, w_6) \in B^6.$$

Let Π be a net of the previous family s. t. $\{w_1 \dots w_6\}$ is $Sing\ D$. Fix coordinates (x, y) on $\mathbf{P}^2 \times \mathbf{P}^5$.One can assume that

$$Sing\ S_n = \{o_1 \dots o_6\}$$

is the set of fundamental points. Then Π is defined by a symmetric matrix of linear forms of the following type:

$$(h_{ij}),$$

where $h_{ii} = 0$ and h_{ij} is an equation of $\overline{w_i w_j}$.

The 3-torsion locus $\mathcal{R}^3_g$ in $\mathrm{Pic}_{0,g}$

Let us restrict to pairs (C, N) such that $N \in Pic^0_3(C)$:

$g = 3$ (Bauer-Catanese)

1. $N \in Pic^0_3(C) \Leftrightarrow l_1, l_2, l_3$ *are rank 2 distinct conic sections.*
2. *A unique Del Pezzo* S_o *exists such that* l_1, l_2, l_3 *is as in 1.*
3. *It follows that* $\mathcal{R}^3_3$ *is birational to the rational quotient*

$$| -2K_{S_o} | / Aut(Sing\ l).$$

- *In genus 3 N is induced from Pic* S_o*:* S_o *contracts to a surface* $\overline{S}_o$ *with 3* A_2*-double points. Sing* $\overline{S}_o$ *defines a 3:1 cover branched on Sing* $\overline{S}_o$*. This induces on* $C \in | -2K_{S_o} |$ *the 3:1 cover defined by N.*

g = 4 (Bauer - —)

No special cubic S in this case: (C, N) defines (S, σ_1) with S general. Indeed let S be general, $\sigma_1 : S \to \mathbf{P}^2$ the contraction of E by $\mid H_1 \mid$:

1. *$N = \mathcal{O}_C(K_S + H_1) \in Pic^0_3(C)$ iff the cup product $\cup_C : H^1(-E) \to H^1(-E + C))$ is not injective.*
2. *The locus $T_3 := \{C \in \mid -2K_S \mid \ / \ N^{\otimes 3} \cong \mathcal{O}_C\}$ is birational to a $\mathbf{P}^1$-bundle over $\mathbf{P}^4 = \mathbf{P}H^1(-E)$.*
3. *$\mathcal{R}^3_4$ is birational to the product $\mathcal{P} \times \mathbf{P}^4 \times \mathbf{P}^1$.*

Let $T'_3 := \{(v, C) \in \mathbf{P}H^1(-E) \times \mid -2K_S \mid \ / \ v \cup s_C = 0\}$. Then $T'_3 \cong T_3$ via p_2, moreover *dim* $T_3 = 5$. One shows that

$$p_1 : T'_3 \to \mathbf{P}H^1(-E)$$

is a $\mathbf{P}^1$-bundle. Proof: $v \in \mathbf{P}H^1(-E)$ defines the extension

$$0 \to \mathcal{O}_S(-E - 2K_S) \to \mathcal{V} \to \mathcal{O}_S(-2K_S) \to 0.$$

The fibre of p_1 turns out to be $P_v := \mathbf{P}H^0(\mathcal{V})$. One has to analyze this extension in detail to deduce that $dim\ P_v = 1$.

1. *Let $E := \sum E_i$, $F := \sum F_i$ be the exceptional divisors of σ_1 and σ_2. The base locus of P_v is a quadratic section of F.*
2. *One shows that P_v is a pencil if P_v has no fixed component.*
3. *Moreover: for a base-point-free pencil P with base locus a quadratic section of F, it follows $P = P_v$ for some $v \in \mathbf{P}H^1(-E)$.*

Then, to prove that the general fibre of $T_3 \to \mathbf{P}H^1(-E)$ is $\mathbf{P}^1$, it suffices to produce a pencil P as above.
One can use the double six. Indeed the pencil generated by the divisors

$$E_1 + E_2 + E_3 + F_4 + F_5 + F_6\ ,\ F_1 + F_2 + F_3 + E_4 + E_5 + E_6$$

is a pencil as required.