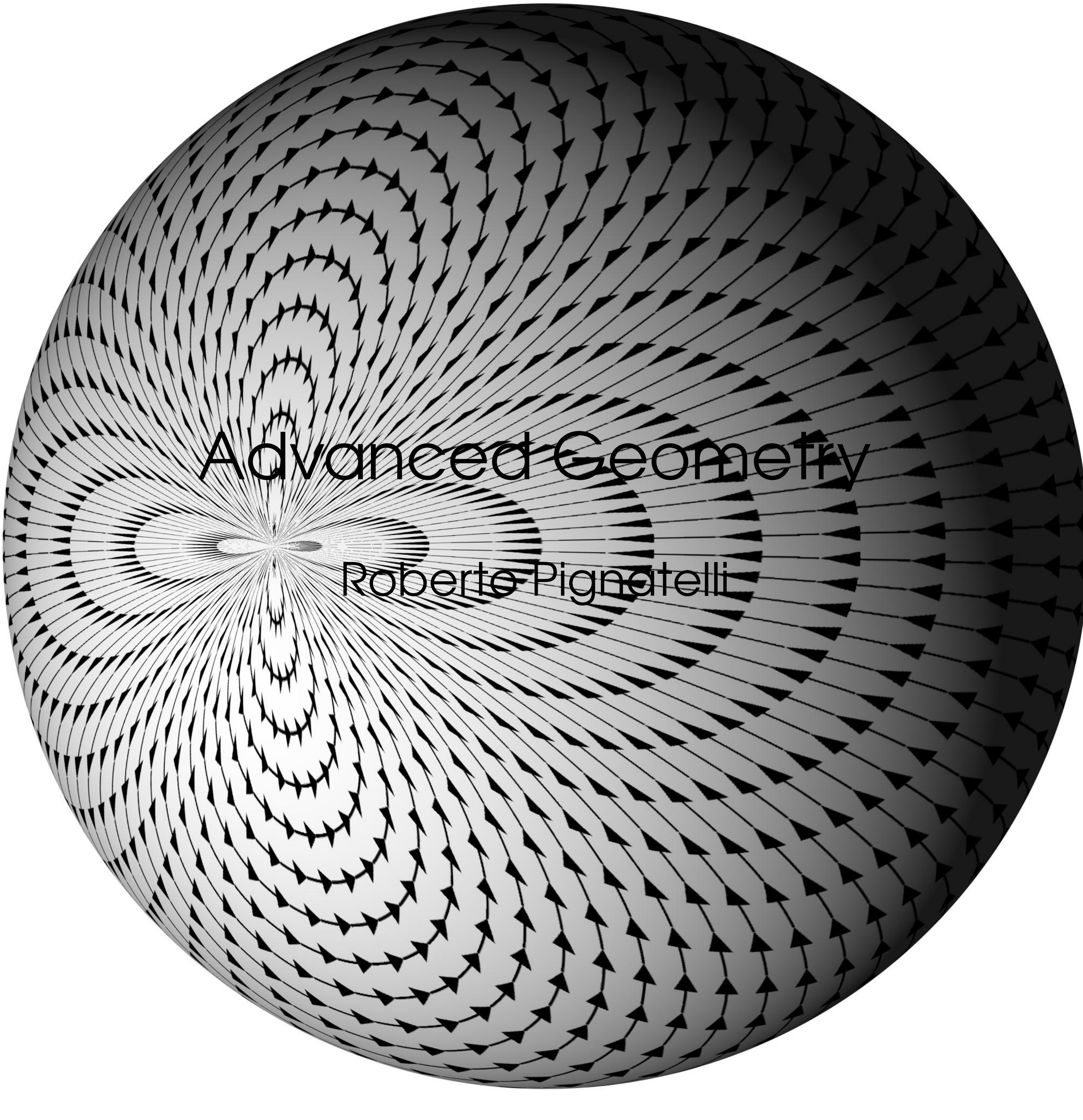
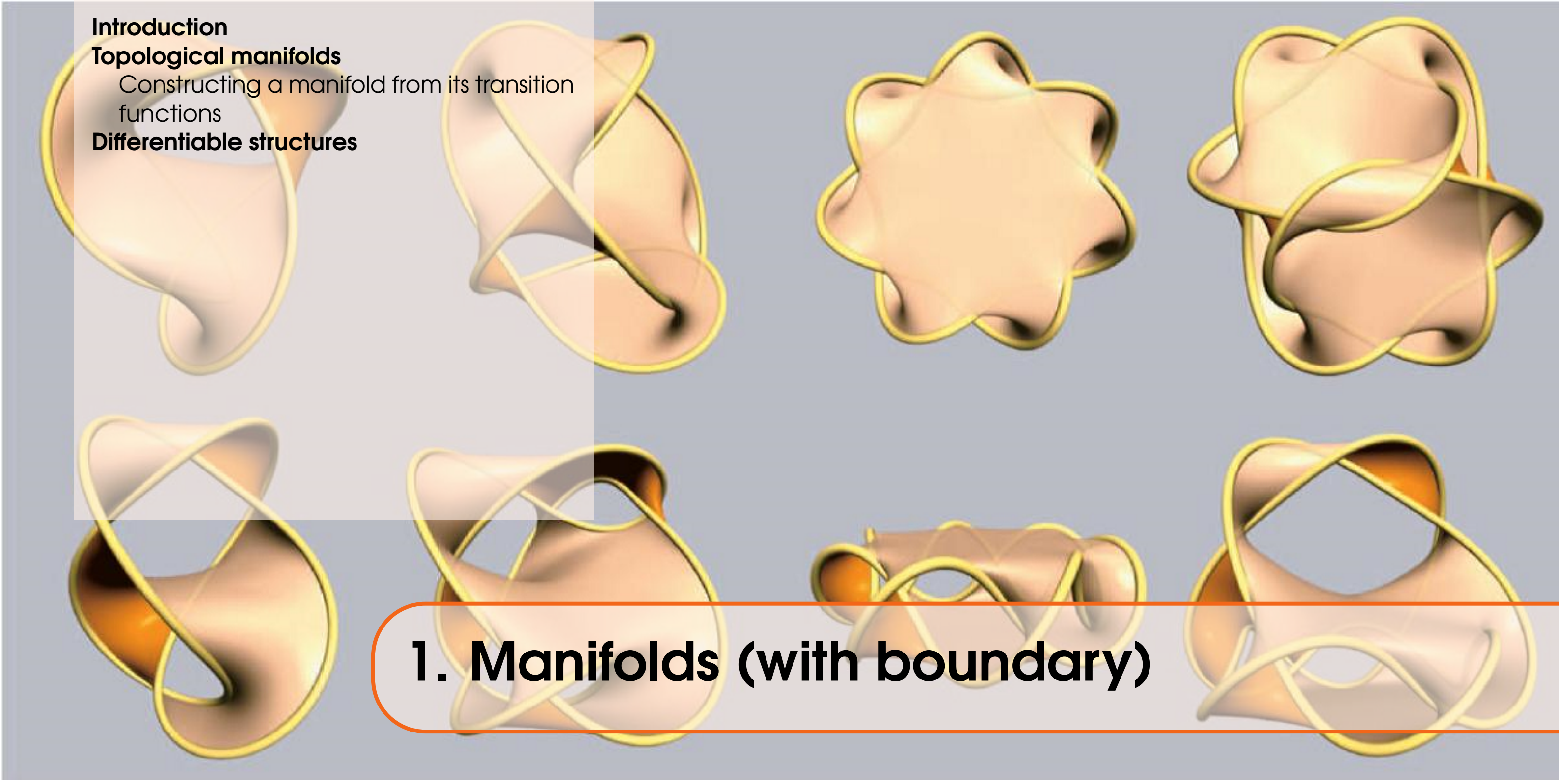
Advanced Geometry

Roberto Pignatelli



Contents



1. Manifolds (with boundary)

1.1 Introduction

Most students of this course have met the differentiable manifolds in the previous years of their undergraduate studies. In this chapter we will develop their theory, considering at the same time complex manifolds and real manifolds with boundary. To be precise, we will mostly discuss the slightly more complicated real case, where we need to consider boundaries, and give indications on how to rewrite everything in the complex case.

Before setting the first formal definition, let us try to give some general ideas. A topological manifold without boundary is a topological space which is locally euclidean: in other words something which "locally" can't be distinguished by $\mathbb{R}^n$. The surface of a sphere, S^2, is a typical example: we know that the surface of the Earth is approximatively a sphere, locally we can't topologically distinguish a sphere from a plane and indeed our ancestors were convinced that the Earth was flat.

We are interested in a slightly more general class of objects: a typical example is the closed ball $B^3 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 \leq 1\}$.

B^3 is not locally euclidean because of its *boundary*. Indeed, if we consider a point $p \in B^3$ of norm 1 there is no neighborhood of p (in the topology of B^3) homeomorphic to an open set of $\mathbb{R}^3$. To include B^3 in our class of objects we need to modify the definitions to allow a boundary.

Note that B^3 may be decomposed as disjoint union of its boundary ∂B^3 and its interior $\mathring{B}^3$ as follows

$$\begin{aligned} \mathring{B}^3 &:= \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 < 1\} \\ \partial B^3 &:= \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}. \end{aligned}$$

We remark that $\mathring{B}^3$ is a topological manifold without boundary (is an open set of $\mathbb{R}^3$!), and ∂B^3 is the sphere S^2 and therefore it is also a topological manifold without boundary, although of different dimension. Similarly we will decompose every manifold with boundary as the disjoint union of two manifolds without boundary: its interior and its boundary.

1.2 Topological manifolds

First, we introduce the model space in the real case.

Notation 1.1. *We will denote by* $\mathbb{R}^n_+$ *the halfspace of the points of* $\mathbb{R}^n$ *whose last coordinate is nonnegative:*

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

Similarly $\mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n \leq 0\}$.
The symbol $\mathbb{R}^n_\pm$ *means:* $\mathbb{R}^n$, $\mathbb{R}^n_+$ *or* $\mathbb{R}^n_-$.

A **topological manifold with boundary** (sometimes just **topological manifold** for short) **of dimension** $\boldsymbol{n}$ is a topological space M which

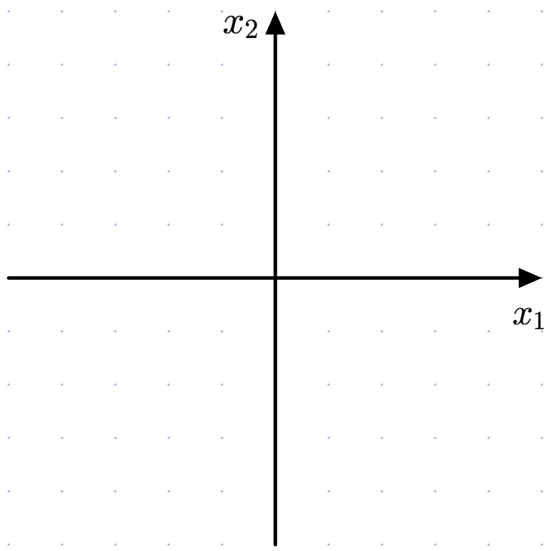
- is locally homeomorphic to $\mathbb{R}^n_\pm$ (that is: $\forall p \in M$, $\exists U$ open set containing p homeomorphic to an open set of $\mathbb{R}^n$, $\mathbb{R}^n_+$ or $\mathbb{R}^n_-$)[1];
- is Hausdorff;
- admits a countable basis of open sets[2].

Example 1.1 Every open set of $\mathbb{R}^n_\pm$ is a topological manifold with boundary.

Recall that an **open covering** of a topological space M is a family $\mathfrak{U} := \{U_i\}_{i \in I}$ of open sets of M with the property that $\bigcup_{i \in I} U_i = M$. The first property in the definition of topological manifold means that there is an open covering of M made by sets that are homeomorphic to open subsets of $\mathbb{R}^n_\pm$.

Example 1.2 The closed interval $B^1 := [-1, 1] \subset \mathbb{R}$ is a topological manifold with boundary of dimension 1. B^1 is Hausdorff, and has a countable basis of open sets, so to prove our statement we need only to construct a covering of M made of open sets homeomorphic to open sets of *e.g.* $\mathbb{R}^1_+ = [0, +\infty)$. The easiest choice seems to be $B^1 = \left[-1, \frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right]$.

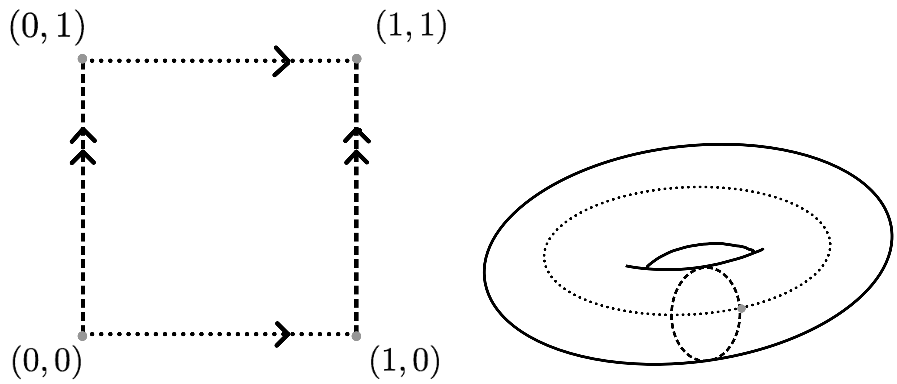
Figure 1.1: The lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$



[1] This is the key property. Unfortunately, this property is not enough to have something that locally "looks like" an affine space, as shown by some of the examples in the Complement 1.2.2.

[2] Some authors remove this assumption too. It is equivalent to requiring the manifold to be embeddable in an affine space (Whitney Embedding Theorem). Without this assumption, the theory becomes more complicated, because the existence of the partitions of unity that we will later use may fail.

Figure 1.2: To the left the square $[0,1]\times[0,1]$, to the right the torus

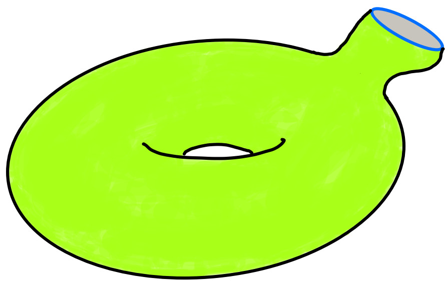


Example 1.3 — The torus. We consider the lattice $\mathbb{Z}^2$ of the points of $\mathbb{R}^2$ with integral coefficients as in Figure 1.1. This is a subgroup with respect to the group structure of $\mathbb{R}^2$ given by the sum. The group quotient T:=$\mathbb{R}^2/\mathbb{Z}^2$ with the quotient topology is a topological manifold, the torus.

In fact, see Figure 1.2, the square $[0,1]\times[0,1]$ maps surjectively to T, sending the 4 vertices to the same point. Moreover each edge is mapped to a circle through that point, and parallel edges map to the same circle. The internal part of the square maps homeomorphically to a dense open subset of T, the complement of the two circles.

Each translated of the internal square $(a,a+1)\times(b,b+1)$ maps homeomorphically on a dense open subset of T. The reader can easily check that 4 of those squares are sufficient to cover the torus.

Figure 1.3: A torus with a "hole"



Example 1.4 — The torus with a hole. Removing the image of a small open disc internal to one of these squares, one obtains a topological manifold with boundary like in Figure 1.3.

Note that since we removed an open disc, the points in the boundary of that disc maps to points of the manifold having a small neighbourhood homeomorphic to an open subset of $\mathbb{R}^2_+$, but not homeomorphic to any open subset of $\mathbb{R}^2$. These points form the blue curve in the picture.

Let M be a topological space. A **chart** (U,φ) on M is given by an open set $U\subset M$ and a homeomorphism $\varphi\colon U\to D$, onto an open set D of $\mathbb{R}^n_+$.

Note the analogy with the road maps, which are functions from a piece of the surface of the Earth to a piece of paper.

A chart allows us to use the coordinates of $\mathbb{R}^n$ to identify a point of the mapped object (U), as when we see on a road map that "Rome is in E7". From now on we will denote by u_i the i-th **coordinate function** on $\mathbb{R}^n$

$$u_i\colon \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R} \\ (w_1,\dots,w_n) & \mapsto & w_i \end{array}$$

Each chart $\{(U,\varphi)\}$ induces **local coordinates** $(x_1,\dots,x_n)$ defined by

$$x_i := u_i \circ \varphi\colon U \to \mathbb{R}.$$

If you have traveled by car, you probably had to "move" from a map to another. For example, using google maps, to compare the map on the screen of your device with the map on the screen of the mobile phone of one of your friends.

To follow your path you need to find the coordinates, in both maps, of the same point, your position in that moment.

Consider for example $M = B^1 = [-1,1]$. Example 1.2 suggests two charts (U_1,φ_1) and (U_2,φ_2) on B^1:

$U_1 := [-1,\frac{1}{2})$, $\varphi_1\colon [-1,\frac{1}{2}) \to [0,\frac{3}{2})$ given by $\varphi_1(t) = t+1$;

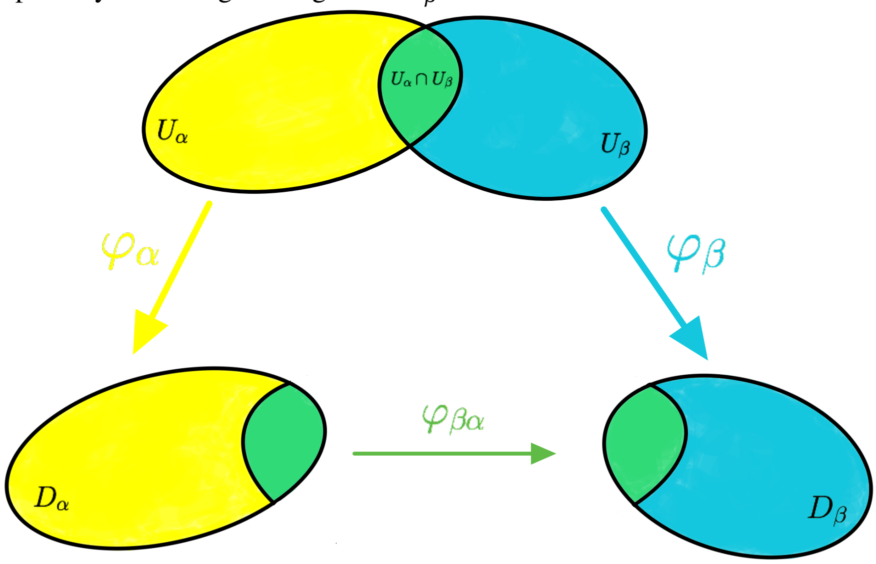
$U_2 := (-\frac{1}{2},1]$, $\varphi_2\colon (-\frac{1}{2},1] \to [0,\frac{3}{2})$ given by $\varphi_2(t) = 1-t$.

The point $p = \frac{1}{4} \in B^1$ has "coordinates" (coordinate: B^1 has dimension 1) $\frac{5}{4}$ for (U_1,φ_1) and $\frac{3}{4}$ for (U_2,φ_2).

How do the coordinates change? For every ordered pair of charts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β) we define the associated **transition function**

$$\varphi_{\beta\alpha} := (\varphi_\beta)_{|U_\alpha\cap U_\beta} \circ (\varphi_\alpha)^{-1}_{|\varphi_\alpha(U_\alpha\cap U_\beta)}\colon \varphi_\alpha(U_\alpha\cap U_\beta) \to \varphi_\beta(U_\alpha\cap U_\beta)$$

Figure 1.4: The transition function $\varphi_{\beta\alpha}$ is defined only on the green region of D_α, mapping it homeomorphically onto the green region of D_β.



In our example $U_1 \cap U_2 = \left(-\frac{1}{2},\frac{1}{2}\right)$ and the transition functions φ_{21} and φ_{12} are easily computed: $\varphi_{21} = \varphi_{12}\colon \left(\frac{1}{2},\frac{3}{2}\right) \to \left(\frac{1}{2},\frac{3}{2}\right)$ is given by $\varphi_{21}(t) = \varphi_{12}(t) = 2-t$. These functions allow us to compute the coordinates of a point in one of the charts from the coordinates in the other chart: $\varphi_{21}\left(\frac{5}{4}\right) = \frac{3}{4}$ and $\varphi_{12}\left(\frac{3}{4}\right) = \frac{5}{4}$.

Notation 1.2. *The definition of $\varphi_{\beta\alpha}$ is heavy, because we had to restrict the domains of all functions to be able to compose them.*

From now on we will use the following convention. Let f and g be functions such that the image of f and the domain of g do not coincide but are subsets of a "common universe". Then by $g \circ f$ we mean the composition of the restriction of f and g to the biggest possible subsets such that the composition is possible.

With this convention, the definition "reduces" to the easier $\varphi_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1}$.

Similarly, if we write an inequality among two functions which do not share the same domain, we mean that the two functions coincide on the points were both are defined.

By definition every topological manifold with boundary M may be covered by a set of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$; in other words $\mathfrak{U} := \{U_\alpha\}_{\alpha\in I}$ is an open covering of M.

Assume you have charts of the whole surface of the Earth, and some glue (I mean, anything one can use to glue two sheets of paper). Then start gluing all your charts in such a way that two points are glued if and only if they represent the same point on the Earth. You will end up with a paper-made sphere: you have constructed something homeomorphic to the surface of the Earth, and the drawings on it make the homeomorphism explicit.

Similarly, we can reconstruct any manifold (well, something homeomorphic to it), by taking the images of the charts, and gluing them using the transition functions. This gives a very concrete method to construct manifolds.

1.2.1 Constructing a manifold from its transition functions

Take a family $\{D_\alpha\}_{\alpha\in I}$ of open sets of $\mathbb{R}^n$, $\mathbb{R}^n_+$ or $\mathbb{R}^n_-$ and denote by N the topological space obtained as disjoint union

$$N := \coprod_{\alpha\in I} D_\alpha.$$

Give, for each pair $\alpha,\beta \in I$, open subsets $D_{\beta\alpha} \subset D_\alpha$, $D_{\alpha\beta} \subset D_\beta$, with $D_{\alpha\alpha} = D_\alpha$ for all α, and a homeomorphism $\varphi_{\beta\alpha} \colon D_{\beta\alpha} \to D_{\alpha\beta}$.

Assume that the set of functions $\varphi_{\beta\alpha}$ has the following properties (see Complement 1.2.3)

- $\forall\alpha \in I$, $\varphi_{\alpha\alpha} = \mathrm{Id}_{D_\alpha}$;
- $\forall\alpha,\beta \in I$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$;
- $\forall\alpha,\beta,\gamma \in I$, $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.

Then we say that $x_\alpha \in D_\alpha$ is equivalent to $x_\beta \in D_\beta$ if and only if $\varphi_{\beta\alpha}(x_\alpha) = x_\beta$.

$$x_\alpha \sim x_\beta \Leftrightarrow \varphi_{\beta\alpha}(x_\alpha) = x_\beta$$

The reader should check that under the above assumptions the equivalence just defined is an equivalence relation. Denote by M the quotient of N by this equivalence relation:

$$M := N/\sim$$

In general, M is neither Hausdorff nor it admits a countable base of open sets. But if these properties are verified, M is a topological manifold with boundary, and $\{(i_\alpha(D_\alpha), i_\alpha^{-1})\}_{\alpha\in I}$ is a set of charts covering M. Here $i_\alpha \colon D_\alpha \to M$ is the composition of the inclusion of D_α in N with the projection of N onto its quotient M.

Example 1.5 — A circumference. We apply the above method to construct a manifold.

Set $I = \{0,1\}$, so our family will be made of exactly two open sets.

Set $D_0 = \mathbb{R}$, $D_1 = \mathbb{R}$. These are then two distinct copies of $\mathbb{R}$. To avoid misunderstandings we will denote by x_0 the natural coordinate of D_0 and by x_1 the natural coordinate of D_1.

Then set

$$D_{10} = D_0 \setminus \{0\} = \{x_0 \in D_0 | x_0 \neq 0\} \qquad D_{01} = D_1 \setminus \{0\} = \{x_1 \in D_1 | x_1 \neq 0\}$$

Then we choose the following transition functions

$$\begin{aligned} \varphi_{00}\colon D_0 &\to D_0 & \qquad \varphi_{11}\colon D_1 &\to D_1 \\ x_0 &\mapsto x_0 & x_1 &\mapsto x_1 \\ \varphi_{10}\colon D_{10} &\to D_{01} & \varphi_{01}\colon D_{01} &\to D_{10} \\ x_0 &\mapsto \frac{1}{x_0} & x_1 &\mapsto \frac{1}{x_1} \end{aligned}$$

Complement 1.2.1 Show that the following topological spaces are topological manifolds with boundary, by checking all the properties in the definition

- the open ball $\overset{\circ}{B^n} := \{(x_1,\ldots,x_n) \in \mathbb{R}^n | \sum_i x_i^2 < 1\}$;
- the closed ball $B^n := \{(x_1,\ldots,x_n) \in \mathbb{R}^n | \sum_i x_i^2 \leq 1\}$;
- the sphere $S^n := \{(x_1,\ldots,x_n) \in \mathbb{R}^n | \sum_i x_i^2 = 1\}$;
- the $n-$dimensional torus $T^n := \mathbb{R}^n/\sim$ where the equivalence relation is the relation

  $$(x_1,\ldots,x_n) \sim (y_1,\ldots,y_n) \Leftrightarrow \forall i \; x_i - y_i \in \mathbb{Z}.$$

  People commonly write this as $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

Complement 1.2.2 The following topological spaces are not topological manifolds with boundary. Determine, for each of them, exactly which of the properties in the definition of topological manifold with boundary fail.

- The cross $\{(x,y) \in \mathbb{R}^2 | xy = 0, \max(|x|,|y|) = 1\}$;
- $\{(x,y) \in \mathbb{R}^2 | x(x^2+y^2-1) = 0\}$;
- The line with two origins $\mathbb{R}\coprod\mathbb{R}/\sim$ where $\sim$ is defined as follows. We write by x_i the point in $\mathbb{R}\coprod\mathbb{R}$ belonging to the $i-$th copy of $\mathbb{R}$ with coordinates x: so -1_1, 5_2, 3_1, 3_2, 0_1, 0_2 are six different points of $\mathbb{R}\coprod\mathbb{R}$. We say that $x_i \sim y_j$ if $x = y \neq 0$; in other words $-1_1 \sim -1_2$, $3_1 \sim 3_2$ but $0_1 \not\sim 0_2$.
- The closed long ray. If X is a totally ordered set, the order topology on X is a topology whose basis is given by the open intervals $(a,b) = \{x | a < x < b\}$. Let ω_1 be the first uncountable ordinal ω_1, with its well ordering. Consider the half-open interval $[0,1)$ with the standard ordering of the real numbers. Take their product $\omega_1 \times [0,1)$ with the lexicographical order, and put the corresponding order topology on it.

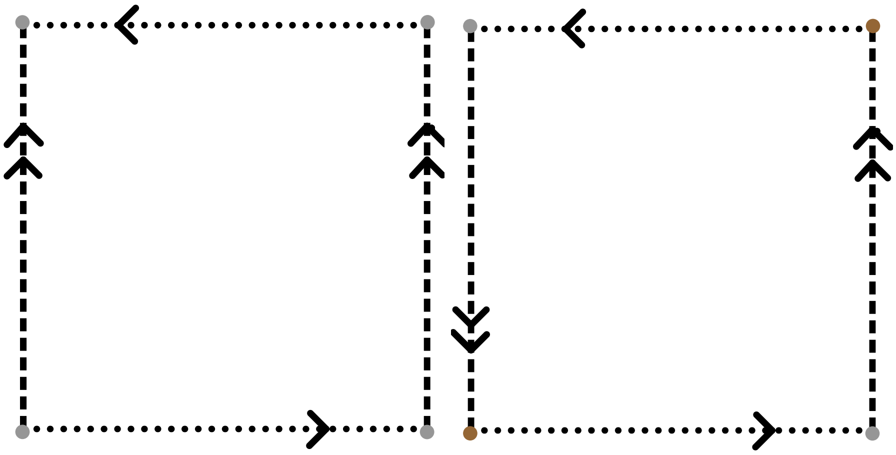
Complement 1.2.3 Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be charts on M. Show

- $\forall \alpha \in I$, $\varphi_{\alpha\alpha} = Id$;
- $\forall \alpha,\beta \in I$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$;
- $\forall \alpha,\beta,\gamma \in I$, $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.

Complement 1.2.4 Show that the equivalence relation in subsection 1.2.1 is an equivalence relation on N since each of the properties in Complement 1.2.3 guarantees one of the properties required by an equivalence relation: reflexivity, symmetry, transitivity.

Complement 1.2.5 Take a topological manifold with boundary M', cover it with a set of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Consider the open sets $D_\alpha := \varphi_\alpha(U_\alpha)$ and the corresponding transition functions $\varphi_{\alpha\beta}$. Note that we can apply to it the construction 1.2.1, to construct a new topological space M. Construct a bijective map from M to M', and show that it is a homeomorphism.

Figure 1.5: More topological manifolds dominated by the square



Exercise 1.2.1 Figure 1.5 shows two ways to identify the opposite sides of a square that are different from the identification presented in Figure 1.2. Show that both quotients are topological manifolds by checking the properties of the definition.

Exercise 1.2.2 For which values[a] of $(p,q) \in \mathbb{N}^2$, is the (p,q)-cusp $\Gamma_{p,q} := \{(x,y) \in \mathbb{R}^2 | x^p = y^q\}$ a topological manifold with boundary? Motivate your answer.

[a] Hint: Look at the map $t \mapsto (t^q, t^p)$

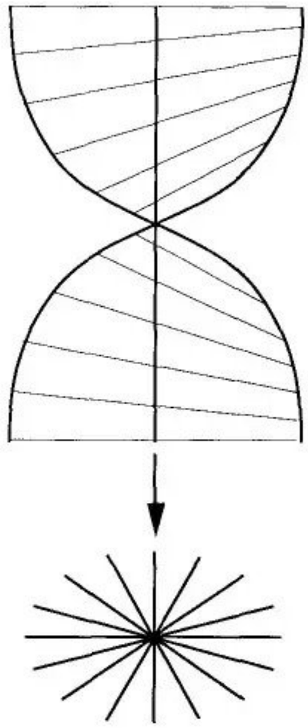
Exercise 1.2.3 Prove that the topological space N constructed in Example 1.5 is a topological manifold homeomorphic to the circumference

$$S_1 = \left\{(u,v) \in \mathbb{R}^2 | u^2 + v^2 = 1\right\}$$

Exercise 1.2.4 — The Blow Up of the real affine plane at the origin. Pick $I = \{0,1\}$, $D_0 \cong D_1 \cong \mathbb{R}^2$. To avoid confusion, $\forall j \in I$ we denote by (x_j, y_j) the coordinates of D_j. Then set

$$D_{10} = D_0 \setminus \{x_0 = 0\}, \qquad D_{01} = D_1 \setminus \{y_1 = 0\}.$$

Figure 1.6: A picture of the blow up of the real affine plane at the origin



Consider the map $\varphi_{10}\colon D_{10} \to D_{01}$ defined by

$$\varphi_{10}(x_0, y_0) = \left(x_0 y_0, \frac{1}{x_0}\right).$$

1. Show that φ_{10} is a homeomorphism by producing an explicit formula for its inverse.
2. Notice that there is a unique set of functions $\varphi_{\beta\alpha}$ as in subsection 1.2.1 containing the given φ_{10} and use it to construct the manifold $Bl_0(\mathbb{R}^2) := (D_0 \coprod D_1)/\sim$.
3. Verify that the functions $F_j\colon D_j \to \mathbb{R}^2$ defined as

$$F_0(x_0, y_0) = (x_0 y_0, y_0) \qquad\qquad F_1(x_1, y_1) = (x_1, x_1 y_1)$$

fulfill $F_0 = F_1 \circ \varphi_{10}$, and use that to *extend both* F_j to a function $F\colon Bl_0(\mathbb{R}^2) \to \mathbb{R}^2$.

4. Show that F is continuous and that, $\forall p \neq 0$, $\#F^{-1}(p) = 1$.
5. Show[a] that $F^{-1}(0) \cong S^1$ (so the point 0 has been "blown up" to a circumference).

[a]Hint: pick any line l through the origin and compute explicitely $F^{-1}(l)$. Look at the result: does it allows you to associate to l a point of $f^{-1}(0)$? If so, this is a map from $\mathbb{P}^1_{\mathbb{R}}$ to $F^{-1}(0)$.

Exercise 1.2.5 — The Blow Up of the complex affine plane at the origin. Construct a topological manifold $Bl_0(\mathbb{C}^2)$ of dimension 4 by substituting, in the construction of the Exercise 1.2.4, D_0 and D_1 with two copies of $\mathbb{C}^2 \cong \mathbb{R}^4$ and considering (x_j, y_j) as complex coordinates. Construct an analogous continous map $F\colon Bl_0(\mathbb{C}^2) \to \mathbb{C}^2$ and prove that $F^{-1}(0) \cong S^2$.

1.3 Differentiable structures

First of all we need to introduce the class of functions we are working with.

Definition 1.3.1 — Smooth and holomorphic functions. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets. A function $F\colon U \to V$ is **smooth** if all its components have continuous partial derivatives of all orders in U.

Similarly, if $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open, we will say that $F\colon U \to V$ is **holomorphic** if all its components have continuous (complex) partial derivatives of all orders in U.

We will denote respectively by $C^\infty(U)$ and $\mathscr{O}(U)$ the set of smooth and holomorphic functions from an open subset U of $\mathbb{K}^n$ to $\mathbb{K}$.

We note that $C^\infty(U)$ and $\mathscr{O}(U)$ have a natural structure of $\mathbb{K}-$algebra where $\mathbb{K}$ is respectively $\mathbb{R}$ or $\mathbb{C}$.

Despite the analogy in our definition, the reader should be aware that smooth functions and holomorphic functions are very different. Standard complex analysis shows the holomorphic functions are automatically analytic, so "rigid" in some sense. For example a holomorphic function that vanishes on an open set will automatically vanish on any connected component of its domain intersecting it. In contrast it is easy to build smooth functions not identically zero that vanish on large open subsets, and they have a major role in the theory of differentiable manifolds.

We will see some occurrences of this phenomenon later on.

We extend the definition of smooth function from $\mathbb{R}^n$ to the other model spaces $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$.

Definition 1.3.2 Let U be an open set of $\mathbb{R}^n_\pm$. A function $F\colon U \to \mathbb{R}^m$ is **smooth** if there is an open set $V \subset \mathbb{R}^n$ with $V \cap \mathbb{R}^n_\pm = U$ and a smooth function $G\colon V \to \mathbb{R}^m$ which extends F, *i.e.* such that $G_{|U} = F$.

A function among manifolds induces many maps between open sets of $\mathbb{R}^n_\pm$ (resp. $\mathbb{C}^n$) by composing it with two charts, one from the domain manifold, one from the codomain manifold. The natural idea for extending the definition of smooth (resp. holomorphic) function to the category of manifolds is to declare a function smooth (resp. holomorphic) if all these compositions are smooth (resp. holomorphic). To ensure that the identity is smooth (resp. holomorphic), we need all transition functions to be smooth (resp. holomorphic), and this motivates all the following definitions.

Definition 1.3.3 An **atlas** (resp. **complex atlas**) for a topological space M is a family of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ on M such that $\bigcup_{\alpha \in I} U_\alpha = M$ and all transition functions $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are smooth (resp. holomorphic).

Two atlases (resp. complex atlases) are **equivalent** if their union is an atlas (resp. complex atlas). A **differentiable structure** (resp. **complex structure**) on M is an equivalence class of atlases for M.

A **real manifold with boundary** (resp. **complex manifold**; in both cases we will sometimes just say **manifold** for short) is given by a topological manifold with boundary M and a differentiable structure (resp. complex structure) on it.

The **maximal atlas** of a manifold is the union of all the atlases in the differentiable structure (resp. complex structure.

A chart (V,ψ) is **compatible** with an atlas (resp. complex atlas) $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ if $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}\cup\{(V,\psi)\}$ is still an atlas (resp. complex atlas).

Note that the maximal atlas of a manifold is an atlas in its differentiable or complex structure. A maximal atlas is obtained by any other atlas in its differentiable (resp. complex) structure by adding all charts compatible with it.

Usually one uses a *small* atlas to determine the differentiable (or complex) structure. For example the two charts for B^1 in the previous section form an atlas and therefore determine a differentiable structure. However, once the differentiable structure is determined, we can use any compatible charts for our computations. So, in the example, we may also use, if convenient, the compatible chart given by the open set $(-1,1)$ with map given by its natural inclusion in $\mathbb{R}$.

Example 1.6 $\mathbb{R}^n$, S^n, B^n and $\mathbb{R}^n_+$ are real manifolds. For example, the atlas $\{(U_i,\varphi_i)\}_{i\in\{1,2\}}$ for $B^1=[-1,1]$ described in the last section gives a differentiable structure on it, since the transition functions are smooth.

Example 1.7 Consider the topological torus, the topological manifold in Example 1.3. The transition functions of the atlas given there are translations, and therefore smooth. This gives a differentiable structure on the torus, making it a differentiable manifold of dimension 2.

Identifying $\mathbb{R}^2$ with the set of the complex numbers $\mathbb{C}$, we can see the topological torus as the group quotient $\mathbb{C}/\{a+bi|a,b\in\mathbb{Z}\}$, a complex manifold of dimension 1.

Definition 1.3.4 — The projective space of a vector space. Let V be a vector space over a field $\mathbb{K}$. Then

$$\mathbb{P}(V):=(V\setminus\{0\})/\!\sim$$

where the equivalence relation is the relation

$$x\sim y\Leftrightarrow \exists\lambda\in\mathbb{K},\ \lambda\neq 0,\ \text{such that } x=\lambda y.$$

Note that $\mathbb{P}(V)$ can be naturally identified with the set of the $1-$dimensional subspaces of V.

If $\dim V$ is finite and $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, it is easy to give a structure of real (resp. complex) manifold over $\mathbb{P}(V)$ of dimension $\dim V-1$ by fixing a basis of V.

We write the charts explicitly for $V=\mathbb{R}^{n+1}$ in the example below.

Example 1.8 The $n-$dimensional real projective space is $\mathbb{P}^n_{\mathbb{R}}:=\mathbb{P}(\mathbb{R}^{n+1})$.

We say that a point $p\in\mathbb{P}^n_{\mathbb{R}}$ has **homogeneous coordinates** $(x_0:x_1:\cdots:x_n)$ if p is the class of the point $(x_0,x_1,\ldots,x_n)\in\mathbb{R}^{n+1}$.

Note that every point can be represented by infinitely many different homogeneous coordinates, pairwise related by the multiplication by a scalar. However

1. The open set $U_j:=\{x_j\neq 0\}$ is well defined[a] for all j.

2. The maps $\varphi_j\colon U_j \to \mathbb{R}^n$ defined via
$$\varphi_j(x_0 : x_1 : \cdots : x_n) = \left(\frac{x_0}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j}\right)$$
are well defined.
3. $\{(U_j, \varphi_j)\}_{j\in\{0,\dots,n\}}$ is an atlas for $\mathbb{P}^n_{\mathbb{R}}$.

Note that the complement $H_j := \{x_j = 0\}$ of U_j (a **reference hyperplane**) is naturally homeomorphic to $\mathbb{P}^{n-1}_{\mathbb{R}}$.

[a]Notice that *e.g.* the set $\{x_j \neq 1\}$ is not well defined.

R Since the sphere $S^n \subset \mathbb{R}^{n+1}$ intersects each $1-$dimensional subspace in two opposite points, $\mathbb{P}^n_{\mathbb{R}} = S^n/\sim$ where the equivalence relation is the relation $x \sim y \Leftrightarrow x = \pm y$.

Example 1.9 The $n-$dimensional complex projective space is $\mathbb{P}^n_{\mathbb{C}} := \mathbb{P}\left(\mathbb{C}^{n+1}\right)$.

It is a complex manifold of dimension n with homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$, reference hyperplanes H_j and atlas $\{(U_j, \varphi_j)\}_{0,\dots,n}$ analogous to those given in Example 1.8.

The complex analogous of the previous remark showing $\mathbb{P}^n_{\mathbb{R}}$ as a quotient of S^n is the Hopf fibration $S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$. We will discuss in the Example 3.4 the case $n = 1$.

Note that every holomorphic function among open sets of $\mathbb{C}^n$ and $\mathbb{C}^m$ can be seen as a smooth function among open sets of $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$. Therefore, every complex manifold has an underlying structure of real manifold, obtained by considering only the real structure of the codomains of its charts. In particular, $\mathbb{P}^n_{\mathbb{C}}$ is also a real manifold of dimension $2n$.

The $n-$dimensional projective spaces parametrize the $1-$dimensional vector subspaces of a vector space V. The Grassmann manifolds are the natural generalization of this idea, parametrizing vector subspaces of fixed dimension.

Definition 1.3.5 — The Grassmann manifolds. Let V be a vector space over a field $\mathbb{K}$. Then
$$\mathrm{Gr}(k,V) := \{\text{ vector subspaces } W \subset V \,|\, \dim W = k\}$$
is the **Grassmann manifold** of the $k-$subspaces of V. Note that $\mathbb{P}(V)$ can be naturally identified with $\mathrm{Gr}(1,V)$.

We set[a] $\mathrm{Gr}_{\mathbb{K}}(k,n) := \mathrm{Gr}_{\mathbb{K}}(k,\mathbb{K}^n)$, so $\mathbb{P}^n_{\mathbb{R}}$ resp. $\mathbb{P}^n_{\mathbb{C}}$ is identified with $\mathrm{Gr}_{\mathbb{R}}(1,n+1)$ (resp. $\mathrm{Gr}_{\mathbb{C}}(1,n+1)$).

If $\dim V$ is finite and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we can extend the structures given to the real and complex projective spaces to give a structure of real resp. complex manifold to all $\mathrm{Gr}_{\mathbb{K}}(k,V)$.

[a]Some authors use $\mathrm{Gr}_{\mathbb{K}}(k,n)$ for the set of the $k-$dimensional linear subspaces of the projective space $\mathbb{P}^n_{\mathbb{K}}$. The linear subspaces of dimension k of $\mathbb{P}^n_{\mathbb{K}}$ are (by definition) the images of the vector subspaces of dimension $k+1$ of $\mathbb{K}^{n+1}$. So their $\mathrm{Gr}(k,n)$ correspond to our $\mathrm{Gr}(k+1,n+1)$.

R By Exercise 5.1.14 $\mathrm{Gr}(q,V)$ may also be interpreted as the subset of $\mathbb{P}(\Lambda^q V)$ corresponding to the tensors of the form $v_1 \wedge \cdots \wedge v_q$.

We do an explicit example.

Example 1.10 — The Grassmann manifolds. Let $W_{k,n}$ be the set of the $k \times n$ matrices of maximal rank k. It is an open subset of $M_{k,n}(\mathbb{K})$, the set of all $k \times n$ matrices, isomorphic to

$\mathbb{K}^{kn}$. Each matrix $A \in W_{k,n}$ determines a point $H_A \in \mathrm{Gr}_{\mathbb{K}}(k,n)$, the subspace generated by its rows, thus defining a surjective map

$$\Phi_{k,n} \colon W_{k,n} \to \mathrm{Gr}_{\mathbb{K}}(k,n).$$

Moreover $H_A = H_B$ if and only if there is an invertible matrix $C \in \mathrm{GL}_k(\mathbb{K})$ such that $A = CB$.

In other words, $\Phi_{k,n}$ is the quotient by the left action of $\mathrm{GL}_k(\mathbb{K})$ on $W_{k,n}$, identifying its orbits with the Grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,n)$.

We give a differentiable (or complex) structure on $\mathrm{Gr}_{\mathbb{K}}(k,n)$ via charts that are (partial) right inverses of $\Phi_{k,n}$ as follows.

Let $D_{1,\dots,k} \subset \mathrm{Gr}_{\mathbb{K}}(k,n)$ be the subset of the matrices whose left $k \times k$ submatrix is the identity, *i.e.* the matrices of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & m_{1,k+1} & \cdots & m_{1,n} \\ 0 & 1 & \cdots & 0 & m_{2,k+1} & \cdots & m_{2,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & m_{k,k+1} & \cdots & m_{k,n} \end{pmatrix}$$

Notice $D_{1,\dots,k} \cong \mathbb{K}^{k(n-k)}$. Notice moreover that the restriction $(\Phi_{k,n})_{|D_{1,\dots,k}}$ is injective, so setting $U_{1,\dots,k} := \Phi_{k,n}(D_{1,\dots,k})$ its inverse gives a map

$$\varphi_{1,\cdots,k} \colon U_{1,\dots,k} \to D_{1,\dots,k} \cong \mathbb{K}^{k(n-k)}$$

thus giving the first chart $(U_{1,\dots,k}, \varphi_{1,\cdots,k})$.

Similarly, for all $1 \leq j_1 \lneqq \cdots \lneqq j_k \leq n$ we consider the subset $D_{j_1,\dots,j_k} \subset \mathrm{Gr}_{\mathbb{K}}(k,n)$ of the matrices containing the identity as the submatrix given by the columns $j_1, \dots, j_k$ and $U_{j_1,\dots,j_k} := \Phi_{k,n}(D_{j_1,\dots,j_k})$. We get a chart $(U_{j_1,\dots,j_k}, \varphi_{j_1,\cdots,j_k})$ by defining $\varphi_{j_1,\cdots,j_k} := ((\Phi_{k,n})_{|D_{1,\dots,k}})^{-1}$. Then

$$\left\{(U_{j_1,\dots,j_k}, \varphi_{j_1,\cdots,j_k}) | 1 \leq j_1 \lneqq \cdots \lneqq j_k \leq n\right\}$$

gives a differentiable or complex structure (depending on $\mathbb{K}$) on $\mathrm{Gr}_{\mathbb{K}}(k,n)$.

We may finally introduce the smooth (resp. holomorphic) functions. Note that the assumption that all transition functions are smooth (resp. holomorphic) is crucial, as it makes the definition of smoothness of f in p independent of the choice of the charts.

Definition 1.3.6 Let M be a manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ and N a manifold with atlas $\{(V_\beta, \psi_\beta)\}_{\beta \in I}$.

A function $F \colon M \to N$ is **smooth (resp. holomorphic) in a point p** $\in M$ if, given a chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$, and a chart (V_β, ψ_β) with $F(p) \in V_\beta$, the function $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth (resp. holomorphic) in $\varphi_\alpha(p)$.

A function $F \colon M \to N$ is **smooth** (resp. **holomorphic**) if it is smooth (resp. holomorphic) in every point $p \in M$.

Definition 1.3.7 A **diffeomorphism** is an invertible smooth function whose inverse is smooth. A **biholomorphism** is an invertible holomorphic function whose inverse is holomorphic.

Two open sets U and V are **diffeomorphic**, respectively **biholomorphic**, if there exists a diffeomorphism, respectively **biholomorphism**, $F \colon U \to V$.

Example 1.11 Let A be an invertible matrix with real coefficients.

Then (see Exercise 1.3.5 for more details) A defines a diffeomorphism from $\mathbb{P}^{n-1}_{\mathbb{R}}$ to itself, mapping a point with homogeneous coordinates v (as column vector) to the point with homogeneous coordinates Av.

Similarly an invertible matrix with complex coefficients define a biholomorphism from $\mathbb{P}^{n-1}_{\mathbb{C}}$ to itself.

Notice that there is natural diffeomorphism between $\mathrm{Gr}_{\mathbb{R}}(1,n)$ and $\mathbb{P}^{n-1}_{\mathbb{R}}$ and a natural biholomorphism between $\mathrm{Gr}_{\mathbb{C}}(1,n)$ and $\mathbb{P}^{n-1}_{\mathbb{C}}$.

Example 1.12 Let M be a manifold, $\varphi\colon U \to \mathbb{K}^n_{\pm}$ a chart, $D := \varphi(U)$.

Then U and D are open sets of the manifolds M and $\mathbb{K}^n_{\pm}$, and therefore they have a natural structure of manifold (see Complement 1.3.3): an atlas for U is given by the single chart $\varphi\colon U \to \mathbb{K}^n_{\pm}$; the differential structure of D has an atlas given also by a single chart, the inclusion $i\colon D \hookrightarrow \mathbb{K}^n_{\pm}$.

Then we can consider φ as function among two manifolds. It is easy to check that it is a diffeomorphism.

An important case is given by the smooth (resp. holomorphic) functions from a manifold M to $\mathbb{K}$. In the real case denote it by

$$C^\infty(M) := \{f\colon M \to \mathbb{R} | f \text{ is smooth}\}.$$

In the complex case the standard notation is $\mathscr{O}(M)$. Note that it is a real (resp. complex) vector space, with the operations induced by those of the codomain $\mathbb{R}$ (resp. $\mathbb{C}$).

Example 1.13 Let M be a manifold, $\varphi\colon U \to D \subset \mathbb{K}^n_{\pm}$ a chart. Consider the local coordinates $x_i := u_i \circ \varphi$. Then $x_i \in C^\infty(U)$ (resp. $\mathscr{O}(U)$).

R This course will never consider two different differentiable structure on the same topological manifold, but the student should be aware that it is possible.

It is indeed easy to construct two different differentiable structures on S^2, but one can prove (although this is not always easy) that the two resulting manifolds are diffeomorphic. Note that the diffeomorphisms will not be the identity: if we consider two different differentiable structures on the domain and on the codomain, the identity map is not smooth!

The situation in higher dimension is more complicated. Kervaire and Milnor constructed 28 different differentiable structures of S^7, which give 28 differentiable manifolds which are pairwise not diffeomorphic. In this course we are going to consider only one differentiable structure on each sphere S^k; we just mention that all other differentiable structures are referred to in the literature as *exotic spheres*.

Fintushel and Stern proved that a certain topological manifold of dimension 4, the Kummer 4-fold, admits at least a countable number of pairwise not diffeomorphic differentiable structures. We will not further investigate this problem in these lectures.

From this point of view the complex case is simpler, since it is not difficult to construct infinitely many pairwise not biholomorphic complex structures on $S^1 \times S^1$, but we will not show this in these lectures.

Complement 1.3.1 Put a differentiable structure on each of the topological manifolds of Complement 1.2.1

Complement 1.3.2 Put two different differentiable structures on $\mathbb{R}$ such that the two resulting manifolds are diffeomorphic and contruct the diffeomorphism between them.

Complement 1.3.3 Let M be a manifold. Prove that every open subset $U \subset M$ has a natural induced differentiable structure.

Exercise 1.3.1 Write an explicit atlas for S^1 formed by two charts obtained considering the stereographic projections $S^1 \setminus P \to \mathbb{R}$ where P is either the north pole $(0,1)$ or the south pole $(0,-1)$. Write all the transition functions and show that they provide a differentiable structure on S^1.

Exercise 1.3.2 Write an explicit atlas for S^2 formed by two charts obtained considering the stereographic projections $S^2 \setminus P \to \mathbb{R}^2$ where P is either the north pole $(0,0,1)$ or the south pole $(0,0,-1)$. Write all the transition functions and show that they provide a differentiable structure on S^2.

Now consider your charts as complex charts identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the natural way. Find a suitable (as simple as possible) modification of these charts giving a complex atlas for S^2.

Exercise 1.3.3 Show that the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ is diffeomorphic as a real manifold to the sphere S^2 with the differentiable structure of Exercise 1.3.2.

Exercise 1.3.4 Let M, N be two real (resp. complex) manifolds, and assume $\partial M = \emptyset$. Then show that $M \times N$ has a natural induced real (resp. complex) structure, by constructing an atlas for $M \times N$ using an atlas of M and an atlas of N. What goes wrong if both manifolds have a boundary?

Exercise 1.3.5 — Projectivities. Fix a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and consider a square matrix $A \in M_{n+1}(\mathbb{K})$. We wish to associate to A a map $\varphi_A : \mathbb{P}^n_{\mathbb{K}} \to \mathbb{P}^n_{\mathbb{K}}$ such that

$$A \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \Longrightarrow \varphi_A(x_0, x_1, \dots, x_n) = (y_0, y_1, \dots, y_n).$$

1. Show that φ_A is a well defined if and only if A is invertible.
2. Show that φ_A is the identity map if and only if A is a multiple of the identity matrix. In particular, since the multiples of the identity matrix form a normal (in fact central) subgroup of $\mathrm{GL}_n(\mathbb{K})$, this defines a faithful action of the quotient
$$\mathbb{P}\mathrm{GL}_n(\mathbb{K}) := \mathrm{GL}_n(\mathbb{K})/\mathbb{K}^*$$
on $\mathbb{P}^n_{\mathbb{K}}$.
3. Show that every element of $\mathbb{P}\mathrm{GL}_n(\mathbb{R})$ (respectively $\mathbb{P}\mathrm{GL}_n(\mathbb{C})$) is a diffeomorphism (respectively biholomorphism) of $\mathbb{P}^n_{\mathbb{R}}$ (respectively $\mathbb{P}^n_{\mathbb{C}}$) .

Exercise 1.3.6 Consider the standard complex torus, the complex manifold of dimension 1 in Example 1.7, the group quotient $\mathbb{C}/\{a+bi|a,b \in \mathbb{Z}\}$.

Show that for every complex number λ the map

$$[z] \mapsto [z+\lambda]$$

is a biholomorphism of the complex torus to itself. Show that it is the identity if and only if λ has integral both the real and the imaginary part.

Finally, show that in all other cases, it has no fixed points.

Exercise 1.3.7 Consider the standard complex torus.

Show that, given a complex number λ, the map

$$[z] \mapsto [z \cdot \lambda]$$

is well defined if and only if λ has integral both the real and the imaginary part.

Determine all $\lambda \in \mathbb{C}$ for which the above map is not constant. Show that they give exactly four maps. Show that they are biholomorphisms and compute the number of fixed points of each of them.

Exercise 1.3.8 Construct a differentiable structure on each of the two topological manifolds constructed in Exercise 1.2.1.

Find out which one is diffeomorphic to $\mathbb{P}^2_{\mathbb{R}}$.

Exercise 1.3.9 Let M be a manifold, and let (U,φ) be a chart on it, $D = \varphi(U)$. Then by Complement 1.3.3 both U and D are manifolds with the differentiable structure respectively induced by M and by $\mathbb{R}^n_+$. Show that φ is a diffeomorphism among them.

Exercise 1.3.10 Prove that if $G\colon M \to N$ and $F : N \to N'$ are smooth, then also $F \circ G\colon M \to N'$ is smooth.

2. Tangent vectors and differentials

2.1 Tangent spaces

We start introducing germs

Definition 2.1.1 — Germs of smooth or holomorphic functions. Let p be a point in the real or complex manifold M.

If $\mathbb{K}=\mathbb{R}$ we consider for each open subset $U\subset M$ the space of the smooth functions $C^\infty(U)=\{f\colon U\to\mathbb{R}\}$ and the space

$$\mathscr{E}_p:=\{f\in C^\infty(U)|U \text{ is open and } p\in U\}/\sim$$

where the equivalence relation is the following: two functions f,g are equivalent if there exists an open set $W\ni p$ contained in the domain of both functions such that $f_{|W}=g_{|W}$. An equivalence class for this relation is a **germ** of smooth function at p. $\mathscr{E}_p$ is the **stalk at p of the sheaf of smooth functions**.

If $\mathbb{K}=\mathbb{C}$ we consider the **stalk at** p **of the sheaf of holomorphic functions**

$$\mathscr{O}_p:=\{f\colon U\to\mathbb{C} \text{ holomorphic } |U \text{ is open and } p\in U\}/\sim$$

with the analogous equivalence relation.

The set $\mathscr{E}_p$ is an $\mathbb{R}$-algebra with the following operations.

sum: Given two germs α, β in $\mathscr{E}_p$ we define their sum $\alpha+\beta\in\mathscr{E}_p$ as follows.
We choose two representatives f, g such that α is the germ of f and β is the germ of g. So $f\in C^\infty(U)$, $g\in C^\infty(V)$ for some open sets $U,V\subset\mathbb{R}^n$ containing p. Then their common domain $W:=U\cap V$ is an open set containing p and we define the sum of α and β as the germ of the sum of the restriction of the representatives to W:

$$\alpha+\beta=\left[f_{|W}+g_{|W}\right]$$

This operation is well defined since, if $\tilde{f}$ and $\tilde{g}$ are different representatives respectively of α and β with common domain $\tilde{W}$, then $f_{|W}+g_{|W}$ and $f_{|\tilde{W}}+g_{|\tilde{W}}$ coincide on $W\cap\tilde{W}$ and therefore $\left[f_{|W}+g_{|W}\right]=\left[f_{|\tilde{W}}+g_{|\tilde{W}}\right]$.

product: Given two germs α, β in $\mathscr{E}_p$ we define their product $\alpha\beta\in\mathscr{E}_p$ in a similar way by using the product of functions instead of the sum of functions

$$\alpha\beta = \left[f_{|W} \cdot g_{|W}\right]$$

The same argument as in the sum shows that product is also well defined.

product by a scalar: Given a scalar $\lambda \in \mathbb{K}$ and a germ α in $\mathscr{E}_p$ we define their product $\lambda\alpha \in \mathscr{E}_p$ as the germ of the function λf for any representative $f \in C^\infty$ of α. The operation is well defined since the germ $\lambda\alpha$ does not depend on the choice of the representative f.

The analogous definitions in the complex case furnish $\mathscr{O}_p$ of a $\mathbb{C}$-algebra structure.

We alert the reader that from now we will often use a letter, such as f or g, for germs as well as for functions.

Note that, given a germ $f \in \mathscr{E}_p$ or $\mathscr{O}_p$, $f(p) \in \mathbb{K}$ is well defined since all the functions in the same equivalence class have the same value at p. This is important in the next definition. On the contrary, $\forall q \neq p$, $f(q)$ is not well defined.

Definition 2.1.2 — Tangent vectors. A **tangent vector** or **derivation** at p is a linear application $v\colon \mathscr{E}_p \to \mathbb{R}$ (in the real case) or $v\colon \mathscr{O}_p \to \mathbb{C}$ (in the complex case) such that for each pair of germs (f,g) the **Leibniz rule**

$$v(fg) = f(p)v(g) + g(p)v(f)$$

holds.

The **tangent space** of M at p is the $\mathbb{K}$-vector space T_pM formed by the derivations at p with the operations defined by

sum: For each pair (v,w) of derivations at p we define their sum $v+w$ by imposing that, for all $f \in \mathscr{E}_p$, $(v+w)(f) = v(f) + w(f)$.

product by a scalar: For each scalar $\lambda \in \mathbb{K}$ and for each derivation v at p we define λv by imposing, for each germ f, $(\lambda v)(f) = \lambda v(f)$.

We leave to the reader the (easy) check that $v+w$ and λv are derivations.

R Recall that if U is an open subset of a manifold M, it has a natural induced differentiable structure by Complement 1.3.3. If p is a point of U, the spaces of germs $\mathscr{E}_p$, if p is considered as a point of U or of M, are naturally canonically isomorphic. So in the following we will identify them. Consequently, we can and will identify T_pU and T_pM.

The first example of derivation comes from the partial derivatives in $\mathbb{K}^n$. If two functions $f,g\colon \mathbb{K}^n \to \mathbb{K}$ coincide in a neighborhood of a point p, then for each i between 1 and n their partial derivatives $\frac{\partial f}{\partial x_i}$ coincide also in the same neighborhood. In particular there is a function

$$\left(\frac{\partial}{\partial x_i}\right)_p : \mathscr{E}_p \to \mathbb{R}$$

($\left(\frac{\partial}{\partial x_i}\right)_p : \mathscr{O}_p \to \mathbb{C}$ in the complex case) well defined by the expression

$$\left(\frac{\partial}{\partial x_i}\right)_p ([f]) = \frac{\partial f}{\partial x_i}(p)$$

The reader should not find it difficult to verify that all $\left(\frac{\partial}{\partial x_i}\right)_p$ are derivations. We will in fact prove that they form a basis of $T_p\mathbb{K}^n$.

We need the following lemma

Lemma 2.1.3 Consider an open subset $U \subset \mathbb{R}^n$, a function $f \in C^\infty(U)$ and a point $p \in U$. Then there exists an open subset $W \subset U$ containing p and functions $f_i \in C^\infty(W)$ such that $f_i(p) = \left(\frac{\partial}{\partial x_i}\right)_p([f])$ and

$$\forall p' \in W \qquad f(p') = f(p) + \sum_1^n (x_i(p') - x_i(p)) f_i(p')$$

The analogous statement obtained by replacing $\mathbb{R}$ by $\mathbb{C}$ and C^∞ by $\mathscr{O}$ holds as well.

Proof. We choose $W = B_\delta(p)$, the ball of radius δ centered at p with δ small enough to ensure $W \subset U$. Having fixed $p' \in W$ we consider the straight path $\gamma\colon [0,1] \to W$ defined by $\gamma(t) = p + t(p' - p)$ connecting p and p'. By the Fundamental Theorem of Calculus and the Chain Rule

$$f(p') - f(p) = (f \circ \gamma)(1) - (f \circ \gamma)(0) = \int_0^1 (f \circ \gamma)'(t)dt =$$
$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\gamma(t))(x_i \circ \gamma)'(t)dt = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\gamma(t))(x_i(p') - x_i(p))dt$$

Then we define, for each i, the function $f_i \in C^\infty(W)$ by

$$f_i(p') = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(p' - p))dt$$

and deduce

$$f(p') - f(p) = \sum_1^n f_i(p')(x_i(p') - x_i(p))$$

The stated expression for $f_i(p)$ is an obvious consequence of the definition of f_i. ■

Now we can prove

Theorem 2.1.4 The set $\left\{ \left(\frac{\partial}{\partial x_i}\right)_p | 1 \le i \le n \right\}$ is a basis for $T_p\mathbb{K}^n$. In particular $\dim T_p\mathbb{K}^n = n$.

Proof. We need to prove that the vectors $\left(\frac{\partial}{\partial x_i}\right)_p$ are linearly independent and generate $T_p\mathbb{K}^n$.

The linear independence is easy. Assume that $\sum_i a_i \left(\frac{\partial}{\partial x_i}\right)_p = 0$. Then by evaluating this expression at every coordinate function x_j we obtain

$$0 = \left(\sum_i a_i \left(\frac{\partial}{\partial x_i}\right)_p \right)(x_j) = \sum_i a_i \left(\frac{\partial x_j}{\partial x_i}\right)_p = a_j$$

and the linear independence is proved.

To prove generation, pick any $v \in T_p\mathbb{K}^n$.

First we notice that v vanishes at the germ of any constant function. In fact

$$v([1]) = v([1] \cdot [1]) = 1 \cdot v([1]) + 1 \cdot v([1]) = 2v([1]) \Rightarrow v([1]) = 0$$

and then $v([c]) = cv([1]) = 0$ for any constant $c \in \mathbb{K}$ as well.

By Lemma 2.1.3, for the germ of any smooth function f,

$$[f] = f(p)[1] + \sum_{i=1}^{n} ([x_i] - [x_i(p)])[f_i]$$

and then

$$v([f]) = \sum_{1}^{n} f_i(p) v([x_i] - [x_i(p)]) = \sum_{1}^{n} \left(\frac{\partial}{\partial x_i}\right)_p ([f]) v([x_i])$$

so

$$v = \sum_{1}^{n} v([x_i]) \left(\frac{\partial}{\partial x_i}\right)_p$$

and the proof is complete. ■

The same proof show the analogous statement for $T_p\mathbb{R}^n_{\pm}$.

Note that $T_p\mathbb{R}^n_{+} \cong \mathbb{R}^n$ even when p lies on the boundary $\{x_n = 0\}$: a common mistake is to consider only half of it.

We will extend this result to general manifolds in the next section.

Exercise 2.1.1 Define the **germs** of the C^r functions analogously to the definition of germs of C^∞ functions and determine if it has a natural structure of $\mathbb{R}-$algebra.

Exercise 2.1.2 In the definition of $\mathscr{E}_p$ can we simplify the equivalence relation as follows?

We could say that $f \sim g$ if and only if f and g and all their first partial derivatives have the same values at p.

Is that equivalent to the definition we gave? Is that an equivalence relation? Is there an $\mathbb{R}-$algebra structure on the quotient?

2.2 The differential of a function

The main tool of this section is the differential of a function.

Definition 2.2.1 Let $F: M \to N$ be a smooth (resp. holomorphic) function among real (resp. complex) manifolds.

The **differential** of F in a point $p \in U$ is the linear application $dF_p\colon T_pM \to T_{F(p)}N$ defined by

$$dF_p(v)(f) = v(f \circ F).$$

for all $f \in \mathscr{E}_{F(p)}$ respectively $\mathscr{O}_{F(p)}$.

Here by $f \circ F$ we mean the germ at p of the composition of any representative of f with F.

We leave to the reader to check the consistency of the definition of dF_p. In other words the reader should show that our definition of $f \circ F \in \mathscr{E}_{F_p}$ does not depend on the choice of the representative of the germ f, then that $v(f \circ F)$ defines a derivation at $F(p)$, so that we defined a map $dF_p\colon T_pM \to T_{F(p)}N$. Then he/she should prove that dF_p is linear.

The following general version of the Chain Rule holds.

Proposition 2.2.2 — Chain Rule. Given two smooth/holomorphic functions $F\colon N \to N'$, $G\colon M \to N$ and a point $p \in M$, then

$$d(F \circ G)_p = dF_{G(p)} \circ dG_p$$

Proof. For all $v \in T_pM$, for all $f \in \mathscr{E}_{F(G(p))}$,

$$(dF_{G(p)} \circ dG_p)(v)(f) = (dF_{G(p)}(dG_p(v))(f) =$$
$$= dG_p(v)(f \circ F) = v(f \circ F \circ G) = d(F \circ G)_p(v)(f)$$

■

The differential of a diffeomorphism (respectively biholomorphism) at a point p of the domain is automatically invertible. In fact, if $F\colon M \to N$ is a diffeomorphism, there is a smooth function $F^{-1}\colon N \to M$ and by Proposition 2.2.2

$$F^{-1} \circ F = \mathrm{Id}_M \quad \Rightarrow \quad d(F^{-1})_{F(p)} \circ dF_p = (d\,\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$$
$$F \circ F^{-1} = \mathrm{Id}_N \quad \Rightarrow \quad dF_p \circ d(F^{-1})_{F(p)} = (d\,\mathrm{Id}_N)_{F(p)} = \mathrm{Id}_{T_{F(p)}N}$$

showing that dF_p is an isomorphism by exhibiting an inverse of it:

$$(dF_p)^{-1} = d(F^{-1})_{F(p)}$$

It follows that if $F\colon M \to N$ is a diffeomorphism then dF_p is an isomorphism between the vector spaces T_pM and $T_{F(p)}N$.

Since the charts are diffeomorphisms, we can use them to give bases of the tangent spaces.

Definition 2.2.3 Let M be a manifold, $p \in M$, (U, φ) a **chart of M in p**, *i.e.* a chart in the differentiable structure such that $p \in U$. Let $u_1, \dots, u_n$ be the coordinate functions of $\varphi(U) \subset \mathbb{R}^n$. Then we define

$$\left(\frac{\partial}{\partial x_i}\right)_p := d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial u_i}\right)_{\varphi(p)}.$$

It follows

Theorem 2.2.4 The set $\left\{\left(\frac{\partial}{\partial x_i}\right)_p |1 \leq i \leq n\right\}$ is a basis for T_pM. In particular $\dim T_pM = \dim M$.

Note that $\left(\frac{\partial}{\partial x_i}\right)_p : \mathscr{E}_p \to \mathbb{R}$ is by definition given by

$$\frac{\partial f}{\partial x_i}(p) := \left(\frac{\partial}{\partial x_i}\right)_p f = \left(\frac{\partial}{\partial u_i}\right)_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial (f \circ \varphi^{-1})}{\partial u_i}(\varphi(p)).$$

Why did we choose the notation $\frac{\partial}{\partial x_i}$? Recall the local coordinates introduced in the Example 1.13: the chart (U, φ) induces coordinates $x_1, \dots, x_n$ on U by $x_i := u_i \circ \varphi$.

Then[1]

$$\left(\frac{\partial}{\partial x_i}\right)_p (x_j) = \left(\frac{\partial}{\partial u_i}\right)_p (x_j \circ \varphi^{-1}) = \left(\frac{\partial}{\partial u_i}\right)_p (u_j) = \delta_{ij}.$$

[1] Here we use the usual *Kronecker* symbol: δ_{ij} equals 1 if $i = j$, whereas it vanishes if $i \neq j$.

We are now able to compute $\frac{\partial f}{\partial x_i}(p)$ for every function $f \in C^\infty(U)$ which we can explicitly write "in coordinates near p".

This means, given a function, we choose a chart (U, φ) in p and consider the induced coordinates $x_1, \ldots, x_n$. If we can express f as combination of the x_i, since $\left(\frac{\partial}{\partial x_i}\right)_p$ is a derivation $\left(\frac{\partial}{\partial x_i}\right)_p x_j = \delta_{ij}$ we can compute $\left(\frac{\partial}{\partial x_i}\right)_p f$ formally as if x_i were coordinates in $\mathbb{R}^n$. For example, if $f = x_1^2 x_2$, then

$$\left(\frac{\partial}{\partial x_1}\right)_p f = (2x_1x_2)(p), \qquad \left(\frac{\partial}{\partial x_2}\right)_p f = x_1^2(p).$$

By linearity, once we have fixed bases of $T_p\mathbb{K}^n$ and $T_{F(p)}\mathbb{K}^m$, we can describe dF_p by the corresponding matrix.

Definition 2.2.5 Let $U \subset \mathbb{K}^n$, $V \subset \mathbb{K}^m$ be open sets, and let $F\colon U \to V$ be a smooth (or holomorphic) function. Let $p \in U$.

The **Jacobi matrix** of F at p is the matrix having as (i, j)-entry (i^{th}th row and j^{th} column) the partial derivative of the i^{th} component $F_i := y_i \circ F$ of F with respect to the j^{th} coordinate, computed at p: $\left(\frac{\partial}{\partial x_j}\right)_p F_i$.

Proposition 2.2.6 Let $U \subset \mathbb{K}^n$, $V \subset \mathbb{K}^m$ be open sets, and consider a smooth/holomorphic function $F\colon U \to V$. We will use coordinates $x_1, \ldots, x_n$ on U and coordinates $y_1, \ldots, y_m$ on V.

Fix a point $p \in U$. Then dF_p is represented, with respect to the bases

$$\left\{\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p\right\} \text{ and } \left\{\left(\frac{\partial}{\partial y_1}\right)_{F(p)}, \ldots, \left(\frac{\partial}{\partial y_m}\right)_{F(p)}\right\}$$

by the Jacobi matrix of F computed in p.

Proof. We denote by $M_{i,j}$ the (i, j)-entry of the matrix of dF_p in the given bases. By definition

$$dF_p\left(\frac{\partial}{\partial x_j}\right)_p = \sum_i M_{i,j}\left(\frac{\partial}{\partial y_i}\right)_{F(p)},$$

and therefore

$$M_{i,j} = \sum_k M_{k,j}\left(\frac{\partial}{\partial y_k}\right)_{F(p)}(y_i) = dF_p\left(\frac{\partial}{\partial x_j}\right)_p(y_i) = \left(\frac{\partial}{\partial x_j}\right)_p(y_i \circ F)$$

and the proof is complete. ■

Now that we have given bases to every T_pM, so we can associate a matrix to every dF_p. The matrix can be computed by exactly the same method used for Proposition 2.2.6. The result is the following.

Proposition 2.2.7 Let M and N be manifolds of respective dimensions n and m, and let $F\colon M \to N$ be a smooth function. Let $p \in M$. Choose charts (U, φ) in p for M and (V, ψ) in $F(p)$ for N, and the respective associated local coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$.

Then the matrix associated to the linear application dF_p with respect to the bases

$$\left\{\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p\right\} \text{ and } \left\{\left(\frac{\partial}{\partial y_1}\right)_{F(p)}, \ldots, \left(\frac{\partial}{\partial y_m}\right)_{F(p)}\right\}$$

is the Jacobi matrix of $\psi \circ F \circ \varphi^{-1}$ computed in $\varphi(p)$. Equivalently, it is the matrix

$$\left(\left(\frac{\partial}{\partial x_j}\right)_p (y_i \circ F)\right) = \left(\left(\frac{\partial}{\partial x_j}\right)_p F_i\right)$$

Proof. In a neighborhood of p, $F = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi$ and therefore $dF_p = d(\psi^{-1})_{\psi(F(p))} \circ d(\psi \circ F \circ \varphi^{-1})_{\varphi(p)} \circ d\varphi_p$.

Therefore the matrix we are looking for equals the product of matrices $M_1 M_2 M_3$ where

- M_1 is the matrix of $d(\psi^{-1})_{\psi(F(p))}$ with respect to the bases $\left\{\left(\frac{\partial}{\partial u_i}\right)_{\psi(F(p))}\right\}$ and $\left\{\left(\frac{\partial}{\partial y_i}\right)_{F(p)}\right\}$;
- M_2 is the matrix of $d(\psi \circ F \circ \varphi^{-1})_{\varphi(p)}$ with respect to $\left\{\left(\frac{\partial}{\partial u_i}\right)_{\varphi(p)}\right\}$ and $\left\{\left(\frac{\partial}{\partial u_i}\right)_{\psi(F(p))}\right\}$;
- M_3 is the matrix of $d\varphi_p$ with respect to $\left\{\left(\frac{\partial}{\partial x_i}\right)_p\right\}$ and $\left\{\left(\frac{\partial}{\partial u_i}\right)_{\varphi(p)}\right\}$;

By Definition 2.2.3, M_1 and M_3 are both identity matrices (resp. $m \times m$ and $n \times n$), and therefore the matrix we are looking for equals M_2, which was computed in Proposition 2.2.6. ■

Now we can give an answer to the following natural question. Why do we call derivations also "tangent vectors"? Tangent to what? Here is an answer by a simple example.

Consider a connected open subset $J \subset \mathbb{K}$ containing 0 and a smooth/holomorphic map (a **path**)

$$\gamma \colon J \to \mathbb{K}^m$$

This induces a derivation γ_* in $T_{\gamma(0)}\mathbb{K}^m$, the **velocity** of γ as follows. To ease the notation we write here only the real case.

For every germ $f \in \mathscr{E}_{\gamma(0)}$ we define

$$\gamma_*(f) = \frac{d(f \circ \gamma)}{dt}(0)$$

Here in the right-hand term we are denoting by t the standard coordinate in $\mathbb{K}$ and implicitly replacing the germ f with one of its representatives: it is obvious that the result does not depend on the choice of the representative. It may not appear immediate that γ_* is a derivation. A quick way to show that is by noticing

$$d\gamma_0\left(\left(\frac{d}{dt}\right)_0\right)(f) = \left(\frac{d}{dt}\right)_0 (f \circ \gamma) = \frac{d(f \circ \gamma)}{dt}(0) \qquad \Rightarrow \qquad \gamma_* = d\gamma_0\left(\left(\frac{d}{dt}\right)_0\right)$$

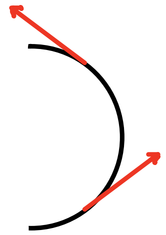
In particular, by Proposition 2.2.6, setting by $y_1, \ldots, y_m$ the coordinates of $\mathbb{K}^m$ and by $\gamma_i := y_i \circ \gamma$ the i^{th} component of γ,

$$\gamma_* = \sum \frac{d\gamma_i}{dt}(0)\left(\frac{\partial}{\partial y_i}\right)_{\gamma(0)}$$

It is usual to draw the tangent vector $\gamma_*^{p,v}$ as an arrow with the rearmost end at the point p and arrowhead at $p+v$, so overlapping the image of the path $\gamma^{p,v}$, see Figure 2.1. So γ_* will be always drawn tangent to the curve image of γ. In the picture below we have consider the velocity of the path $\gamma(t-t_0) = (\cos(t-t_0), \sin(t-t_0))$ for two different values of t_0. The image of the paths (they have the same image!) is drawn in black, the velocity vectors in red.

It is easy to show that every tangent vector can be obtained in this way. In fact, for every pair p, v of points of $\mathbb{K}^m$ we can consider the "straight" path $\gamma^{p,v}(t) = p + tv$ and easily compute $\gamma_*^{p,v}$ as follows.

Figure 2.1: The velocity vectors look very tangent



We know by Theorem 2.1.4 that $\gamma_*^{p,v} = \sum a_i \left(\frac{\partial}{\partial y_i}\right)_p$ for some constants a_i. We compute the a_j by evaluating $\gamma_*^{p,v}$ on the coordinate functions:

$$a_j = \sum a_i \left(\frac{\partial}{\partial y_i}\right)_p (x_j) = \gamma_*^{p,v}(y_j) = \frac{d(y_j \circ \gamma)}{dt}(0) = v_j$$

where v_j is the j^{th} component of $v = (v_1, \dots, v_m)$.

So every tangent vector can be obtained by a straight path. Different paths give the same tangent vector if and only if they share at zero the same value and the same first derivatives, in other words if the paths are tangent.

R Consider, $\forall q \in N$ the inclusion map $i_q \colon N \to M \times N$ defined by $i_q(p) = (p,q)$.
By Exercise 2.2.1, i_q is smooth and the map $d(i_q)_p$ is injective.
We can then identify T_pM with its image $d(i_q)_p(T_pM)$ in $T_{(p,q)}(M \times N)$. With this abuse of notation the last equality in Exercise 2.2.1 can be written

$$T_{(p,q)}(M \times N) = T_pM \oplus T_qN.$$

Complement 2.2.1 Let M be a manifold and let $U \subset M$ be an open subset with the differentiable structure induced by the differentiable structure of M. Let $i \colon U \to M$ be the inclusion map, and fix a point $p \in U$.

Prove that $di_p \colon T_pU \to T_pM$ is an isomorphism.

Complement 2.2.2 Use Corollary 3.0.1 to prove that $\{(TU, d\varphi) | (U, \varphi) \text{ is a chart for } M\}$ is an atlas for TM, so giving to TM a differentiable structure of manifold of dimension $2\dim M$.

Exercise 2.2.1 Let M, N be manifolds, M without boundary. Consider $M \times N$ with the differentiable structure in Exercise 1.3.4. Then

- Show that the projections $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ are smooth.
- Consider, $\forall q \in N$ the inclusion map $i_q \colon N \to M \times N$ defined by $i_q(p) = (p,q)$. Similarly consider, $\forall p \in M$ the inclusion map $i_p \colon N \to M \times N$ defined by $i_p(q) = (p,q)$. Prove that both maps i_p, i_q are smooth.
- Show that $\forall p \in M$, $\forall q \in N$, $d(\pi_1)_{(p,q)}$, $d(\pi_1)_{(p,q)}$ are surjective whence $d(i_p)_q$, $d(i_q)_p$ are injective
- Show that $\forall p \in M$, $\forall q \in N$, the image of $d(i_p)_q$ equals $\ker d(\pi_1)_{(p,q)}$. Similarly the image of di_q equals $\ker d(\pi_2)_{(p,q)}$.

- Show that $\forall p \in M$, $\forall q \in N$, $d(\pi_1)_{(p,q)} \circ (di_q)_p = Id_{T_pM}$, $d(\pi_2)_{(p,q)} \circ (di_p)_q = Id_{T_qN}$
- Show that $T_{(p,q)}(M \times N) = d(i_q)_p\,(T_pM) \oplus d(i_p)_q\,(T_qN)$.

Exercise 2.2.2 Under the assumptions of Proposition 2.2.2 write, for all i and j, an explicit expression for

$$\frac{\partial (F \circ G)_i}{x_j}(p)$$

in terms of the partial derivatives of F and G.

2.3 Submanifolds

In this section we discuss injective maps among manifolds $M \hookrightarrow N$ with good properties, called *embeddings*. We will need to use some results which we will state without proof.

Definition 2.3.1 Let M, N be manifolds and let $F: M \to N$ be a smooth function. Then

- a **critical point** of F is a point $p \in M$ such that the rank of the linear application dF_p is different from $\min(\dim M, \dim N)$, the maximal possible rank.
- a **critical value** is a point $q \in N$ that is the image $q = F(p)$ of a critical point p.
- a **regular value** is a point $q \in N$ that is not a critical value.

Note that by definition every point which is not in the image of F is a regular value. Indeed, Sard's Lemma shows that the set of regular values $\mathrm{Reg}(F)$ is very big, and more precisely it is an open dense subset of N.

A very important special case is the case $\mathrm{Reg}(F) = N$, when no point is a critical point. This is the case of the immersions (when $\dim M \le \dim N$), the submersions (when $\dim M \ge \dim N$) and the embeddings (when moreover the map is a homeomorphism among of M with its image).

Definition 2.3.2 Let M, N be manifolds and let $F: M \to N$ be a smooth function. Then

- F is an **immersion** if $\forall p \in M$, dF_p is injective.
- F is a **submersion** if $\forall p \in M$, dF_p is surjective.
- F is a **local diffeomorphism** if $\forall p \in M$, dF_p is invertible.
- F is an **embedding** if F is an immersion and a homeomorphism among M and $F(M)$, where $F(M)$ is considered with the topology induced by N.
  If F is an embedding then we often identify M with its image $F(M) \subset N$, and say that $M \subset N$ is a **submanifold**.

The following is a famous simple example of embedding.

Example 2.1 — The rational normal curves. The map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{d+1}$ defined by

$$(x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_0 x_1^{d-1} : x_1^d) \tag{2.1}$$

is an embedding, whose image is called rational normal curve of degree d.

Let us prove that (2.1) is an embedding.

We first note that the definition (2.1) is well posed.

In fact if we change the homogeneous coordinates of a point in the domain, replacing $(x_0 : x_1)$ with, say, $(\lambda x_0 : \lambda x_1)$ for some $\lambda \in \mathbb{K} \setminus \{0\}$, then the homogeneous coordinates of the image get multiplied by the same constant λ^d, and therefore define the same point; moreover it is not possible that simultaneously $x_0^d = x_0^{d-1}x_1 = \cdots = x_0 x_1^{d-1} = x_1^d = 0$ since that would imply $x_0 = x_1$, a contradiction.

Then we show that (2.1) is an injective immersion. Let us first consider the chart $U_0 = \{x_0 \neq 0\}$ of $\mathbb{P}^1$ with local coordinate $\bar{x} = \frac{x_i}{x_0}$. If we call the homogenous coordinates of the codomain $y_0, \dots y_d$ the image of $\{x_0 \neq 0\}$ is contained in the affine open subset $\{y_0 \neq 0\}$ having local coordinates $\bar{y}_1 = \frac{y_1}{y_0}, \dots, \bar{y}_d = \frac{y_d}{y_0}$ and the map is locally

$$\bar{x} \mapsto (\bar{x}, \bar{x}^2, \dots, \bar{x}^d)$$

that is obviously an injective immersion. The analogous computation on the chart $U_1 = \{x_1 \neq 0\}$ shows that (2.1) is an injective immersion too.

Finally, recalling that every continous bijective map from a compact space to a Hausdoff space is a homeomorphism, we conclude that (2.1) is an embedding.

It is not difficult to show that the rational normal curve of degree 2 is an irreducible conic, the conic $y_1^2 = y_0 y_2$ n the coordinates above. See Exercise 2.3.2 for a description of the rational normal curves of higher degree.

We give now few other classical examples of embeddings. In all cases it is possible to show that they are embeddings by arguments similar to those in Example 2.1. We leave the details to the reader.

Example 2.2 — The Veronese surface. The map $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ defined by

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0 x_1 : x_1^2 : x_0 x_2 : x_1 x_2 : x_2^2)$$

is an embedding, whose image is called Veronese surface.

The two previous examples are special cases of the following general construction.

Example 2.3 — d−Veronese embeddings. Consider the homogeneous coordinates of $\mathbb{P}^n_{\mathbb{K}}$ as $n+1$ "variables" $x_0, \dots, x_n$.

Fix an integer $d \geq 1$ and set $N := \binom{n+d}{d} + 1$. Fix a bijection among the "variables" $y_0, \dots, y_N$ of $\mathbb{P}^N_{\mathbb{K}}$ and the monomials of degree d in the variables x_j: $x_0^{d_0} \cdots x_n^{d_n}$ with $\sum_{j=0}^n d_j = d$. This defines a map

$$V_{n,d} \colon \mathbb{P}^n_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}.$$

The map is well defined since

- if you change the homogeneous coordinates of a point p, they are multiplied by the same factor $\lambda \in \mathbb{K} \setminus \{0\}$: then the values of all chosen monomials are multiplied by the same factor λ^d and then they define the same point in $\mathbb{P}^N_{\mathbb{K}}$;
- for every point $p \in \mathbb{P}^n_{\mathbb{K}}$, chosen homogeneous coordinates of it, the monomials of degree d evaluated in them cannot vanish all simultaneously.

It is not difficult to prove that it is an embedding.

R If $d = 1$ the maps $V_{n,1}$ are (uninteresting) diffeomorphisms.
The rational normal curves are the images of $V_{1,d}$.
The Veronese surface is the image of $V_{2,2}$.

The following two examples of embeddings come by a similar idea.

Example 2.4 — Segre varieties. Consider two projective space $\mathbb{P}^h$ and $\mathbb{P}^k$ with respective homogeneous coordinates $(x_0 : \cdots : x_h)$, $(y_0 : \cdots : y_k)$, The monomials $x_i y_j$ define a map

$$\sigma_{h,k}\colon \mathbb{P}^h \times \mathbb{P}^k \to \mathbb{P}^{(h+1)(k+1)-1}$$

that is called Segre embedding of $\mathbb{P}^h \times \mathbb{P}^k$. Its image is the Segre variety $\Sigma_{h,k}$.

Example 2.5 Consider k copies of $\mathbb{P}^1$ with respective homogeneous coordinates $(x_{h0} : x_{h1})$, $h = 0, \dots, k$ .The 2^k monomials $x_{0i_0} x_{1i_1} \cdots x_{ki_k}$ define a map

$$S_k\colon \left(\mathbb{P}^1\right)^k \to \mathbb{P}^{2^k - 1}$$

that is called Segre embedding of $\left(\mathbb{P}^1\right)^k$.

Example 2.6 — Plücker embeddings. We have seen in the Remark after Definition 1.3.5 that the Grassmann manifold $\mathrm{Gr}(k,V)$ is in bijection with the elements of $\mathbb{P}(\Lambda^k V)$ corresponding to tensors of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_k$. This defines the Plücker embedding

$$P\colon \mathrm{Gr}(k,V) \to \mathbb{P}(\Lambda^k V).$$

We conclude this section by considering the boundary of a manifold.

Definition 2.3.3 Let M be a manifold. Then
- the **interior** of M is the open subset $M^\circ := \{p \in M$ such that $\exists$ a chart (U,φ), $p \in U$, with $\varphi(U)$ open in $\mathbb{K}^n\}$.
- the **boundary** of M is $\partial M := M \setminus M^\circ$.

M is **without boundary** if $\partial M = \emptyset$.

It is easy to show that M° is an open subset of M, so it is a manifold of the same dimension of M. Moreover M° is without boundary. Note that all complex manifolds are without boundary, so this definition really makes sense only in the real case.

Something similar holds for the boundary. Note that $\partial\mathbb{R}^n = \emptyset$, $\partial\mathbb{R}^n_+ = \partial\mathbb{R}^n_- = \mathbb{R}^{n-1}$.

We will use the following lemma without proving it.

Lemma 2.3.4 Let U, V be open subsets of $\mathbb{R}^n_\pm$, and let $F\colon U \to V$ be a diffeomorphism. Consider $U_0 := U \cap \partial\mathbb{R}^n_\pm$, $V_0 := V \cap \partial\mathbb{R}^n_\pm$. Then $F(U_0) \subset V_0$ and $F_{|U_0}\colon U_0 \to V_0$ is a diffeomorphism.

Note that in particular, if U_0 is not empty, then also V_0 is not empty. Let now M be a manifold, $p \in M$. Then assume that there is a chart (U,φ) such that $\varphi(p) \in \varphi(U) \cap \partial\mathbb{R}^n_\pm$. Then, since every transition function is a diffeomorphism, by Lemma 2.3.4 for each other chart (V,ψ) with $p \in V$, $\psi(p) \in \partial\mathbb{R}^n_\pm$. It follows

$$\begin{aligned}\partial M &= \{p \in M \text{ such that there exists a chart } (U,\varphi) \text{ with } \varphi(p) \in \partial\mathbb{R}^n_\pm\}\\ &= \{p \in M \text{ such that for all chart } (U,\varphi) \text{ with } p \in U, \varphi(p) \in \partial\mathbb{R}^n_\pm\}\end{aligned}$$

The boundary ∂M has a natural differentiable structure making it a real manifold of dimension $n-1$ as follows. Take an atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$. Let $I' \subset I$ be the subset of the indices α such that $U_\alpha \cap \partial M \neq \emptyset$. Then $\varphi_\alpha(U_\alpha \cap \partial M)$ is a nonempty open subsets of $\partial\mathbb{R}^n_\pm = \mathbb{R}^{n-1}$. Then we can consider the maps $(\varphi_\alpha)_{|U_\alpha\cap\partial M}$ as maps onto open subsets of $\mathbb{R}^{n-1}$.

It is now easy to see that $\{(U_\alpha \cap \partial M, (\varphi_\alpha)_{|U_\alpha\cap\partial M})\}_{\alpha\in I'}$ is an atlas for ∂M, making ∂M a manifold of dimension $\dim M - 1$. Since the images of all charts in the atlas are open subsets of $\mathbb{R}^{n-1}$, ∂M has no boundary. So $\partial\partial M = \emptyset$.

Example 2.7 We have just seen two examples of embeddings, the inclusions $M^\circ \hookrightarrow M$ and $\partial M \hookrightarrow M$. Similarly, if $U \subset M$ is an open subset, the inclusion $U \hookrightarrow M$ is an embedding.

Exercise 2.3.1 Consider the torus $T := \mathbb{R}^2/\mathbb{Z}^2$ with the differentiable structure in Example 1.7 and observe that the quotient map $\pi\colon \mathbb{R}^2 \to T$ is a local diffeomorphism.

Consider a line $l = l_{a,b,c} := \{ax+by+c=0\} \subset \mathbb{R}^2$ and observe that l is a submanifold of $\mathbb{R}^2$. Consider the induced map $F := \pi_{|l}\colon l \to T$. Show that

1. F is an immersion.
2. If $\frac{a}{b} \notin \mathbb{Q}$, F is injective and its image is dense in T.
3. If $\frac{a}{b} \in \mathbb{Q}$, F is not injective and its image is a compact embedded submanifold of dimension 1 of T diffeomorphic to S^1.

Observe that in particular F is never an embedding.

Exercise 2.3.2 Show that the rational normal curve in $\mathbb{P}^{d+1}$, with homogeneous coordinates $(y_0 : y_1 : \cdots : y_d)$, is the locus where the 2×2 minors of the matrix

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{k-1} \\ y_1 & y_2 & \cdots & y_k \end{pmatrix}$$

vanish.

Exercise 2.3.3 Show that the image of the Segre embedding of $\left(\mathbb{P}^1\right)^2$ is a quadric of $\mathbb{P}^3$, *i.e.* is the set of points where a homogenous polynomial of degree 2 in the coordinates of $\mathbb{P}^3$ vanish.

Exercise 2.3.4 Show that the Veronese surface in $\mathbb{P}^5$ is the zero locus of six quadrics, and precisely the 2×2 minors of a symmetric 3×3 matrix with entries the homogeneous coordinates of $\mathbb{P}^5$.

Exercise 2.3.5 Show that the image of the Plücker embedding of $\mathrm{Gr}(2,4)$ is a quadric[a] of $\mathbb{P}^5$.

[a]Hint: Restrict first to a chart computing the image of a point of $\mathrm{Gr}(2,4)$ corresponding to a matrix of the form

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

If you choose the natural homogeneous coordinates in $\mathbb{P}\left(\Lambda^2\mathbb{K}^4\right)$ you will find out that the image of such point is always contained in a specific chart, and fulfill an obvious (inhomogeneous) relation of degree 2 among its affine coordinates.

Exercise 2.3.6 Show that

$$\varphi(x_0 : x_1 : x_2) = \frac{1}{\sum_{j=0}^2 x_j^2}\left(x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2\right)$$

defines an embedding of $\mathbb{P}^2_{\mathbb{R}}$ in $\mathbb{R}^6$.

Exercise 2.3.7 1. Show that

$$\varphi(x_0 : x_1 : x_2) = \frac{1}{\sum_{j=0}^{2} x_j^2}\left(x_0^2 - x_1^2, x_0x_1, x_0x_2, x_1x_2\right) \tag{2.2}$$

defines an embedding of $\mathbb{P}^2_{\mathbb{R}}$ in $\mathbb{R}^4$.

Exercise 2.3.8 Since $\mathbb{P}^2_{\mathbb{C}}$ is compact, every holomorphic function on $\mathbb{P}^2_{\mathbb{C}}$ is constant and therefore $\mathbb{P}^2_{\mathbb{C}}$ cannot be holomorphically embedded in any affine space $\mathbb{C}^n$.

Take your solution of the previous two exercises, substitute $\mathbb{R}$ with $\mathbb{C}$ (and then smooth with holomorphic) and find out where exactly it becomes wrong.

Exercise 2.3.9 Construct a smooth function $F \in C^\infty(\mathbb{R})$ such that $\mathrm{Reg}(F)$ is exactly the complement of the image of F.

Exercise 2.3.10 Show that the map $F : \mathbb{R} \to \mathbb{R}^2$ defined by $F(t) = (\cos t, \sin t)$ is an immersion and is not an embedding.

Exercise 2.3.11 Consider the function $F : (0, 2\pi) \to \mathbb{R}^2$ defined by $F(t) = (\sin t, \sin 2t)$. Show that it is injective immersion but it is not an embedding.

Exercise 2.3.12 Show that $\{(x_1^2 + x_2^2 + x_3^2 + 3)^2 - 16(x_1^2 + x_2^2) = 0\}$ is a manifold without boundary embedded in $\mathbb{R}^3$. Can you recognize the underlying topological manifold?

2.4 The local diffeomorphism Theorem

We have noticed that the differential of a diffeomorphism is invertible.

This remark is inverted, in some sense, by the famous Inverse Function Theorem stating that, if the differential of a smooth (respectively holomorphic) function is invertible, then it is a **local diffeomorphism** (respectively **local biholomorphism**). In other words, we can *shrink* suitably domain and codomain to obtain a diffeomorphism (respectively biholomorphism). We give now a precise statement, considering only the real case for sake of simplicity. The complex version of the statement is completely analogous.

Theorem 2.4.1 — Inverse Function Theorem. Let U be an open subset of $\mathbb{R}^n$ and let $F : U \to \mathbb{R}^m$ be a smooth function.

Assume that $p \in U$ is a point such that dF_p is an isomorphism of vector spaces.

Then there exists an open subset W of U containing p such that the induced map

$$F_{|W} : W \to F(W)$$

is a diffeomorphism.

We will not prove the Inverse Function Theorem but we will prove two less famous corollaries showing that, if the differential of a smooth (resp. holomorphic) function has maximal rank, then *locally up to a coordinate change* the function is given by some coordinate functions.

Corollary 2.4.2 Let U be an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^m$ be a smooth function.

Assume that $p \in U$ is a point such that dF_p is a surjective.

Then there exists an open subset W of U containing p and a diffeomorphism $G\colon W \to W'$ such that

$$F \circ G^{-1}\colon W' \to \mathbb{R}^m$$

is the projection on the first coordinates

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$$

Proof. We denote by $K \subset T_p\mathbb{R}^n$ the kernel of dF_p and choose any projection $\pi\colon T_p\mathbb{R}^n \to K$. So π is a linear surjective map that is the identity on K.

Choose a basis $k_1, \ldots, k_{n-m}$ of K. We introduce a map $F_1\colon \mathbb{R}^n \to \mathbb{R}^{n-m}$ by setting

$$\pi\left(\sum_{i=1}^n a_i \left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_{j=1}^{n-m} b_j k_j \qquad \Leftrightarrow \qquad F_1(a_1, \ldots, a_n) = (b_1, \ldots, b_{n-m})$$

Then, using the natural identification $\mathbb{R}^m \oplus \mathbb{R}^{n-m} = \mathbb{R}^n$, we define the map

$$G := F \oplus (F_1)_{|U}\colon U \to \mathbb{R}^n$$

We can then write $G(x) = (F(x), F_1(x))$.

The matrix of dG_p with respect to the natural bases is, by Proposition 2.2.6, the Jacobi matrix of G at p.

The first m rows form the Jacobi matrix of F at p.

The remaining $n-m$ rows form the Jacobi matrix of F_1 at p. Since F_1 is linear, its Jacobi matrix (at any point) coincides with the matrix of F_1 with respect to the standard bases, which equals the matrix of π with respect to the bases $\left\{\left(\frac{\partial}{\partial x_i}\right)_p\right\}$ and v_j. Then

$$\ker dG_p = \ker dF_p \cap \ker \pi = K \cap \ker \pi = \{0\}.$$

So dG_p is injective and therefore, by a dimension count, an isomorphism of vector spaces.

Applying the Inverse Function Theorem 2.4.1 we find an open subset W containing p such that $G_{|W}$ is a diffeomorphism onto the image $W' := G(W)$. Since $F \circ (G_{|W})^{-1}$ is the projection on the first m coordinates by definition of G, we conclude the proof by renaming $G_{|W}$ as G. ■

Corollary 2.4.3 Let U be an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^m$ be a smooth function.

Assume that $p \in U$ is a point such that dF_p is a injective.

Then there exist an open subsets W of U containing p and a diffeomorphism $G\colon F(W) \to V$ such that

$$G \circ F\colon W \to V$$

is the injective map given by the first coordinates

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$$

Proof. We choose a basis $h_1, \dots, h_{m-n}$ of a subspace $H \subset T_{F(p)}\mathbb{R}^m$ supplementary to the image of dF_p and define a map $F_1 \colon \mathbb{R}^{m-n} \to \mathbb{R}^m$ by

$$\sum_{i=1}^{m} b_i \left(\frac{\partial}{\partial y_i}\right)_{F(p)} = \sum_{j=1}^{n-m} a_j h_j \qquad \Leftrightarrow \qquad F_1(a_1, \dots, a_{m-n}) = (b_1, \dots, b_m)$$

Then, considering the open subset $U' := U \oplus \mathbb{R}^{m-n}$ of $\mathbb{R}^n \oplus \mathbb{R}^{m-n} = \mathbb{R}^m$, we define the map $G' \colon U' \to \mathbb{R}^m$ by

$$G'(x, v) = F(x) + F_1(v)$$

We note that the first n columns of the Jacobi matrix of G' at p form the Jacobi matrix of F at p and the remaining columns form the Jacobi matrix of F_1 at p, so the image of dG'_p is the sum of the images of dF_p and $d(F_1)_p$. Since by definition the latter equals H, a supplementary of the first, then dG'_p is surjective and therefore by a dimension count an isomorphism. We conclude then by the Inverse Image Theorem 2.4.1 choosing as G the inverse of a suitable restriction of G_1. ■

By the Inverse Function Theorem 2.4.1 and its corollaries it easily follows the following results.

Theorem 2.4.4 — Local diffeomorphism theorem. Let M, N be manifolds, let $F \colon M \to N$ be a smooth (resp. holomorphic) function and fix a point $p \in M^\circ$.

Assume that dF_p is invertible.

Then there exists open subsets $U \subset M$, $V \subset N$ such that $p \in U$, $F(U) = V$, and $F_{|U} \colon U \to V$ is a diffeomorphism (resp. biholomorphism).

If dF_p is surjective, in local coordinates F is the projection on the first coordinates *up to change the chart in the domain.*

Proposition 2.4.5 Let M, N be manifolds, $F \colon M \to N$ a smooth (resp. holomorphic) function, $p \in M^\circ$. Assume that dF_p is surjective. Then for every chart (V, ψ) in $F(p)$ there exists a chart (U, φ) in p such that $\psi \circ F \circ \varphi^{-1}$ is the projection on the first coordinates:

$$\psi \circ F \circ \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_m).$$

If dF_p is injective, in local coordinates F is the inclusion of a coordinate subspace *up to change the chart in the codomain.*

Proposition 2.4.6 Let M, N be manifolds, $F \colon M \to N$ a smooth (resp. holomorphic) function, $p \in M^\circ$. Assume that dF_p is injective. Then there exists a chart (U, φ) in p and a chart (V, ψ) in $F(p)$ such that $\psi \circ F \circ \varphi^{-1}$ is the immersion of the first coordinates:

$$\psi \circ F \circ \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0).$$

A couple of not trivial consequences of Proposition 2.4.6 can be used to construct manifolds. The following holds only in the real case.

Theorem 2.4.7 — Regular Value Theorem 1. Let M be a real manifold with $\partial M = \emptyset$, and choose a function $f \in C^\infty(M)$. Let $y \in \mathbb{R}$ be a regular value of f and set $N := f^{-1}((-\infty, y])$. Then either N is empty or N has a differentiable structure such that the inclusion $N \hookrightarrow M$ is an embedding, $\dim N = \dim M$ and $\partial N = f^{-1}(y)$.

Note that N may have several connected components.

Example 2.8 The function $f = \sum x_i^2 \in C^\infty(\mathbb{R}^n)$ has only one critical point, the origin, so it has only one critical value, zero. Theorem 2.4.7 induces then a differentiable structure on each closed ball of positive radius.

Choosing $y = 1$ we obtain then differential structures on B^n and S^{n-1} such that the respective inclusion maps in $\mathbb{R}^n$ are embeddings.

Note that, since composition of embeddings is an embedding, by Example 2.7, in the situation of Theorem 2.4.7, $f^{-1}(y)$ is embedded in M.

The complex version of Theorem 2.4.7 is

Theorem 2.4.8 — Regular Value Theorem 2. Let M be a complex manifold, and choose a function $f \in \mathscr{O}(M)$. Let $y \in \mathbb{C}$ be a regular value of f. Then $f^{-1}(y)$ has a complex structure such that the inclusion in M is an embedding and $\dim f^{-1}(y) = \dim M - 1$.

If we construct a manifold as preimage of a regular value, we can represent its the tangent spaces as hyperplanes of the corresponding tangent spaces of M. This is what we do when we *draw* the tangent line of a plane curve, for example. The next proposition show that this hyperplane is the kernel[2] of the differential of the function.

Proposition 2.4.9 Let M be a manifold. In the real case we assume $\partial M = \emptyset$. Let $f \in C^\infty(M)$ (in the complex case: $\mathscr{O}(M)$), $y \in \mathrm{Reg}(f)$. Set $X := f^{-1}(y)$ with the differentiable structure induced by Theorem 2.4.7 (in the complex case: 2.4.8), $i\colon X \hookrightarrow M$ the corresponding embedding and choose a point $p \in X$. Then di_p is injective and $di_p(T_pX) = \ker df_p$.

Proof. The function $f \circ i \in C^\infty(X)$ is the constant function, assuming in each point the same value y. Therefore $df \circ di = d(f \circ i) = 0$, so the image of di is contained in the kernel of df: $di_p(T_pX) \subset \ker df_p$.

Since i is an embedding, di_p is injective. Since $\dim X = \dim M - 1$, $di_p(T_pX)$ has codimension 1. On the other hand, since y is a regular value, p is not a critical point, and therefore df_p has maximal rank 1, so $\ker df_p$ has codimension 1 too. Since the first space is contained in the second one, they coincide. ■

We will usually write $T_pX \subset T_pM$, identifying each vector of T_pX with its image in T_pM. This gives an embedding $TX \hookrightarrow TM$.

We can then construct vector fields on X if we know how to construct vector fields on M. Take a vector field $v\colon M \to TM$ with the property that $\forall p \in X$, $v_p \in T_pX$. Then the image of $v_{|X}$ is contained in TX, so $v_{|X}(X) \subset TX$. It is not difficult to show using Proposition 2.4.6 that if $v \in \mathfrak{X}(M)$ then $v_{|X} \in \mathfrak{X}(X)$ (if v is smooth, its restriction to X is smooth too).

If $M = \mathbb{R}^n$ we can then see the tangent space of X as the orthogonal of the gradient of f. Using the function in example 2.8, we see that for each point $p = (p_1, \dots, p_n) \in S^{n-1}$,

$$T_p(S^{n-1}) = \left\{ \sum v_i \left(\frac{\partial}{\partial u_i} \right)_p | \sum p_i v_i = 0 \right\}$$

There is a different version of the regular value theorem, which applies to manifolds with boundary.

[2] Some authors prefer to describe it as the orthogonal to the gradient, but in these notes (since we are doing differential topology and not Riemannian geometry) we do not find it convenient to insert a scalar product since it is not strictly necessary.

Theorem 2.4.10 — Regular Value Theorem 3. Let M,N be real manifolds, $\dim N < \dim M$, $F\colon M \to N$ a smooth function, $y \in \mathrm{Reg}(F) \cap \mathrm{Reg}(F_{|\partial M})$. Then $F^{-1}(y)$ has a differentiable structure such that the inclusion $F^{-1}(y) \hookrightarrow M$ is an embedding, and $\partial F^{-1}(y) = \partial M \cap F^{-1}(y)$.

It is not difficult to show, exactly as in the other case, that the differential of the inclusion identify T_pX with $\ker dF_p$. In particular $\dim X = \dim M - \dim N$.

The regular value theorems can be also used to prove the existence of a structure of embedded submanifold of loci that may be only *locally* represented as preimage of a regular value for a smooth/holomorphic function, as in Exercise 2.4.2.

Definition 2.4.11 — (Smooth) Cartier divisors. Let M be a real (resp. complex) manifold without boundary.

A Cartier divisor in M is a closed subset X such that for all $x \in X$ there exists an open subset $U \subset M$ containing x and a smooth (resp. holomorphic) function f_U on U such that $X \cap U$ equals $f_U^{-1}(0)$.

If 0 is a regular value for all f_U then X is a closed submanifold of codimension 1. In this case we say that X is a smooth Cartier divisor.

Exercise 2.4.1 Prove that every closed submanifold of dimension 1 is a smooth Cartier divisor.

Exercise 2.4.2 — The Fermat hypersurface. Show that for all $d,n \in \mathbb{N}$ the locus

$$\left\{ (x_0 : x_1 : \ldots , x_n) \in \mathbb{P}^n_{\mathbb{C}} | \sum_{j=0}^{n} x_j^d = 0 \right\}$$

is a smooth[a] Cartier divisor in $\mathbb{P}^n_{\mathbb{C}}$.

[a] Warning: the expression $\sum_{j=0}^n x_j^d$ does NOT define a function from $\mathbb{P}^n_{\mathbb{C}}$ to $\mathbb{C}$.

Exercise 2.4.3 For $\mathbb{K} = \mathbb{R},\mathbb{C}$ consider the locus $Bl_0(\mathbb{K}^2) \subset \mathbb{K}^2 \times \mathbb{P}^1_{\mathbb{K}}$ defined by the equation $x_0y_1 = x_1y_0$ where (x_0,x_1) are the standard coordinates of $\mathbb{K}^2$ and $(y_0 : y_1)$ are the induced homogeneous coordinates in $\mathbb{P}^1_{\mathbb{K}}$.

Prove that $Bl_0(\mathbb{K}^2)$ is a smooth Cartier divisor in $\mathbb{K}^2 \times \mathbb{P}^1_{\mathbb{K}}$.

Prove that $Bl_0(\mathbb{K}^2)$ is homeomorphic to the topological manifold of Exercise 1.2.4 or 1.2.5, depending on the choice of $\mathbb{K}$.

3. The tangent bundle(s)

Now we introduce the vector fields.

Roughly speaking, a vector field on a manifold M is the datum, for every point $p \in M$, of a tangent vector $v_p \in T_pM$. A natural way to do it (locally) is by choosing a chart (U, φ), denoting by $x_1, \ldots, x_n$ the induced local coordinates and finally by writing something of the form $\sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ for some functions $f_i \colon U \to \mathbb{R}$: this associates to each point $p \in U$ the vector $\sum_{i=1}^n f_i(p) \left(\frac{\partial}{\partial x_i}\right)_p$. We would like to say that the vector field is smooth at p if all f_i are smooth. Is that independent from the choice of the chart?

If we have two charts containing the same point $p \in M$, they induce two different bases of T_pM. We need to understand the relation between them. It can be computed applying Proposition 2.2.7.

Corollary 3.0.1 Let M be a manifold, $p \in M$, and let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two charts with $p \in U_\alpha \cap U_\beta$. We denote with $(x_{1\alpha}, \ldots, x_{n\alpha})$ and $(x_{1\beta}, \ldots, x_{n\beta})$ the respective local coordinates.

Consider a vector $v \in T_pM$, and let $v_{i\alpha}$, resp. $v_{j\beta}$ be the coordinates of v in the basis $\left\{\left(\frac{\partial}{\partial x_{i\alpha}}\right)_p\right\}$, resp. $\left\{\left(\frac{\partial}{\partial x_{j\beta}}\right)_p\right\}$, so that

$$v = \sum_{i=1}^n v_{i\alpha} \left(\frac{\partial}{\partial x_{i\alpha}}\right)_p = \sum_{i=1}^n v_{j\beta} \left(\frac{\partial}{\partial x_{j\beta}}\right)_p.$$

Then

$$\begin{pmatrix} v_{1\beta} \\ \vdots \\ v_{n\beta} \end{pmatrix} = J(\varphi_{\beta\alpha})_{\varphi_\alpha(p)} \begin{pmatrix} v_{1\alpha} \\ \vdots \\ v_{n\alpha} \end{pmatrix} = \left(\left(\frac{\partial}{\partial x_{i\alpha}}\right)_p x_{j\beta} \right) \begin{pmatrix} v_{1\alpha} \\ \vdots \\ v_{n\alpha} \end{pmatrix}$$

where $J(\varphi_{\beta\alpha})_{\varphi_\alpha(p)}$ denotes the Jacobi matrix of the application $\varphi_{\beta\alpha}$ at the point $\varphi_\alpha(p)$.

Proof. Obviously

$$\begin{pmatrix} v_{1\beta} \\ \vdots \\ v_{n\beta} \end{pmatrix} = M \begin{pmatrix} v_{1\alpha} \\ \vdots \\ v_{n\alpha} \end{pmatrix}$$

for the matrix M representing the identity map of T_pM in the bases $\left\{ \left(\frac{\partial}{\partial x_{i\alpha}} \right)_p \right\}$ in the domain and $\left\{ \left(\frac{\partial}{\partial x_{j\beta}} \right)_p \right\}$ in the codomain. Since $Id_{T_pM} = d(Id_M)_p$ for the identity map $Id_M \colon M \to M$, we can compute M by Proposition 2.2.7, obtaining the Jacobi matrix of the map $\varphi_\beta \circ Id_M \circ \varphi_\alpha^{-1} = \varphi_{\beta\alpha}$ in $\varphi_\alpha(p)$. ■

We could now define the smoothness of our "roughly defined" vector fields using their expression in local coordinates, using Corollary 3.0.1 to prove that our definition does not depend on the choice of coordinates.

We will instead follow a longer way, putting them in the more general contest of vector bundles.

3.1 Fibre bundles

Definition 3.1.1 Let F, B be topological spaces.

A **fibre bundle** over a **base** B with **fibre** F is a pair (E, π) where E is a topological space, the **total space**, and $\pi \colon E \to B$ is a continous map, the **projection**, such that there exists an open cover $\{U_\alpha\}_{\alpha \in I}$, and homeomorphisms $\phi_\alpha \colon E_{|U_\alpha} := \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that the diagrams

$$\begin{tikzcd} E_{|U_\alpha} \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd} \tag{3.1}$$

commute, where $\pi_1 \colon U_\alpha \times F \to U_\alpha$ is the projection on the first factor.

In other words we ask $\pi = \pi_1 \circ \phi_\alpha$.

The set $\{\phi_\alpha \colon E_{|U_\alpha} \to U_\alpha \times F\}_{\alpha \in I}$ is a **trivialization** of the bundle.

We denote, $\forall p \in B$, by E_p or F_p the "fibre over p": $\pi^{-1}(p)$.

By (3.1) all E_p are homeomorphic to F.

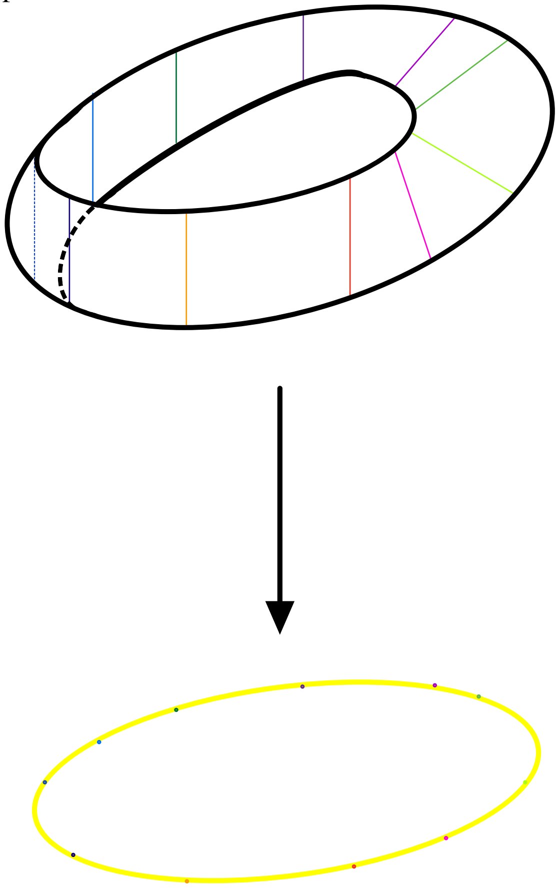
Example 3.1 The simplest example is the product of topological spaces, with the projection on one factor: $E := B \times F$, $\pi = \pi_1 \colon B \times F \to B$ the map $\pi(b, f) = b$.

Example 3.2 — The "closed" Moebius Band. Consider the closed square $Q := [0,1]^2$ and identify the two vertical edges by the equivalence relation $(0,t) \sim (1, 1-t))$. The closed (Moebius) band is the quotient topological space $M := Q/\sim$. Note that it has a natural differentiable structure.

The map $(x,y) \mapsto x$ induce a map $M \to [0,1]/(0 \sim 1) \cong S^1$. This is a fibre bundle with fiber $[0,1]$.

An interesting case is the case when F is a discrete set, leading to the theory of the *topological coverings*. For example

Figure 3.1: A representation of the closed Moebius band as fiber bundle over S^1



Example 3.3 Fix $d \in \mathbb{N}$ and take $E = B = S^1 := \{(\cos\theta, \sin\theta)\} \subset \mathbb{R}^2$ and $\pi\colon E \to B$ defined by

$$\pi(\cos\theta, \sin\theta) = (\cos d\theta, \sin d\theta).$$

This is a fibre bundle with fibre a discrete set of cardinality d.

Similarly the natural maps $S^n \to \mathbb{P}^n_{\mathbb{R}}$ are fibre bundles with fibre a discrete set of cardinality 2. The following interesting example is a complex analogous of that.

Example 3.4 — The Hopf fibration. Consider the sphere S^3 as the subset of $\mathbb{C}^2$ of the pairs (x_0, x_1) of complex numbers such that $||x_0||^2 + ||x_1||^2 = 1$. The map $\pi\colon S^3 \to \mathbb{P}^1_{\mathbb{C}}$ defined by

$$\pi(x_0, x_1) = (x_0 : x_1)$$

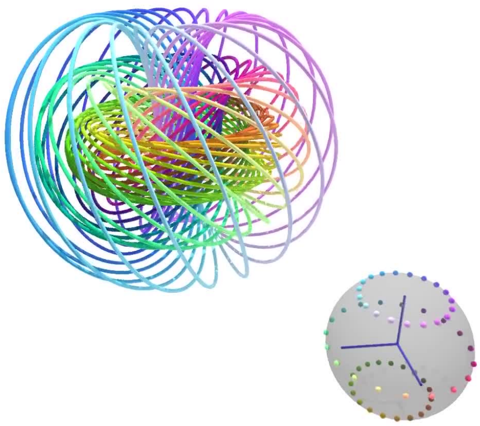
is a fibre bundle with fibre S^1.

Indeed, recall that the sets $U_j = \{x_j \neq 0\}$ define an open cover $\{U_0, U_1\}$ of $\mathbb{P}^1_{\mathbb{C}}$. Then, identifying S^1 with the subset of $\mathbb{C}$ of the complex numbers of norm 1 a trivialization is given by the maps $\Phi_j\colon \pi^{-1}(U_j) \to U_j \times S^1$ defined by

$$\Phi_j(x_0, x_1) = \left((x_0 : x_1), \frac{x_j}{||x_j||}\right).$$

The stereographic projection of center p mapping $S^3 \setminus \{p\}$ to $\mathbb{R}^3$ sends every fibre of the

Figure 3.2: A representation of the Hopf fibration through the stereographic projection of S^3 to $\mathbb{R}^3$



Hopf fibration not containing p to a circle in the 3-space. This gives a graphical representation of the Hopf fibration as in figure 3.2.

Roughly speaking, a fibre bundle is *locally a product* as well as manifolds are *locally affine spaces*. So, as for the theory of the manifolds, also the theory of the fibre bundles have its transition functions as follows.

Definition 3.1.2 The **transition functions** of the fibre bundle are[a] the maps, $\forall \alpha, \beta \in I$,

$$\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F.$$

They are, by (3.1), of the form $\phi_{\alpha\beta}(p, f) = (p, g_{\alpha\beta}(p)(f))$ for some maps $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Aut}(F)$ where $\mathrm{Aut}(F)$ is the group of self-homeomorphisms of F. $\{g_{\alpha\beta}\}$ is a **cocycle** of the bundle, and verifies the three **cocycle conditions**:

i) $\forall \alpha \in I, \forall p \in U_\alpha$, $g_{\alpha\alpha}(p) = \mathrm{Id}_F$;
ii) $\forall \alpha, \beta \in I, \forall p \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(p) = g_{\beta\alpha}(p)^{-1}$;
iii) $\forall \alpha, \beta, \gamma \in I, \forall p \in U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$.

[a] Note the similarities with the transition functions of a differentiable structure

In the following we will often identify the bundle with its total space or its projection, speaking about "a bundle E" or "a bundle $\pi \colon E \to B$".

As in every category, once determined the objects, we have to choose which maps among them we want to consider.

Definition 3.1.3 Consider two fibre bundles $\pi \colon E \to B$, $\pi' \colon E' \to B'$.

Let $g \colon B \to B'$ be a continous map. A **morphism of bundles covering *g*** or **morphism of bundles over *g*** is a continous map $f \colon E \to E'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

commutes. In other words, such that $g \circ \pi = \pi' \circ f$.

If $B = B'$, *i.e.* if the two bundles have the same base, a **morphism of bundles over *B*** $f\colon E \to E'$ is a morphism of bundles covering the identity Id_B of B. Then a morphism of bundles over B may be seen as a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd} .$$

An **isomorphism of bundles** is a morphism $f\colon E \to E'$ of bundles over B that is also a homeomorphism. If an isomorphism of bundles $f\colon E \to E'$ exists we say that E is **isomorphic** to E'. A bundle is **trivial** if it is isomorphic to the bundle $\pi_1\colon B \times F \to B$.

A first remark is that the cocycle determines the bundle up to isomorphisms; this follows essentially by the same argument used in subsection 1.2.1 to show the analogous property of the transition functions of a differentiable structure.

There are few more definitions we need.

Definition 3.1.4 Let $\pi\colon E \to B$ be a fibre bundle. A **section** of E is a continous map $s\colon B \to E$ such that $\pi \circ s = \mathrm{Id}_B$.

Definition 3.1.5 Let G be a subgroup of $\mathrm{Aut}(F)$. A ***G*−bundle** is a fibre bundle with fibre F provided with a trivialization whose cocycle is contained in G:

$$\forall \alpha, \beta \in I, \forall p \in U_\alpha \cap U_\beta, g_{\alpha\beta}(p) \in G.$$

If E, B and F are all real (resp. complex) manifolds we will, unless differently specified, consider all the above definitions moved to the corresponding category. So all continous map will be implicitly supposed smooth (resp. holomorphic).

We conclude this section by an important construction, the base change, also known as fibre product.

Definition 3.1.6 Consider two functions with the same codomain $f\colon A \to C$, $g\colon B \to C$.

The **fibre product** of f and g, usually denoted by $A \times_C B$ is the subset of the product $A \times B$ of the elements that "agree on C" in the following sense:

$$A \times_C B := \{(a,b) \in A \times B | f(a) = g(b)\}.$$

We denote by g', f' the restrictions to $A \times_C B$ of the two natural projections $A \times B$. This gives a diagram

$$\begin{tikzcd} A \times_C B \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

that is commutative by definition of $A \times_C B$.

Note that, for all $p \in A$, $(g')^{-1}(p) = \{(p,b) | g(b) = f(p)\} = \{p\} \times g^{-1}(f(p))$. In this sense we can say that g' and g (and similarly f' and f) *have the same fibres*.

Indeed, it is not difficult to show that if B is a $G-$bundle over C with fibre F and projection g, then also $A \times_C B$ is a $G-$bundle over A with fibre F and projection g'.

Definition 3.1.7 The **pull-back bundle** of a bundle $g\colon B \to C$ by a continous map $f\colon A \to C$ is the bundle $g'\colon f^{-1}B := A \times_C B \to A$. This is a **base change** in the sense that the pull-back bundle is a bundle with the same fibre but different base.

If f is the inclusion of a subset $A \subset C$, then $A \times_C B$ is naturally homeomorphic to $g^{-1}(A)$. Therefore in this case $f^{-1}B$ is called **restriction** of B to A, and denoted by $B_{|A}$.

Complement 3.1.1 Prove that the cocycle determines the bundle up to isomorphism. In other words, reconstruct E and π from $(B, F, \{U_\alpha\}, \{g_{\alpha\beta}\})$.

Complement 3.1.2 Let E be a $G-$bundle and E' be a G'-bundle on the same base B (the fibres may be different). Show that they admit two trivializations $\{\phi_\alpha\}$ of E and $\{\phi'_\alpha\}$ of E' which share the same open cover $\{U_\alpha\}_{\alpha \in I}$ of B.

Exercise 3.1.1 Write a trivialization of the fibre bundle given by the natural map of the Moebius band onto S^1 and the corresponding cocycle.

Show that the smallest of order of a subgroup G of $\mathrm{Aut}(S^1)$ for which this is a$G-$bundle is respectively 2.

Exercise 3.1.2 Let $S^1 \subset \mathbb{C}^2$ be the subset of the complex numbers z having norm $||z|| = 1$. Note that for each $\lambda \in S^1$ the map $z \mapsto \lambda z$ is an automorphism, so defining an injective map $S^1 \hookrightarrow \mathrm{Aut}(S^1)$. Set $H \subset \mathrm{Aut}(S^1)$ for its image. Note that H is a subgroup of $\mathrm{Aut}(S^1)$.

Show that the Hopf fibration is a H-bundle.

Exercise 3.1.3 Show that, if (E, π) is a trivial bundle, then $f^{-1}E$ is trivial for every continous function f.

Exercise 3.1.4 Consider the Moebius band M as bundle on S^1 as in the previous exercise. Let $f\colon S^1 \to S^1$ be defined by $f(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$. Show that $f^{-1}M$ is trivial as fibre bundle over S^1.

Exercise 3.1.5 Consider, for each fixed $d \in \mathbb{N}$, the map $S^1 \to S^1$ in Example 3.3.

Determine for which $d \in \mathbb{N}$ the pull-back of the Moebius band by it is a trivial bundle.

3.2 Vector bundles

Definition 3.2.1 A **real** (resp. **complex) vector bundle over B of rank r** is a $G-$bundle with fibre $\mathbb{R}^r$ (resp. $\mathbb{C}^r$) where G is the group of the invertible linear applications $\mathrm{GL}(\mathbb{R}^r)$ (resp. $\mathrm{GL}(\mathbb{C}^r)$). A **line bundle** is a vector bundle of rank 1.

Again, the simplest example of a vector bundle is the product $B \times V$ where V is a vector space.

The following is more interesting

Example 3.5 — The tautological bundle over $\mathbb{P}^n$. Let E be the subset of $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1}$ union of $\mathbb{P}^n_{\mathbb{K}} \times \{0\}$ with the points of the form

$$((x_0 : \cdots : x_n), (x_0, \ldots, x_n)).$$

The restriction of the first projection to E is a map $\pi \colon E \to \mathbb{P}^n_{\mathbb{K}}$ such that the pair (E, π) is a (real or complex according to the choice of $\mathbb{K}$) line bundle.

Note that the fibre $\pi^{-1}(x_0 : \cdots : x_n)$ is the line generated by $(x_0, \ldots, x_n)$.

R Similarly there is a tautological rank k vector bundle over each Grassmannian $\mathrm{Gr}(k,n)$ (see Example 1.10).

For all α, $\forall p \in U_\alpha$, ϕ_α induces a bijection $\varphi_{\alpha,p} \colon F_p \to \mathbb{K}^r$ via $\phi_\alpha(v) = (p, \varphi_{\alpha,p}(v))$. This gives a structure of vector space on F_p via $\forall v, w \in F_p$, $\forall c \in \mathbb{K}$, $v + w := \varphi^{-1}_{\alpha,p}(\varphi_{\alpha,p}(v) + \varphi_{\alpha,p}(w))$, $cv := \varphi^{-1}_{\alpha,p}(c\varphi_{\alpha,p}(v))$.

The given vector space structure on F_p does not depend on the choice of α. Indeed, if $p \in U_\alpha \cap U_\beta$, since $g_{\alpha\beta}(p) = \varphi_{\alpha,p} \circ \varphi^{-1}_{\beta,p} \in \mathrm{GL}(\mathbb{K}^r)$, $c\varphi_{\alpha,p}(v) = g_{\alpha\beta}(p)\left(c\varphi_{\beta,p}(v)\right)$ and then $\varphi^{-1}_{\alpha,p}(c\varphi_{\alpha,p}(v)) = \varphi^{-1}_{\beta,p}\left(c\varphi_{\beta,p}(v)\right)$. Similarly one shows $\forall v, w \in F_p$, $\varphi^{-1}_{\alpha,p}(\varphi_{\alpha,p}(v) + \varphi_{\alpha,p}(w)) = \varphi^{-1}_{\beta,p}\left(\varphi_{\beta,p}(v) + \varphi_{\beta,p}(w)\right)$.

So we can see a vector bundle as a way to attach to each point of B a vector space of fixed dimension r "in a continous way".

In particular we can consider the neutral element of the sum, 0_p, on each E_p. This defines a smooth section $s_0 \colon B \to E$, **the zero section** , by $s_0(p) = 0_p$.

The group $\mathrm{GL}(\mathbb{R}^r)$ (resp. $\mathrm{GL}(\mathbb{C}^r)$) of the invertible operators on $\mathbb{R}^r$ (resp. $\mathbb{C}^r$) is naturally identified with the set of the square matrices $\mathrm{GL}(r, \mathbb{R})$ with real (resp. complex) coefficients of order r whose determinant differs from zero. This gives a differentiable (resp. complex) structure on $\mathrm{GL}(\mathbb{R}^r)$ (resp. $\mathrm{GL}(\mathbb{C}^r)$) as open subset of $\mathbb{R}^{r^2}$ (resp. $\mathbb{C}^{r^2}$). So if B is a real (resp. complex) manifold one may inquire whether the maps $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^r)$(resp. $\mathrm{GL}(\mathbb{C}^r)$) are smooth (resp. holomorphic); in other words that the vector space varies "in a smooth (resp. holomorphic) way".

We have seen in Complement 3.1.1 that every fibre bundle is determined by its cocycle. The same idea gives the following.

Proposition 3.2.2 Let B be a manifold, let $\mathfrak{U} := \{U_\alpha\}_{\alpha \in I}$ be an open cover of B, $r \in \mathbb{N}$. Assume we have $\forall \alpha, \beta \in I$, a smooth (resp. holomorphic) map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^r)$ (resp. $\mathrm{GL}(\mathbb{C}^r)$) such that

i) $\forall \alpha \in I$, $\forall p \in U_\alpha$, $g_{\alpha\alpha}(p) = Id$;

ii) $\forall \alpha, \beta \in I$, $\forall p \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(p) = g_{\beta\alpha}(p)^{-1}$;

iii) $\forall \alpha, \beta, \gamma \in I$, $\forall p \in U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\alpha\beta}(p) g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$.

Then there is a unique, up to isomorphisms, real (resp. complex) vector bundle E of rank r over B having a trivialization with cocycle $\{g_{\alpha\beta}\}$. Moreover E has a natural structure of (complex) manifold such that the projection $\pi \colon E \to B$ and the zero section $s_0 \colon B \to E$ are smooth (resp. holomorphic).

Moreover $\dim E = \dim B + \mathrm{rk}\, E = \dim B + r$, the differential of π is surjective at every point and the differential of s_0 is injective at every point.

Proof. The total space E is defined, as topological space, as the quotient of the disjoint union X of all products $U_\alpha \times \mathbb{R}^r$ by the equivalence relation naturally induced by the $g_{\alpha\beta}$: a point $(p_\alpha, v_\alpha) \in U_\alpha \times \mathbb{R}^r$ and a point $(p_\beta, v_\beta) \in U_\beta \times \mathbb{R}^r$ are equivalent if and only if $p_\alpha = p_\beta$ (in B) and $g_{\alpha\beta}(p_\alpha)(v_\beta) = v_\alpha$.

The map $\pi\colon E \to B$ associating to the class of $(p_\alpha, v_\alpha) \in U_\alpha \times \mathbb{R}^r$ its first component p_α is well defined and give a fibre bundle structure on E.

For fixed α, consider an open subset $\Omega \subset U_\alpha \times \mathbb{R}^r$. For every β, consider the subset Ω_β of $U_\beta \times \mathbb{R}^r$ of the pairs (p_β, v_β) that are equivalent to some (p_α, v_α) in Ω. Then the function $(p,v) \mapsto (p, g_{\alpha\beta}(p)(v))$ maps Ω_β homeomorphically onto the intersection of Ω and $\left(U_\alpha \cap U_\beta\right) \times \mathbb{R}^r$ in $U_\alpha \times \mathbb{R}^r$. Then, considering the quotient map $\bar\pi\colon X \to E$, $\bar\pi^{-1}\left(\bar\pi(\Omega)\right)$ is open in X. This implies that $\bar\pi$ is an open map.

Since B is a manifold, its topology has a countable basis of open subsets $\{U'_n\}_{n\in\mathbb{N}}$. Without loss of generality, we can assume[1] that the covering $\{U'_n\}_{n\in\mathbb{N}}$ is a refinement of $\{U_\alpha\}_{\alpha\in I}$. In other words, for all $n \in \mathbb{N}$ $\exists \alpha(n)$ such that $U'_n \subset U_{\alpha(n)}$. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis of $\mathbb{R}^r$. Then for all pairs of natural numbers i, j, $U'_i \times V_j$ is an open subset $\Omega_{i,j}$ of $U_{\alpha(i)} \times \mathbb{R}^r$. Since $\bar\pi$ is an open map, all subsets $U_{i,j} := \bar\pi\left(\Omega_{i,j}\right)$ are open in E. It is easy to prove that $U_{i,j}$ is a basis for the topology of E. So E admits a countable basis of open subsets.

Now consider the equivalence relation as a subset R of $X \times X$:

$$R := \left\{((p_\alpha, v_\alpha),(p_\beta, v_\beta)) \in X \times X | (p_\alpha, v_\alpha) \sim (p_\beta, v_\beta)\right\}$$

We show that R is closed by proving that its complement is open. In fact if (p_α, v_α) and (p_β, v_β) are not equivalent,

- either $p_\alpha \neq p_\beta$ in B
- or $p_\alpha = p_\beta$ in B and $g_{\alpha\beta}(p_\alpha)(v_\beta) \neq v_\alpha$.

In the first case we can find two disjoint open subsets U'_α of U_α and U'_β of U_β such that $p_\alpha \in U_\alpha$ and $p_\beta \in U_\beta$. Then $(U'_\alpha \times \mathbb{R}^r) \times \left(U'_\beta \times \mathbb{R}^r\right)$ is an open subset of $(U_\alpha \times \mathbb{R}^r) \times \left(U_\beta \times \mathbb{R}^r\right)$ containing $((p_\alpha, v_\alpha),(p_\beta, v_\beta))$ disjoint from R. In the second case we obtain the same result by considering two disjoint open subsets V_α and V_β of $\mathbb{R}^r$ containing respectively v_α and $g_{\alpha\beta}(p_\alpha)(v_\beta)$ and considering the open subset $\left((U_\alpha \cap U_\beta) \times V_\alpha\right) \times \Omega$ of $(U_\alpha \times \mathbb{R}^r) \times \left(U_\beta \times \mathbb{R}^r\right)$ where Ω is the image of $(U_\alpha \cap U_\beta) \times V_\beta$ via the obvious map.

So E is the quotient of a Hausdorff space via a closed equivalence relation such that the quotient map is open. This implies that E is Hausdorff.

Set b for the dimension of B. We can assume, up to substitute $\mathfrak{U}$ by a refinement of it, that all open sets U_α come from charts. In other words, we can assume that for each α there is a homeomorphism $\varphi_\alpha\colon U_\alpha \to D_\alpha$ where D_α is an open subset of $\mathbb{R}^b_\pm$. Denoting by V_α the open subset $\bar\pi(U_\alpha \times \mathbb{R}^r)$, we obtain homeomorphisms $\psi_\alpha\colon V_\alpha \to \mathbb{R}^r \times D_\alpha \subset \mathbb{R}^{r+b}_\pm$.

Then $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in I}$ gives a differentiable structure on E. Note that we need the assumption of smoothness of the $g_{\alpha\beta}$ to ensure the smoothness of the transition functions $\psi_\alpha \circ \psi_\beta^{-1}$.

The projection π maps V_α onto U_α. In the local coordinates given by the charts (V_α, ψ_α) and $(U_\alpha, \varphi_\alpha)$, $\pi(x_1, \dots, x_{r+b}) = (x_{r+1}, \dots, x_{r+b})$ and $s_0(x_1, \dots, x_b) = (0, \dots, 0, x_1, \dots, x_b)$: it follows that both maps are smooth, the differential of π is surjective at every point and the differential of s_0 is injective at every point. ■

This is the situation we are interested in. So, for sake of simplicity, from now on we are implicitly assuming that B is a real (resp. complex) manifold, all maps $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^r)$ (resp. $\mathrm{GL}(\mathbb{C}^r)$) are smooth (resp. holomorphic) and E has the differentiable (resp. complex) structure in Proposition 3.2.2. Moreover all morphisms of bundles and sections are implicitely assumed to be smooth (resp. holomorphic).

[1] Replacing $\{U'_n\}$ with the set of all the intersections $\{U'_n \cap U_\alpha\}$ we obtain a new basis of the same topology that is a refinement of $\{U_\alpha\}$ from which we can extract a countable subfamily that is also a basis for the given topology, see John L. Kelley, *General topology*, Exercise F at page 57.

Definition 3.2.3 Let E, E' two vector bundles on respective bases B and B'.

A **morphism of vector bundles** (over g resp. over B) is a morphism of fibre bundles $f\colon E \to E'$ (over g resp. over B) such that $\forall p \in B$, the map $f_{|E_p}\colon E_p \to E'_{g(p)}$ is a linear application.

An **isomorphism of vector bundles** is a morphism of vector bundles that is an isomorphism of fibre bundles. Two vector bundles are **isomorphic as vector bundles** if there is an isomorphism of vector bundles among them.

A **trivial** vector bundle is a vector bundle isomorphic as vector bundle to the vector bundle $\pi_1 : B \times \mathbb{K}^r \to B$.

The concepts of vector subspace and quotient vector space naturally extend to vector bundles:

Definition 3.2.4 Let $\pi'\colon E' \to B$ be a vector bundle. A **subbundle** E of E' is the datum of a vector bundle $\pi\colon E \to B$ with an injective morphism of vector bundles $f\colon E \to E'$ over B.

Let $\pi\colon E \to B$ be a subbundle of a vector bundle $\pi'\colon E' \to B$. The **quotient bundle** $\overline{\pi}\colon E'/E \to B$ is the vector bundle whose total space is the quotient of E' by the equivalence relation

$$v_1 \sim v_2 \Leftrightarrow \pi'(v_1) = \pi'(v_2) =: p \text{ and } v_1 - v_2 \in E_p$$

Here the difference $v_1 - v_2$ is the difference in the vector space E'_p.

Setting $\tilde{\pi}\colon E' \to E'/E$ for the projection on the quotient, then $\overline{\pi}$ is defined by $\pi' = \overline{\pi} \circ \tilde{\pi}$.

So, if E is a subbundle of E', for all $p \in B$ it is naturally to identify E_p with the subspace $f(E_p)$ of E'_p and $(E'/E)_p$ with the quotient of vector space $E'_p/E_p = E'_p/f(E_p)$.

To a Cartier divisor one associates naturally a line bundle

Definition 3.2.5 — Line bundle associated to a smooth Cartier divisor. Let X be a Cartier divisor in M.

By definition, there is a family of open subsets $\{U_j\}$ of M whose union contains X and smooth/holomorphic functions f_j on U_j such that $X \cap U_j$ is $f_j^{-1}(0)$ and 0 is a regular value. By Proposition 2.4.5 we may then assume that U_j is a chart with local coordinates

$$z_1^j, \ldots, z_{n-1}^j, z_n^j$$

and $f_j = z_n^j$.

Without loss of generality we may assume that $\{U_j\}$ is an open cover of M by adding to the family the open subset $M \setminus X$ with the constant function 1.

Now consider the ratio $\frac{f_i}{f_j}$. This defines a smooth/holomorphic function on $(U_i \cap U_j) \setminus X$ that never vanishes. Choose any point in $X \cap U_i \cap U_j$. Writing the Taylor expansion of f_i with respect to the variables $z_1^j, \ldots, z_{n-1}^j, z_n^j$ we notice that, since f_i vanish along $z_n^j = 0$, z_n^j divides f_i. So the function f_i/f_j extends to a smooth/holomorphic function on $U_i \cap U_j$. Reversing the role of i and j we deduce similarly that f_i/f_j never vanishes on $U_i \cap U_j$.

We have than a family of maps

$$g_{ij} := \frac{f_i}{f_j} : U_i \cap U_j \to \mathrm{GL}_1(\mathbb{K}).$$

that obviously respects the cocycle conditions, so they define a (real respectively complex) line bundle on M.

Example 3.6 — The hyperplane bundle. The hyperplane bundle of $\mathbb{P}^n_{\mathbb{K}}$ is the line bundle associated to the smooth Cartier divisor H_0.

Complement 3.2.1 Write the missing details of the proof of Proposition 3.2.2.

Exercise 3.2.1 Show that the tautological bundle over $\mathbb{P}^n_{\mathbb{K}}$ is a line bundle by writing a suitable trivialization and the corresponding cocycle.

Exercise 3.2.2 Show that a line bundle E over B is trivial if and only if it has a section $s\colon B \to E$ which never vanishes: in other words $\forall p \in B$, $s(p) \neq 0 \in E_p$.

Exercise 3.2.3 Show that every section of the tautological bundle over $\mathbb{P}^1_{\mathbb{R}}$ vanishes at least at a point. In other words $\exists p \in \mathbb{P}^1_{\mathbb{R}}$ such that $s(p) = 0 \in E_p$. In particular, the tautological bundle over $\mathbb{P}^1_{\mathbb{R}}$ is not trivial.

Exercise 3.2.4 Show that the tautological bundle over $\mathbb{P}^1_{\mathbb{R}}$ is homeomorphic to a Moebius band.

Exercise 3.2.5 Show that the zero section is the only holomorphic section of the tautological bundle over $\mathbb{P}^n_{\mathbb{C}}$.

Exercise 3.2.6 — Frames and triviality. Show that a vector bundle E of rank r over B is trivial if and only if it has r sections $s\colon B \to E$ forming, $\forall p \in B$, a basis of E_p.

Such sections are sometimes called **a frame**.

Exercise 3.2.7 Show that the line bundle associated to a Cartier divisor X has a smooth (resp. holomorphic) section vanishing exactly at X.

Exercise 3.2.8 Let H be a hyperplane of $\mathbb{P}^n_{\mathbb{K}}$, the locus

$$\sum a_j x_j = 0$$

where the $a_j \in \mathbb{K}$ are not all zero. Show that

1. H is a smooth Cartier divisor;
2. the line bundle associated to H is isomorphic to the hyperplane bundle.

3.3 Tangent and normal bundles

We can now define the tangent bundle $TM \xrightarrow{\pi} M$ through its cocycle.

Definition 3.3.1 Let M be a manifold of dimension n. Choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$.

Then the tangent bundle $TM \xrightarrow{\pi} M$ is the vector bundle of rank n given by the cocycle.

$$g_{\alpha\beta}(p) = J(\varphi_{\alpha\beta})_{\varphi_\beta(p)}. \tag{3.2}$$

if M is a complex manifold, we may consider it as real manifolds, which gives two different definition of tangent bundle on it. When we will need to distinguish them we will call them respectively **holomorphic tangent bundle** and **real tangent bundle**.

There is a natural way to identify each tangent space T_pM to the fibre $(TM)_p$.

The construction in Definition 3.3.1 gives a trivialization $\{\phi_\alpha\}_{\alpha\in I}$ of TM.

Let $p \in M$, $v \in (TM)_p$. Choose a chart $(U_\alpha, \varphi_\alpha)$ from the atlas used in Definition 3.3.1 such that $p \in U_\alpha$. Then $\phi_\alpha(v) = (p,(v_1,\ldots,v_n)) \in U_\alpha \times \mathbb{K}^n$. Set then $x_1,\ldots,x_n$ for the induced local coordinates near $\pi(v)$. We associate to v the derivation $\sum v_i \left(\frac{\partial}{\partial x_i}\right)_p \in T_pM$.

This gives an isomorphism of vector spaces among $(TM)_p$ and T_pM that, by Corollary 3.0.1, does not depend on the choice of the chart containing p. Note that here the choice of the cocycle (3.2) is crucial: no other cocycle would have worked!

By a similar argument one proves that the definition does not depend, up to isomorphisms, on the choice of the atlas.

R Let $F\colon M \to N$ be a smooth function.

The differentials $dF_p\colon T_pM \to T_{F(p)}N$ naturally glue to a morphism of vector bundles over F

$$\begin{array}{ccc} TM & \xrightarrow{dF} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array}$$

defined by $dF(v) := dF_{\pi(v)}(v) \in T_{F(\pi(v))}N \subset TN$.

Moreover, if G is a further smooth function from N to another manifold, then $d(G\circ F) = dG\circ dF$.

There is an important construction related to embeddings and tangent bundles, the normal bundle. Let $f\colon X \to M$ be an embedding. Geometrically, if we think to X as a subset of M, to each point of X we have associated the vector space T_pM, and its subspace T_pX, identifying every tangent vector to X with its image by df.

Then we can consider the quotient space $(\mathscr{N}_{X|M})_p := T_pM/T_pX$. This is the normal space of X in M at p.

All these spaces naturally glue to a bundle on X: we start from the vector bundle $TM_{|X} := f^{-1}TM$, which is a vector bundle on X such that every fibre is canonically isomorphic to the tangent space of M at that point. Then TX is naturally a subbundle of $TM_{|X}$ via df and the construction of the quotient bundle produces

Definition 3.3.2 The **normal bundle** of X in M is the quotient bundle $\mathscr{N}_{X|M} := TM_{|X}/TX$.

For Cartier divisor it holds the following

Proposition 3.3.3 Let X be a smooth Cartier divisor in a manifold M. Then the normal bundle of X in M is isomorphic to the restriction to X of the line bundle on M associated to X.

Proof. Choose on each U_i a coordinate system $(z_1^i,\ldots,z_n^i)$ such that $U_i\cap X$ is given by the equation $z_n^i = 0$, so that $f_i = z_n^i$. The normal bundle $\mathscr{N}_{X|M}$ is then given by the cocycle

$$\left\{ n_{ij}\colon U_i\cap U_j\cap X \to \mathrm{GL}_1(\mathbb{K}) | n_{ij} = \frac{\partial z_n^i}{\partial z_n^j} \right\}.$$

The line bundle on M associated to X is defined by the cocycle

$$\left\{ g_{ij} \colon U_i \cap U_j \to \mathrm{GL}_1(\mathbb{K}) | g_{ij} = \frac{z_n^i}{z_n^j} \right\}.$$

Taking partial derivatives with respect to z_n^j of the equality $z_n^i = z_n^j g_{ij}$ yields $\frac{\partial z_n^i}{\partial z_n^j} = g_{ij} + z_n^j \cdot \frac{\partial g_{ij}}{\partial z_n^j}$. Restricting this to X, recalling that $z_n^j = 0$ on $X \cap U_j$ we get the cocycle $\{n_{ij}\}$. This proves the claim. ■

This has the following interesting consequence.

Corollary 3.3.4 Let M be a manifold, $f \in C^\infty(M)$, $y \in Reg(f)$, $X = f^{-1}(y)$. Then the normal bundle of X in M is trivial.

Proof. The line bundle over M associated to X is trivial, and therefore so is its restriction to X. ■

We will later need the following result, which we state without proof.

Theorem 3.3.5 — Tubular neighbourhood theorem. Let X, M be manifolds without boundary, and let $i \colon X \hookrightarrow M$ be an embedding.

Then there is a neighbourhood W of $i(X)$ in M and a diffeomorphism $\nu \colon \mathcal{N}_{X|M} \to W$, such that $i = \nu \circ s_0$.

Exercise 3.3.1 Show that the group quotient[a] $\mathbb{R}^n/\mathbb{Z}^n$ has a natural structure of manifold such that the differential $d\pi_p$ of the quotient map $\pi \colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is invertible at each point.

Show that with this differentiable structure $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomorphic to $(S^1)^n$.

Show that its tangent bundle is trivial[b].

[a] with respect to the sum

[b] Notice that it follows that all $\Lambda^q(T(S^1)^n)^*$ are trivial for all q.

Exercise 3.3.2 A *lattice* $\Lambda \subset \mathbb{C}^n$ is a subgroup, with respect to the sum, generated by $2n$ vectors that are linearly independent over $\mathbb{R}$.

The group quotient $\mathbb{C}^n/\Lambda$ has a complex structure such that the differential of the quotient map $\pi \colon \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ is invertible at each point. These complex manifold are called *complex tori*.

Show that the complex tangent bundle of a complex torus is trivial.

Exercise 3.3.3 Show that the holomorphic tangent bundle of $\mathbb{P}^1_{\mathbb{C}}$ is not trivial.

Exercise 3.3.4 Prove that $\sum_{i=1}^k \left(u_{2i} \frac{\partial}{\partial u_{2i-1}} - u_{2i-1} \frac{\partial}{\partial u_{2i}} \right)$ defines a smooth vector field on S^{2k-1} which never vanishes. We have *combed flat* all spheres of odd dimension.

Exercise 3.3.5 Show that the normal bundle of a Moebius band embedded in $\mathbb{R}^3$ is not trivial.

3.4 Real and holomorphic tangent bundles

Let M be now a complex manifold of dimension n, $p \in M$. Then there is a tangent space T_pM which is a complex vector space of dimension n. Since M has an induced real structure of dimension $2n$, it has also a tangent space as real manifold, which we denote (to distinguish it from the other one) by $T_p^{\mathbb{R}}M$, of dimension $2n$.

We have then constructed two different tangent bundles for M, the one as complex manifold, the **holomorphic tangent bundle** TM, a complex bundles of rank n, and the one as real manifold (the **real tangent bundle**), say $T^{\mathbb{R}}M$, a real vector bundle of rank $2n$. One is naturally tempted to try to find some canonical isomorphism of real vector bundles among them.

Set local coordinates $z_j = x_j + iy_j$, $j = 1, \ldots, n$ at a point $p \in M$, so defined on a chart $U \ni p$. Then the real tangent space $T_p^{\mathbb{R}}M$ is generated by the partial derivatives $\left(\frac{\partial}{\partial x_j}\right)_p, \left(\frac{\partial}{\partial y_j}\right)_p$.

Note that the action of the vectors in $T_p^{\mathbb{R}}M$ on $\mathscr{E}_p$ may be naturally extended to complex valued function $f = g + ih$, g, h smooth, by setting

$$v(g+ih) := v(g) + iv(h) \in \mathbb{C}.$$

Then define

$$\left(\frac{\partial}{\partial \bar{z}_j}\right)_p = \frac{1}{2}\left(\left(\frac{\partial}{\partial x_j}\right)_p + i\left(\frac{\partial}{\partial y_j}\right)_p\right).$$

Wait, this does not make any sense! We can't multiply an element of a real vector space by i! In fact this makes sense as an element of the complexification of $T_p^{\mathbb{R}}M$.

Definition 3.4.1 — Complexification. Let V be a real vector space. We define its complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ (or just $V \otimes \mathbb{C}$ for short) as follows.

As real vector space, $V \otimes_{\mathbb{R}} \mathbb{C}$ is the abstract direct sum of two copies of V. A general element of $V \otimes_{\mathbb{R}} \mathbb{C}$ is given by two vectors $v_1, v_2 \in V$. We will write it[a] as $v_1 + iv_2$. This is also summarized by the notation

$$V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$$

Please note that the notation induces a natural inclusion of V in $V \otimes_{\mathbb{R}} \mathbb{C}$ mapping each vector v on... v.

$V \otimes_{\mathbb{R}} \mathbb{C}$ has a natural canonical structure of complex vector space induced by the multiplication

$$i \cdot (v_1 + iv_2) = -v_2 + iv_1$$

Note that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} (V \otimes_{\mathbb{R}} \mathbb{C})$.

[a] We will later see the notation $v_1 \otimes 1 + v_2 \otimes i$ for $v_1 + iv_2$.

Then $\left(\frac{\partial}{\partial \bar{z}_j}\right)_p$ is an element of the complex vector space $T_p^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$.

Note that

$$\left(\frac{\partial}{\partial \bar{z}_j}\right)_p (g+ih) = \frac{1}{2}\left(\left(\frac{\partial g}{\partial x_j} - \frac{\partial h}{\partial y_j}\right) + i\left(\frac{\partial h}{\partial x_j} + \frac{\partial g}{\partial y_j}\right)\right)(p).$$

Then, by definition[2] of holomorphic function, $g + ih$ is holomorphic if and only if $\forall j$,

$$\frac{\partial(g+ih)}{\partial \bar{z}_j} = 0.$$

[2] The reader that has not done any complex analysis in several variables should take this as definition of holomorphic

Now define

$$\left(\frac{\partial}{\partial z_j}\right)_p = \frac{1}{2}\left(\left(\frac{\partial}{\partial x_j}\right)_p - i\left(\frac{\partial}{\partial y_j}\right)_p\right) \in T_p^{\mathbb{R}} M \otimes_{\mathbb{R}} \mathbb{C}.$$

If $f = g + ih$ is holomorphic then

$$\left(\frac{\partial}{\partial z_j}\right)_p (g+ih) = \frac{1}{2}\left(\left(\frac{\partial g}{\partial x_j} + \frac{\partial h}{\partial y_j}\right) + i\left(\frac{\partial h}{\partial x_j} - \frac{\partial g}{\partial y_j}\right)\right)(p) = \left(\frac{\partial g}{\partial x_j} + i\frac{\partial h}{\partial x_j}\right)(p)$$

showing that $\left(\frac{\partial}{\partial z_j}\right)_p$ coincides with the complex derivative in the direction of the variable z_j. Indeed

$$\left(\frac{\partial z_k}{\partial z_j}\right)_p = \left(\frac{\partial (x_k + iy_k)}{\partial z_j}\right)_p = \frac{1}{2}\left(\left(\frac{\partial x_k}{\partial x_j} + \frac{\partial y_k}{\partial y_j}\right) + i\left(\frac{\partial y_k}{\partial x_j} - \frac{\partial x_k}{\partial y_j}\right)\right) = \delta_{jk}.$$

The complexification, as most of the construction in linear algebra, may be extended to bundles.

Definition 3.4.2 — Complexification of a real vector bundle. If E is a real vector bundle with cocycle $g_{\alpha\beta}$, then, since every matrix with real coefficients is also a matrix with complex coefficients, the same cocycle $g_{\alpha\beta}$ gives also a complex vector bundle[a] $E_{\mathbb{C}}$.

[a] In this case, even if E is a smooth manifold, we are not claiming that $E_{\mathbb{C}}$ has any structure of complex manifold.

Then, for every $p \in B$, the fibre $(E_{\mathbb{C}})_p$ is canonically isomorphic to the complex vector space $E_p \otimes_{\mathbb{R}} \mathbb{C}$.

We define the **complexified real tangent bundle** $(T^{\mathbb{R}}M)_{\mathbb{C}}$ as the complexification $T^{\mathbb{R}}M \otimes \mathbb{C}$ of the real tangent bundle $T^{\mathbb{R}}M$. Note that it naturally contains $T^{\mathbb{R}}M$ as (real) subbundle.

It contains all $\left(\frac{\partial}{\partial \bar{z}_j}\right)_p$, $\left(\frac{\partial}{\partial z_j}\right)_p$. This identifies the complex subbundle $T' \subset (T^{\mathbb{R}}M)_{\mathbb{C}}$ generated pointwise by the $\left(\frac{\partial}{\partial z_j}\right)_p$ with the holomorphic tangent bundle TM of M. The complex conjugation is well defined on $(T^{\mathbb{R}}M)_{\mathbb{C}}$. Since $\overline{\left(\frac{\partial}{\partial z_j}\right)_p} = \left(\frac{\partial}{\partial \bar{z}_j}\right)_p$, $\overline{T'} \subset (T^{\mathbb{R}}M)_{\mathbb{C}}$ is pointwise generated by the $\left(\frac{\partial}{\partial \bar{z}_j}\right)_p$. $\overline{T'}$ is the **antiholomorphic** tangent bundle.

It follows

$$(T^{\mathbb{R}}M)_{\mathbb{C}} = T' \oplus \overline{T'} \tag{3.3}$$

in the sense that T' and $\overline{T'}$ are subbundles of $(T^{\mathbb{R}}M)_{\mathbb{C}}$ such that, for all $p \in M$, $\left((T^{\mathbb{R}}M)_{\mathbb{C}}\right)_p$ is the direct sum of its vector subspaces T'_p and $\overline{T'}_p$.

We deduce

function. The reader that has done complex analysis in one variable will recognize the Cauchy-Riemann relations

$$\begin{cases} \frac{\partial g}{\partial x} &= \frac{\partial h}{\partial y} \\ \frac{\partial g}{\partial y} &= -\frac{\partial h}{\partial x} \end{cases}$$

Note that

$$\left(\frac{\partial z_k}{\partial \bar{z}_j}\right)_p = \left(\frac{\partial (x_k + iy_k)}{\partial \bar{z}_j}\right)_p = \frac{1}{2}\left(\left(\frac{\partial x_k}{\partial x_j} - \frac{\partial y_k}{\partial y_j}\right) + i\left(\frac{\partial y_k}{\partial x_j} + \frac{\partial x_k}{\partial y_j}\right)\right) = 0:$$

the local coordinates are holomorphic functions.

Proposition 3.4.3 The real tangent bundle, the holomorphic tangent bundle and the antiholomorphic tangent bundle of a complex manifold M are isomorphic as real vector bundles.

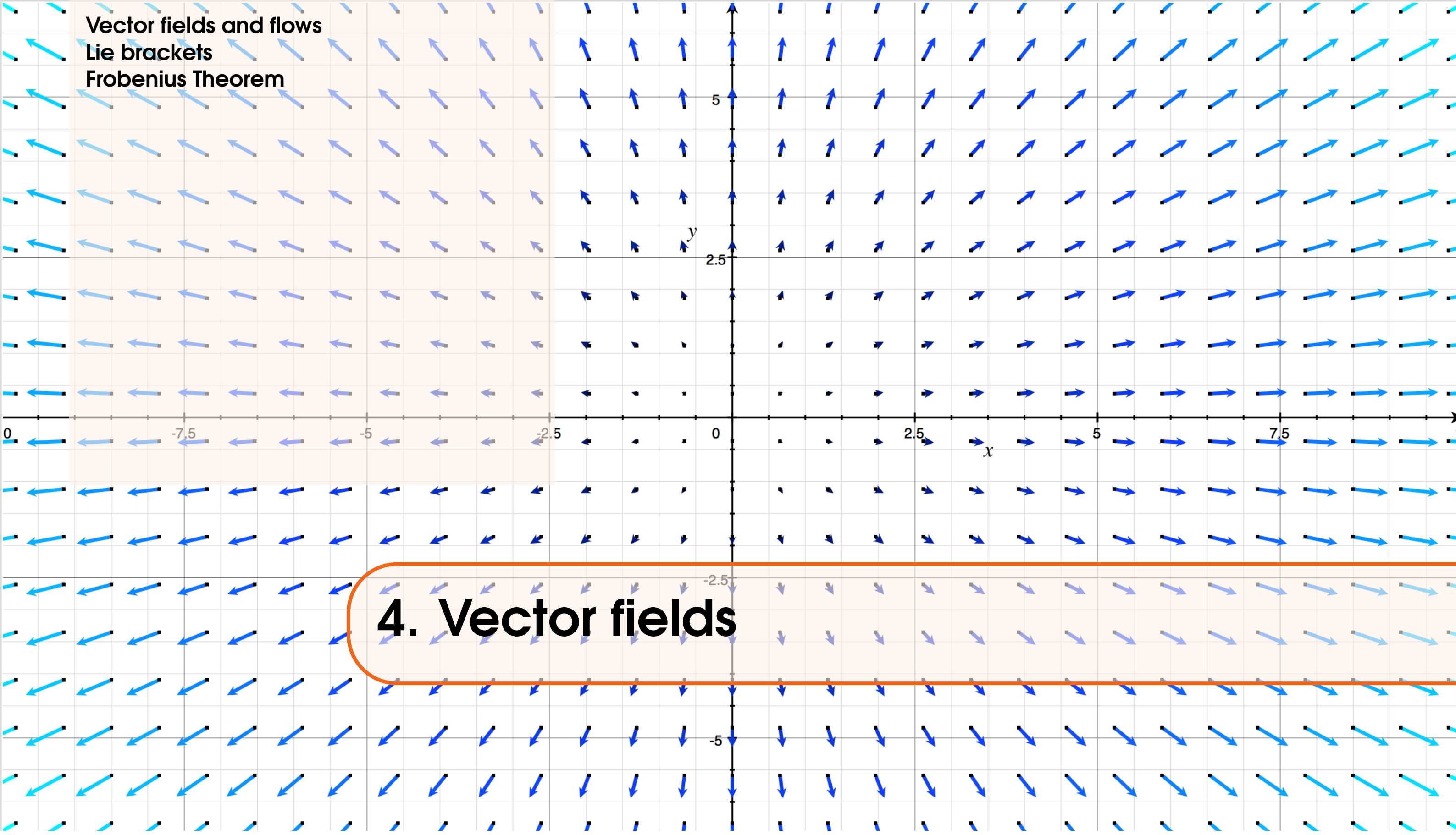
Proof. The decomposition (3.3) $(T^{\mathbb{R}}M)_{\mathbb{C}} = T' \oplus \overline{T'}$ induces surjective morphisms of vector bundles $(T^{\mathbb{R}}M)_{\mathbb{C}} \to T'$ and $(T^{\mathbb{R}}M)_{\mathbb{C}} \to \overline{T'}$ whose restrictions to the real tangent bundle $T^{\mathbb{R}}M$, since $T^{\mathbb{R}}M \cap T' = T^{\mathbb{R}}M \cap \overline{T'} = \{0\}$ are injective and therefore, by a dimension count, isomorphisms. ■

Complement 3.4.1 Prove that the definition of the tangent bundle does not depend, up to automorphisms, on the choice of the atlas.

Exercise 3.4.1 Compute, $\forall j,k$

$$\left(\frac{\partial}{\partial z_j}\right)(z_k), \quad \left(\frac{\partial}{\partial \bar{z}_j}\right)(z_k), \quad \left(\frac{\partial}{\partial z_j}\right)(\bar{z}_k), \quad \left(\frac{\partial}{\partial \bar{z}_j}\right)(\bar{z}_k).$$

Deduce that $\left(\frac{\partial}{\partial z_j}\right), \left(\frac{\partial}{\partial \bar{z}_j}\right)$ is the local frame (compare Exercise 3.2.6) dual to the functions $z_j, \bar{z}_j$.



4. Vector fields

To ease the notation we consider in this chapter only the real case.

4.1 Vector fields and flows

Definition 4.1.1 A **vector field** on a manifold M is a section $v\colon M \to TM$ of the tangent bundle. A vector field is **smooth** if it is smooth as a map among manifolds. The smooth vector fields form the vector space $\mathfrak{X}(M)$.

We are only interested in smooth vector fields, so, unless differently specified, all vector fields are from now on assumed to be smooth.

For every vector field v, every chart (U,φ) for M may be used to represent the vector field v on U: $v_{|U}$. If $x_1,\dots,x_n$ are the local coordinates given by the chart, there are functions $v_i\colon U \to \mathbb{R}$ such that $\forall p \in U$, $v(p) = \sum v_i(p)\left(\frac{\partial}{\partial x_i}\right)_p$. We will write $v_{|U}$ as $\sum v_i \frac{\partial}{\partial x_i}$.

Vector fields bring naturally to the definition of integral curve.

We first allow the vector fields to vary in time.

Definition 4.1.2 — Variable vector fields. Consider an open subset $J \subset \mathbb{R}$ with $0 \in J$.

A smooth **variable vector field** is a function $v\colon J \to \mathfrak{X}(M)$ such that, given any chart (U,φ), writing its local coordinates as $x_1,\dots,x_n$, then for all $p \in U$

$$v(t)_p = \sum_{i=1}^{n} v_i(p,t)\left(\frac{\partial}{\partial x_i}\right)_p$$

with all functions $v_i\colon U \times J \to \mathbb{R}$ smooth.

Vector fields can be seen as a special case of variable vector fields by taking constant functions v. In other words, they correspond to functions v_i that do not depend on t.

Definition 4.1.3 — Integral curves. Consider an open subset $J \subset \mathbb{R}$ containing 0 and a smooth variable vector field $v\colon J \to \mathfrak{X}(M)$.

We say that a smooth path $\gamma\colon J \to M$ is an **integral curve** of v if

$$\forall t_0 \in J \quad d\gamma_{t_0}\left(\frac{d}{dt}\right)_{t_0} = v(t_0)_{\gamma(t_0)}$$

The **starting point** of γ is $\gamma(0)$.

In other words the **velocity vector** of the path γ at each time equals the value of the vector field at the point where the path is at that same time.

The integral curves of a variable vector field are locally the solutions of a system of differential equations. In fact, restricting to a chart and writing the definition of integral curve we obtain that the integral curves are solutions of Cauchy problems, as follows.

Definition 4.1.4 — Cauchy problem. Consider two open subsets $J \subset \mathbb{R}$, $U \subset \mathbb{R}^n$ and assume $0 \in J$. For all $\delta > 0$ we denote by J_δ the open interval $(-\delta, \delta)$. If δ is small, $J_\delta \subset J$.

We consider a smooth function $V : U \times J \to \mathbb{R}^n$ and write its components as v_j as follows

$$V((x_1,\dots,x_n),t) = (v_1((x_1,\dots,x_n),t),\dots,v_n((x_1,\dots,x_n),t))$$

Having fixed δ such that $J_\delta \subset J$ and a point $p \in U$ we say that a function $\gamma\colon J_\delta \to U$ **satisfies the Cauchy problem for** V **with initial point** p if

$$\gamma(0) = p \qquad \text{and} \qquad \frac{d\gamma_i}{dt} = v_i(\gamma(t),t) \quad \forall i = 1,\dots,n$$

The functions γ that satisfies the Cauchy problem for V are the integral curves of the vector field $v(t) = \sum_{i=1}^n v_i(\cdot,t)\frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$.

An important result in this framework is the following

Theorem 4.1.5 — Existence and uniqueness of solutions of Cauchy problems. For every smooth function $V : U \times J \to \mathbb{R}^n$ and for every $p \in U$ there is a $\delta > 0$, an open subset $U' \subset U$ containing p and a smooth function $g\colon U' \times J_\delta \to U$ such that for all $q \in U'$ the following holds: the function $\gamma(t) = g(q,t)$ satisfies the Cauchy problem for V with initial point q.

The solution is unique in the sense that, if $g\colon U' \times J_\delta \to U$ and $g' : U'' \times J_{\delta'} \to U$ are two such functions, they coincide in the common domain $(U' \times J_\delta) \cap (U'' \times J_{\delta'})$.

We explain the roles of U' and δ by an elementary example, which will illustrate in particular that there is no hope of generalizing Theorem 4.1.5 with $U' = U$.

Example 4.1 Set $J = (-1,1)$ and $U = J \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$. So $U = \{(x_1,x_2,\dots,x_n) | |x_1| < 1\}$.

Choose as V the constant function $(1,0,0,\dots,0)$. Then we claim that the function

$$g(q,t) = q + (t,0,\dots,0)$$

gives the unique solution of the Cauchy problem for V with initial point q.

This claim is "essentially" correct, as the reader can easily check by computing the right derivatives.

However we haven't specified the domain of g, since we haven't specified neither δ nor U'. If we try to set $U' = U$ then, doesn't matter how small we choose δ, the image of g will not be contained in U, and so the claim becomes wrong since V is defined only on its domain, $U \times J$.

To make our claim correct we ned to get a well defined g: we need then to fix $\delta > 0$ and choose $U' \subset (-1, 1-\delta) \times \mathbb{R}^{n-1}$.

Different choices of δ and U' give formally different solutions, coinciding in the common domains as stated in theorem 4.1.5.

Since all solutions coincide in the common domains, we may glue all of them to get a "main", solution g, that we will denote as **flow** of V (resp. v). The domain Ω of the flow will no more be of the product form $U' \times J_\delta$.

Proposition 4.1.6 — Flows. Consider open subsets $J \subset \mathbb{R}$, $U \subset \mathbb{R}^n$ and assume $0 \in J$.

Consider a smooth function $V\colon U \times J \to \mathbb{R}^n$ and the corresponding variable vector field $v\colon J \to \mathfrak{X}(U)$.

Let $\Omega \subset U \times J$ be the union of all possible open subsets $U' \times J_\delta$ arising in Theorem 4.1.5. Then there is a unique function $g\colon \Omega \to U$ such that for all $p \in U$, $g(p,0) = p$ and the curve $\gamma(t) = g(p,t)$ is integral for v.

Moreover g is smooth.

Note that Ω is open and it contains $U \times \{0\}$.

Lifting these results to manifolds, we get naturally the following definitions.

Let U be an open subset of M and let Ω be an open subset of $U \times \mathbb{R}$ containing $U \times \{0\}$. Consider a smooth map

$$g\colon \Omega \to M$$

We induce, for all $t \in \mathbb{R}$, for all $p \in U$, smooth functions

$$g_t\colon \Omega_t \to M \qquad\qquad g_p\colon J_p \to M$$

by

$$g_t(p) = g_p(t) = g(t,p)$$

We haven't specified the domains Ω_t and J_p of the maps g_t and g_p yet. They are the biggest possible domains for which the definition makes sense, defined by:

$$\Omega \cap (U \times \{t\}) = \Omega_t \times \{t\} \qquad\qquad \Omega \cap (\{p\} \times \mathbb{R}) = \{p\} \times J_p$$

Note that the Ω_t are open subsets of M whereas the J_p are open subsets of $\mathbb{R}$ containing 0.

Definition 4.1.7 — Local one parameter groups. Let U be an open subset of M and let Ω be an open subset of $U \times \mathbb{R}$ containing $U \times \{0\}$. A smooth map

$$g\colon \Omega \to M$$

is a **local one parameter group of smooth transformations of** U **into** M if all maps $g_t\colon \Omega_t \to M$ are embeddings (so diffeomorphism with their image) and moreover

$$g_{s+t}(p) = g_s(g_t(p))$$

whenever possible, so when $(p,s+t),(p,t),(g_t(p),s)$ all belong to Ω.

We note that if g is a local one parameter group automatically $g(p,0) = g_0(p)$ equals p for all p. In other words g_0 is the identity of $\Omega_0 = U$.

We associate to every local one parameter group g of smooth transformations of U into M a smooth vector field on U as follows: it is the unique vector field $v \in \mathfrak{X}(U)$ such that, for all $p \in U$ and $f \in C^\infty(U)$

$$v_p(f) = \frac{d(f \circ g_p)}{dt}(0)$$

In other words, if $x_1,\dots,x_n$ are local coordinates at p for M, we consider natural induced local coordinates $x_1,\dots,x_n,t$ at $(p,0)$ for Ω and

$$v_p = \sum_{i=1}^{n} \frac{\partial(x_i \circ g)}{\partial t}(p,0) \left(\frac{\partial}{\partial x_i}\right)_p$$

Shortly, writing $g_i(x_1,\dots,x_n,t)$ for the i^{th} component of g,

$$v = \sum_{i=1}^{n} \frac{\partial g_i}{\partial t}(0) \frac{\partial}{\partial x_i}$$

In particular, the maps g_p are integral curves of v with starting point p.

Now we see that conversely every vector field is induced as above by a local one parameter group, and precisely by its flow.

Proposition 4.1.8 Let M be a manifold without boundary, and consider a point $p \in M$ and a vector field $v \in \mathfrak{X}(M)$.

There is a neighborhood U of p, a $\delta > 0$ and a local one parameter group $g \colon U \times J_\delta \to M$ of smooth transformations of U into M such that v is induced by g as above.

If g, g' are two local one parameter groups inducing the same vector field, then $g = g'$ in the common domain.

Proof. The statement being local, we can restrict to a chart in p giving local coordinates $x_1,\dots,x_n$. In other words, we can assume M to be an open subset of $\mathbb{R}^n$.

Then we can write

$$v = \sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}$$

By Theorem 4.1.5 there is $\delta > 0$, a neighborhood U of p and a smooth map

$$g \colon U \times J_\delta \to M$$

such that for all $q \in U$, $g(q,0) = q$ and

$$\frac{\partial g}{\partial t}(q,t) = (v_1(g(q,t)),\dots,v_n(g(q,t)))\,.$$

We claim that g is a local one parameter group. For fixed $s \in J_\delta$ set $g_1(q,t) = g(q,t+s)$ and $g_2(q,t) = g(g_s(q),t)$. Then both g_1 and g_2 satisfy the same Cauchy problem

$$\frac{\partial u(q,t)}{\partial t} = (v_1(u(q,t)),\dots,v_n(u(q,t))) \qquad\qquad u(q,0) = g_s(q)$$

and then by the uniqueness assertion in Theorem 4.1.5 $g_1(q,t) = g_2(q,t)$. So $g_{t+s}(q) = g_1(q,t) = g_2(q,t) = g_t \circ g_s(q)$:

$$g_{t+s} = g_t \circ g_s.$$

In particular, for all $t \in J_\delta$, $g_t \circ g_{-t} = g_{-t} \circ g_t = g_0$ is the identity, so all g_t are embeddings.

It is obvious that g induces v. The uniqueness assertion follows by the uniqueness assertion in Theorem 4.1.5. ■

It is natural to ask, if, given a vector field, one can obtain a "global" one parameter group, a local parameter group with $U = M$ and $\Omega = M \times \mathbb{R}$.

Definition 4.1.9 — One parameter groups. We say that a smooth map

$$g\colon M\times\mathbb{R}\to M$$

is a one parameter group of smooth transformations if all g_t are diffeomorphisms of M and moreover for all $s,t\in\mathbb{R}$

$$g_{s+t}=g_s\circ g_t$$

Note that in particular g_0 is the identity of M and $g_s^{-1}=g_{-s}$. This leads to the natural interpretation of one parameter groups as maps $t\mapsto g_t$: group homomorphisms from $\mathbb{R}$ (seen as abelian group with the operation $+$) and the group $\operatorname{Aut}M$ of the diffeomorphisms of M into itself (seen as a group with the operation $\circ$).

The example 4.1 shows that in general integrating a vector field does not give a (global) one parameter group, but only a local one. However, if the support of the vector field is compact one can prove the following.

Theorem 4.1.10 Let v be a smooth vector field and assume that its support

$$\operatorname{supp}v=\overline{\{p\in M|v_p\neq 0\}},$$

the closure of the complement of the vanishing locus of v, is compact.

Then there exists a unique one parameter group $g\colon M\times\mathbb{R}\to M$ of transformations of M which induces v. Moreover $g(p,t)=p$ for all t when p does not belong to $\operatorname{supp}v$.

Proof. Set $K:=\operatorname{supp}v$.

By Proposition 4.1.8, for any $p\in K$ there is neighborhood U_p of p, a positive number $\delta(p)>0$ and a local one parameter group

$$g^{(p)}\colon U_p\times J_{\delta(p)}\to M$$

inducing v on U_p. Since K is compact, there are finitely many points $p_1,\dots,p_r$ such that

$$K\subset U:=\bigcup_{j=1}^{r}U_{p_j}.$$

We set $\delta:=\min\delta(p_j)$ Then, by the uniqueness assertion in Proposition 4.1.8, $g^{(p_j)}$ and $g^{(p_k)}$ coincide on $\left(U_{p_j}\cap U_{p_k}\right)\times J_\delta$. Hence we can define

$$g\colon U\times J_\delta\to M$$

by

$$g(q,t)=g^{(p_j)}(q,t)$$

for any p_j such that $q\in U_{p_j}$.

We note that if $q\in U\setminus K$, X vanishes in a neighborhood of q and then by the uniqueness assertion in Proposition 4.1.8, $g(q,t)=q$ for all t. Hence extending g to a function

$$g\colon M\times J_\delta\to M$$

by setting $g(q,t)=q$ for all $q\notin K$ we obtain a smooth function.

Now recall that if $t,s,t+s \in J_\delta$ then $g_{t+s} = g_t \circ g_s$. We use it to extend g to a one parameter group

$$g\colon M \times \mathbb{R} \to M$$

by choosing, for arbitrary $t \in \mathbb{R}$, a positive integer k such that $|t| < k\delta$, and then define

$$g_t = g_{t/k} \circ g_{t/k} \circ \cdots \circ g_{t/k} = g_{t/k}^k$$

It is immediately seen that g is a well defined[1] one parameter group inducing v. ■

Exercise 4.1.1 Let v be a variable vector field and let g be its flow. Prove

1. $dg_{(p,0)} \left(\frac{\partial}{\partial x_i}\right)_{(p,0)} = \left(\frac{\partial}{\partial x_i}\right)_p$
2. $dg_{(p,0)} \left(\frac{\partial}{\partial t}\right)_{(p,0)} = v(0)_{g(p,0)}$

Exercise 4.1.2 Consider the matrices

$$A_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in GL_1(\mathbb{R})$$

1. Show that

$$g_t(x,y) = (u,v) \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = A_t \begin{pmatrix} x \\ y \end{pmatrix}$$

defines a map

$$g_t\colon \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$$

that is a diffeomorphism.
2. Show that the map $g\colon M \times \mathbb{R} \to M$ defined by $g(p,t) := g_t(p)$ is a one parameter group.
3. Set $v \in \mathfrak{X}\left(\mathbb{P}^1_{\mathbb{R}}\right)$ for the vector field associated to g. Compute the locus

$$\left\{p \in \mathbb{P}^1_{\mathbb{R}} | v_p = 0\right\} \subset \mathbb{P}^1_{\mathbb{R}}$$

Exercise 4.1.3 Compute the one parameter groups g of the vector fields in Exercise 3.3.4.
Find all $t \in \mathbb{R}$ such that $g_t = \mathrm{Id}_{S^{2k-1}}$.
Find all $t \in \mathbb{R}$ such that g_t is the antipodal map $p \mapsto -p$. In other words for which t is $g_t(-p) = p$ for all p in S^{2k-1}?

4.2 Lie brackets

The vector fields act on $C^\infty(M)$: if v is a vector field and $f \in C^\infty(M)$, the function $v(f)$ is naturally defined by

$$v(f)(p) = v_p([f])$$

[1] It does not depend on the choice of k since different choices give solutions to the same Cauchy problem, so they coincide in the common domain

In local coordinates, if $v_{|M} = \sum v_i \frac{\partial}{\partial x_i}$, then $v(f)(p) = \sum v_i(p) \left(\frac{\partial}{\partial x_i}\right)_p f$, which we shortly write

$$v(f) = \sum v_i \frac{\partial f}{\partial x_i}. \tag{4.1}$$

If $v \in \mathfrak{X}(U)$ then $v(f)$ is obviously smooth and therefore we have defined a map

$$\begin{array}{rcl} \mathfrak{X}(U) \times C^\infty(U) & \to & C^\infty(U) \\ (v, f) & \mapsto & v(f) \end{array}$$

This associates to each vector field a map from $C^\infty(M)$ to itself. The image is the space of **derivations** of $C^\infty(M)$, special operators on $C^\infty(M)$. If $v\colon C^\infty(M) \to C^\infty(M)$ is a derivation then

$$v(fg) = v(f)g + fv(g) \tag{4.2}$$

This map is obviously linear and injective since a non trivial vector field yields a nontrivial derivation. So it gives an isomorphism among the space of vector fields and the space of derivations. By abuse of notation we are going to use the same notation for a vector field $v \in \mathfrak{X}(M)$ and the corresponding derivation $v\colon C^\infty(M) \to C^\infty(M)$.

Given two derivations v and w we can consider the composition $v \circ w\colon C^\infty(M) \to C^\infty(M)$. This is in general not a derivation. For example, if $M = \mathbb{R}^n$ and $v = w = \frac{\partial}{\partial x_i}$ then $v \circ w = \frac{\partial^2}{\partial x_i^2}$ is not a derivation since it does not respect (4.2).

On the contrary, $v \circ w - w \circ v$ is a derivation!

To show that, we prove that, for all $p \in M$, $v \circ w - w \circ v$ acts on $\mathscr{E}_p$ as an element of T_pM. So let us choose local coordinates near p and compute $v \circ w - w \circ v$ in these coordinates. We write $v = \sum v_i \frac{\partial}{\partial x_i}$, $w = \sum w_j \frac{\partial}{\partial x_j}$. Then

$$v \circ w = \left(\sum_i v_i \frac{\partial}{\partial x_i}\right) \circ \left(\sum_j w_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j} v_i w_j \frac{\partial^2}{\partial x_i \partial x_j}$$

Since by Schwarz Theorem $\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}$

$$\begin{aligned} v \circ w - w \circ v &= \sum_{i,j} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j} v_i w_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i,j} w_i \frac{\partial v_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{i,j} w_i v_j \frac{\partial^2}{\partial x_i \partial x_j} = \\ &= \sum_j \left(\sum_i \left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \right) \frac{\partial}{\partial x_j} \end{aligned} \tag{4.3}$$

The second derivatives have canceled! The expression we have found shows that $v \circ w - w \circ v$ is a vector field.

Definition 4.2.1 — Lie bracket of vector fields. Given two vector fields $v, w \in \mathfrak{X}(M)$ we denote by $[v, w]$ their **Lie bracket**

$$[v, w] := v \circ w - w \circ v \in \mathfrak{X}(M)$$

Consider now two manifolds M and N and a diffeomorphism $F\colon M \to N$ among them.

Definition 4.2.2 Let v be a vector field on M. Then F induces a vector field $F_*v \in \mathfrak{X}(N)$ by

$$F_*(v)_p = dF_{F^{-1}(p)}(v_{F^{-1}(p)})$$

Note that it is crucial for the definition that F is supposed to be invertible.

If $f \in C^\infty(N)$ then $F_*v(f) = v(f \circ F) \circ F^{-1}$.

Proposition 4.2.3 If $F\colon M \to N$ is a diffeomorphism then for each $v, w \in \mathfrak{X}(M)$

$$F_*[v,w] = [F_*v, F_*w]$$

Proof. For every open subset $U \subset N$, for every $f \in C^\infty(U)$

$$\begin{aligned}[F_*v, F_*w](f) &= F_*v(F_*w(f)) - F_*w(F_*v(f)) = \\ &= F_*v(w(f \circ F) \circ F^{-1}) - F_*w(v(f \circ F) \circ F^{-1}) = \\ &= (v(w(f \circ F) - w(v(f \circ F)) \circ F^{-1} = F_*([v,w])(f)\end{aligned}$$

■

Now consider a manifold M and a local one parameter group $g\colon U \times J_\delta \to M$ induced by a vector field $v \in \mathfrak{X}(U)$. Consider a diffeomorphism $F\colon U \to U'$, an open relatively compact[2] subset $W \subset U$ and set $W' = F(W)$. By the same argument as in the first part of the proof of Theorem 4.1.10, if $\varepsilon > 0$ is small enough $g(W \times J_\varepsilon) \subset U$ and we obtain a local one parameter group $g'\colon W' \times J_\varepsilon \to U'$ by

$$g'_t = F \circ g_t \circ F^{-1}$$

For all $f \in C^\infty(W')$ and $p \in W'$

$$F_*v(f)(p) = v(f \circ F) \circ F^{-1}(p) = \frac{d(f \circ F \circ g_t \circ F^{-1}(p))}{dt}(0)$$

so that g' induces F_*v on W'.

In the special case when U and U' are both open subsets of M (*e.g.* $F = g_t$) we get the following

Proposition 4.2.4 Let $F\colon M \to M$ be a diffeomorphism. Consider a vector field $v \in \mathfrak{X}(M)$ and its induced local one parameter group g.

Assume that both W and $F(W)$ are open relatively compact subsets of M and set $\varepsilon > 0$ as above.

Then we have

$$F \circ g_t(p) = g_t \circ F(p)$$

for all p in W, for all $t \in J_\varepsilon$ if and only if, for all $p \in W$

$$dF_p(v_p) = v_{F(p)}.$$

This leads us to the following definition

[2] A subset of a topological space is relatively compact if its closure is compact.

Definition 4.2.5 Let $g\colon \Omega \to M$ be a local one parameter group of smooth transformations of U into M and let v be a vector field on M.

We say that g **leaves** v **invariant** if for any $(p,t) \in \Omega$

$$d(g_t)_p(v_p) = v_{g_t(p)}$$

It is easy to show that any local one parameter group of smooth transformations of U into M, $g\colon \Omega \to M$, leaves its associated vector field invariant.

Let U' be an open subset relatively compact in U and choose a vector field $w \in \mathfrak{X}(U)$.

We can *perturbate* w using g as follows. For all small t we define vector field w_t and dw_t/dt on U' by

$$w_t = (g_t)_* w \qquad\qquad \frac{dw_t}{dt}(f) = \frac{dw_t(f)}{dt}$$

This allows us to give the following geometrical interpretation of the Lie bracket as derivative of one vector field with respect to the one parameter group of transformations induced by the other vector field.

Proposition 4.2.6 For t small enough, on U', if v is the vector field induced by g, then

$$\frac{dw_t}{dt} = [w_t, v]$$

Proof. Set $Z_t := \frac{dw_t}{dt}$. We first show $Z_0 = [w,v]$.

Fix a point $p \in U'$. For small t, $(w_t)_p$ describes a path in T_pU' whose velocity at $t = 0$ is Z_0. Since $(w_t)_p = d(g_t)_q(w_q)$ for $q = g_{-t}(p)$ then

$$(Z_0)_p = \frac{d(w_t)_p}{dt}(0) = \lim_{t\to 0} \frac{(w_t)_p - w_p}{t} = \lim_{t\to 0} \frac{d(g_t)_{g_{-t}(p)}(w_{g_{-t}(p)}) - w_p}{t}$$

Then, for every $f \in C^\infty(U)$ we have

$$(Z_0)_p(f) = \lim_{t\to 0} \frac{d(g_t)_{g_{-t}(p)}(w_{g_{-t}(p)}) - w_p}{t}(f) = \lim_{t\to 0} \frac{(w_{g_{-t}(p)})(f\circ g_t) - w_p(f)}{t}$$

so

$$\begin{aligned}
Z_0(f) &= \lim_{t\to 0} \frac{w(f\circ g_t)\circ g_{-t} - w(f)}{t} \\
&= \lim_{t\to 0} \frac{(w(f\circ g_t) - w(f)\circ g_t)\circ g_{-t}}{t} \\
&= \lim_{t\to 0} \frac{w(f\circ g_t) - w(f)\circ g_t}{t} \\
&= \lim_{t\to 0} \frac{w(f\circ g_t) - w(f) - w(f)\circ g_t + w(f)}{t} \\
&= \lim_{t\to 0} \frac{w(f\circ g_t) - w(f)}{t} - \lim_{t\to 0} \frac{w(f)\circ g_t - w(f)}{t}
\end{aligned}$$

if the last two limits exist uniformly on U'.

The second limit is easy. By definition of v

$$v_p(w(f)) = dg_p\left(\frac{d}{dt}\right)_0 (w(f)) = \frac{d(w(f)\circ g_p)}{dt}(0)$$

so

$$v(w(f)) = \lim_{t\to 0} \frac{w(f)\circ g_t - w(f)}{t}$$

uniformly on U'.

For the first limit consider the smooth function $h = f\circ g$. Then $h \in C^\infty(U' \times J_\delta)$ for δ small enough.

Hence the function

$$H(p,t) = \begin{cases} \frac{f\circ g_t - f}{t} = \frac{h(p,t)-h(p,0)}{t} \text{ for } t \neq 0 \\ \frac{\partial h}{\partial t}(p,0) = v_p(f) \text{ for } t = 0 \end{cases}$$

belongs to $C^\infty(U' \times J_\delta)$. Hence

$$\lim_{t\to 0} \frac{w(f\circ g_t) - w(f)}{t} = w\left(\lim_{t\to 0} \frac{f\circ g_t - f}{t}\right) = w(v(f))$$

This implies then

$$Z_0 = [w,v] = [w_0, v]$$

on U'.

Finally, since g leaves v invariant, by definition of w_t it follows $(g_t)_* Z_0 = Z_t$ and then

$$(g_t)_*[w,v] = [(g_t)_* w, (g_t)_* v] = [w_t, v]$$

on U'. ■

Exercise 4.2.1 — Properties of the Lie bracket. Let $a,b \in \mathbb{R}$, $f \in C^\infty(U)$, $u,v,w \in \mathfrak{X}(U)$ Prove

1. $[v,w] = -[w,v]$
2. $[v,[w,u]] + [w,[u,v]] + [u,[v,w]] = 0$ **[Jacobi identity]**
3. $[au+bv,w] = a[u,w] + b[v,w]$
4. $[fv,w] = f[v,w] - w(f)v$

The first three properties give a structure of **Lie algebra** to $\mathfrak{X}(U)$.

Exercise 4.2.2 Prove that any local one parameter group of smooth transformations leaves its associated vector field invariant.

Exercise 4.2.3 — The bracket measures if local one parameter groups commute. Consider two local one parameter groups g,h of smooth transformations of U in M inducing respectively vector fields $v,w \in \mathfrak{X}(U)$. Show that $[v,w] = 0$ if and only if for every open subset U' relatively compact in U, for all $p \in U'$, for small enough t,s

$$g_t(h_s(p)) = g_s(h_t(p))$$

4.3 Frobenius Theorem

Consider a smooth vector field v on a manifold M and a point $p \in M$ such that $v_p \neq 0$. Then, in a neighborhood U of p, v does not vanish. Its local one parameter group produces a family of integral curves $g_p \colon J \to M$, embedded submanifolds of dimension 1 covering U (a **foliation** in curves) whose tangent spaces as subspaces of T_qM are generated by v_q for all $q \in U$.

We will in fact see that one can always choose as U a chart in p with local coordinates $x_1, \dots, x_n$ such that $v = \frac{\partial}{\partial x_1}$ and the foliation is given by the *lines* obtained fixing the values of all variables x_j, $j \geq 2$.

R So our foliation gives *locally* a partition of M in submanifold of dimension 1. This does not mean that we have a (global) partition of M in submanifolds of dimension 1. See Exercise 4.3.1 for an example.

Can we do this simultaneously for two vector fields v_1 and v_2? We note that, if there are local coordinates such that $v_1 = \frac{\partial}{\partial x_1}$ and $v_2 = \frac{\partial}{\partial x_2}$,then their Lie bracket vanish: $[v_1, v_2] = 0$ on U. So, if for example we have local coordinates such that $v_1 = \frac{\partial}{\partial x_1}$ and $v_2 = e^{x_1} \frac{\partial}{\partial x_1}$, since $[v_1, v_2] = e^{x_1} \frac{\partial}{\partial x_2} \neq 0$, then we have no hope to find coordinates $y_1, \dots, y_n$ such that *simultaneously* $v_j = \frac{\partial}{\partial y_j}$, $j = 1, 2$. Still, if we consider the *foliation* given by the surfaces obtained fixing the values of all variables x_j, $j \geq 3$ their tangent spaces, as subspaces of T_qM are generated by $v_1(q)$ and $v_2(q)$.

Now consider local vector fields $v_1 = \frac{\partial}{\partial x_1}$ and $v_2 = e^{x_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$. We compute $v_3 := [v_1, v_2] = e^{x_1} \frac{\partial}{\partial x_2}$ and note that $w_3(p)$ is not in the subspace of T_pM generated by $w_1(p)$ and $w_2(p)$. This implies that there is no embedded surface S in M such that $w_1(p)$ and $w_2(p)$ lie both in T_pS! In fact, if such a surface S would exist, then w_1 and w_2 would define vector fields on S whose bracket at any $p \in S$ would coincide, see the forthcoming proposition 4.3.2, with $w_3(p)$, so $w_1(p)$, $w_2(p)$ and $w_3(p)$ would stay in the plane T_pS, a contradiction.

To be more precise we need to introduce some definitions.

Definition 4.3.1 Let M be a manifold.

A **distribution** $\mathscr{D}$ of rank r on M is an embedded subbundle of T_pM of rank r.

In particular a distribution $\mathscr{D}$ of rank r gives, for all $p \in M$, a vector subspace $\mathscr{D}_p$ of M of dimension r. Since subbundles are locally trivial, for each $p \in M$ there is a neighborhood U of p and smooth vector fields $v_1, \dots, v_r \in \mathfrak{X}(U)$ generating $\mathscr{D}_q$ at each $q \in U$.

We say that a vector field $v \in \mathfrak{X}(M)$ **lies** on $\mathscr{D}$ if $v_p \in \mathscr{D}_p$ for all $p \in M$.

A distribution $\mathscr{D}$ is **involutive** if for every pair of vector fields v_1, v_2 lying on $\mathscr{D}$, their Lie bracket $[v_1, v_2]$ lies on $\mathscr{D}$ as well.

An embedded subvariety $\iota \colon S \hookrightarrow M$ is an **integral submanifold** of $\mathscr{D}$ if for all $p \in S$, $d\iota_p(T_pS) = \mathscr{D}_p$.

A distribution is **integrable** if all points of M are contained in an integral submanifold.

A chart (U, φ) inducing local coordinates $x_1, \dots, x_n$ is **flat** for a distribution $\mathscr{D}$ of rank r if $\mathscr{D}_p$ is generated by $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_r}\right)_p$ at all points $p \in U$.

A distribution $\mathscr{D}$ is **completely integrable** if for each point $p \in M$ there is chart at p flat for $\mathscr{D}$.

It is easy to see that

$$\text{completely integrable} \Rightarrow \text{integrable} \Rightarrow \text{involutive}$$

The first implication is obvious, the second is following

Proposition 4.3.2 If a distribution is integrable, then for any two vector fields lying on it their Lie bracket lies on it too.

Proof. Let $S \subset M$ be an integral manifold of $\mathscr{D}$. Suppose that v, w are vector fields lying on $\mathscr{D}$. The restrictions $v_{|S}$, $w_{|S}$ as section of $T_{S|N}$ are sections of the subbundle T_S and then they define vector fields on S.

By definition, the restriction of the Lie Bracket $[v,w]$ to S, as section of $T_{S|N}$, equals the Lie Bracket $[v_{|S}, w_{|S}]$ of vector fields on S, and therefore it lies on $\mathscr{D}$ as well. ∎

The main result of this section, the Frobenius Theorem 4.3.6, says that these three properties are equivalent.

R This is related with the problem of solving the following type of differential equations. Consider an open subset $U \subset \mathbb{R}^n$, and look for functions $u \in C^\infty(U)$ solving a system of r differential equations of the form

$$\sum_{k=1}^{n} v_i^k(x) \frac{\partial u}{\partial x_k}(x) = 0 \quad i = 1, \ldots, r$$

Consider the r vector fields $v_1, \ldots, v_r \in \mathfrak{X}(U)$ defined as $v_i = \sum_k v_i^k \frac{\partial}{\partial x_k}$, and assume that they are all contained in a completely integrable distribution $\mathscr{D}$ of rank $h \neq n$. Then we have local coordinates $y_1, \ldots, y_h, y_{h+1}, \ldots, y_n$ such that all v_k belong to the subspace generated by $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_h}$ and every function depending only on the last coordinates $y_{h+1}, \ldots, y_n$ solve the given differential equations. One can in fact prove that every solution is of this form, and therefore that if the only involutive distribution containing our vector fields is the whole tangent bundle, then no non trivial solution exists.

To prove Frobenius Theorem 4.3.6 we need a more general version of Theorem 4.1.5, considering, roughly speaking, Cauchy problems depending on m parameters. The following propositions 4.3.3 and 4.3.4 are proven in the book of Narasimhan *Analysis on real and complex manifolds*, Chapter 1, Section 8.

Proposition 4.3.3 Consider open subsets $J \subset \mathbb{R}$, $V \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$.

Consider a smooth map $F\colon U \times J \times V \to \mathbb{R}^n$.

Then for all $t_0 \in J$, $p \in U$, for each relatively compact open subset $V' \subset V$, there exists a $\delta > 0$ and a smooth function

$$u\colon J_\delta \times V' \to U$$

such that

$$u(t_0, y) = p \qquad\qquad F(u(t,y),t,y) = \frac{\partial u}{\partial t}(t,y)$$

Moreover two such functions coincide in the common domain.

The "solution" u depends on "starting point" p. In fact u is smooth also as a function of the starting point. Namely Narasimhan proves

Proposition 4.3.4 Consider J, V, U, F as in Proposition 4.3.3.

Then for every $t_0 \in J$, $p \in U$, $y_0 \in V$ there are open subsets $t_0 \in J' \subset J$, $p \in U' \subset U$, $y_0 \in V' \subset V$ and a smooth function

$$\tilde{u}\colon U' \times J' \times J' \times V' \to U$$

such that, for all $q \in U'$, for all $y \in V'$, for all $t,s \in J'$,

$$\tilde{u}(q,s,s,y) = q \qquad F(\tilde{u}(q,t,s,y),t,y) = \frac{\partial \tilde{u}}{\partial t}(q,t,s,y)$$

Proposition 4.3.3 may be seen as a special case of Proposition 4.3.4 by setting

$$u(t,y) := \tilde{u}(p,t,t_0,y)$$

In fact Proposition 4.3.4 may be seen as a generalization of Proposition 4.3.3 obtained replacing the "fixed" point $p \in U$ by a smooth function $p \colon V \to U$. In other words, if we let the starting point to vary with the parameters. In fact, in that case, setting

$$u(t,y) = \tilde{u}(p(y),t,t_0,y)$$

we obtain

$$u(t_0,y) = p(y) \qquad F(u(t,y),t,y) = \frac{\partial u}{\partial t}(t,y)$$

We can now prove the following lemma.

Lemma 4.3.5 Consider open subsets $V \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ with respective coordinates $x_1,\dots,x_m$ and $y_1,\dots,y_n$.

Consider smooth functions $v_i \colon V \times U \to \mathbb{R}^n$, $i = 1,\dots,m$ and set v_i^k for the k^{th} component $y_k \circ v_i$ of v_i.

Then, for each $\bar{x} \in V$ and $\bar{y} \in U$ there exists an open subset $W \subset V$ containing $\bar{x}$ and a smooth function $u \colon W \to U$ such that

$$\begin{cases} u(\bar{x}) = \bar{y} \\ \frac{\partial u}{\partial x_i}(x) = v_i(x,u(x)) \text{ for all } x \in W \text{ and } i = 1,\dots,m \end{cases} \tag{4.4}$$

if and only if

$$\frac{\partial v_i}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial v_i}{\partial y_k} v_j^k = \frac{\partial v_j}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial v_j}{\partial y_k} v_i^k \text{ for all } i,j \in \{1,\dots,m\} \tag{4.5}$$

in $V \times U$.

If $u \colon W \to U$ and $u' \colon W' \to U$ are two functions as above, then they coincide on the connected component of $W \cap W'$ containing $\bar{x}$.

Proof. Assume that there exists a function u solving Problem (4.4). Then

$$\frac{\partial^2 u}{\partial x_j \partial x_i}(x) = \frac{\partial}{\partial x_j} v_i(x,u(x)) = \frac{\partial v_i}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial v_i}{\partial y_k} v_j^k$$

The conditions (4.5) follow immediately by Schwarz Theorem.

Conversely assume (4.5). We assume moreover for sake of simplicity $\bar{x} = 0$.

Several arguments of the forthcoming proof hold only *up to shrinking* V, up to substituting V with a smaller open neighborhood of $\bar{x}$. After all these shrinkings we obtain the open subset W of V in the statement of the Lemma. We are going to deliberately ignore this technical point of the proof to make more evident the idea behind the proof.

We first construct u on the curve $x_2 = \cdots = x_m = 0$.

We consider the Cauchy problem

$$\begin{cases} y(0) = \bar{y} \\ \frac{dy}{dt}(t) = v_1(t,0,\dots,0,y(t)). \end{cases}$$

By Theorem 4.1.5 we get a unique smooth solution β_1 for $|t| < \delta_1$. We want to extend it to u in the sense that $u(x_1,0,\cdots,0) = \beta_1(x_1)$: we notice that for any such extension (4.4) would hold for $i = 1$ on $x_2 = \cdots = x_m = 0$.

Now we proceed inductively on m. Assume that we have a smooth function β_{k-1} on $x_k = \cdots = x_m = 0$. By Proposition 4.3.3 in the generalized form discussed in the remark after Proposition 4.3.4 there is a unique solution β_k of the system of differential equations

$$\begin{cases} y(0;x_1,\dots,x_{k-1}) = \beta_{k-1}(x_1,\dots,x_{k-1}) \\ \frac{\partial y}{\partial t}(t;x_1,\dots,x_{k-1}) = v_k(x_1,\dots,x_{k-1},t,0,\dots,0,y(t;x_1,\dots,x_{k-1})) \end{cases}$$

We want to extend u on $x_{k+1} = \cdots = x_m = 0$ by $u(x_1,\dots,x_k,0,\dots,0) = \beta_k(x_k;x_1,\dots,x_{k-1})$. We notice that for any such extension (4.4) would hold for $i = k$ on $x_{k+1} = \cdots = x_m = 0$:

$$\frac{\partial u}{\partial x_k}(x_1,\dots,x_k,0,\dots,0) = v_k((x_1,\dots,x_k,0,\dots,0),u((x_1,\dots,x_k,0,\dots,0)))$$

However, we need to prove that (4.4) holds also for $i < k$ on $x_{k+1} = \cdots = x_m = 0$; in other words we need to prove the analogous equality $\frac{\partial u}{\partial x_i} = v_i$ holds, even when $x_k \neq 0$.

To a fixed such i we associate the function h

$$\begin{aligned} &h(t;x_1,\dots,x_{k-1}) := \\ &\quad = \frac{\partial u}{\partial x_i}(x_1,\dots,x_{k-1},t,0,\dots,0) - v_i(x_1,\dots,x_{k-1},t,0,\dots,0,u(x_1,\dots,x_{k-1},t,0,\dots,0)) \end{aligned}$$

By Schwarz Theorem and (4.5)

$$\begin{aligned} \frac{d}{dt}h(t;x_1,\dots,x_{k-1}) &= \frac{\partial^2 u}{\partial x_k \partial x_i} - \frac{\partial v_i}{\partial x_k} - \sum_l \frac{\partial u_l}{\partial x_k}\frac{\partial v_i}{\partial y_l} \\ &= \frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial v_i}{\partial x_k} - \sum_l v_k^l \frac{\partial v_i}{\partial y_l} \\ &= \frac{\partial v_k}{\partial x_i} + \sum_l \frac{\partial u_l}{\partial x_i}\frac{\partial v_k}{\partial y_l} - \frac{\partial v_i}{\partial x_k} - \sum_l \frac{\partial v_i}{\partial y_l} v_k^l \\ &= \frac{\partial v_k}{\partial x_i} + \sum_l \left(h_l + v_i^l\right)\frac{\partial v_k}{\partial y_l} - \frac{\partial v_i}{\partial x_k} - \sum_l \frac{\partial v_i}{\partial y_l} v_k^l \\ &= \sum_l h_l(t;x_1,\dots,x_{k-1})\frac{\partial v_k}{\partial y_l} \end{aligned}$$

exhibiting h as solution of an ordinary differential equation. Moreover, by the inductive assumption $h(0;x_1,\dots,x_{k-1}) = 0$.

We can deduce by Proposition 4.3.3 that $h(t;x_1,\dots,x_{k-1}) = 0$, since both h and the zero function solve the same differential equations with the same data. This proves the claim that (4.4) holds also for $i < k$ on $x_{k+1} = \cdots = x_m = 0$.

Then by induction we obtain a function u as required. The uniqueness follows again by the uniqueness assertion in Proposition 4.3.3. ■

We can now use it to prove Frobenius Theorem

Theorem 4.3.6 — Frobenius Theorem. A distribution is completely integrable if and only if it is involutive.

Proof. We have only to prove that every involutive distribution $\mathscr{D}$ is completely integrable.

So let $\mathscr{D}$ be an involutive distribution on M. We fix a point p in M. We will construct a chart in p flat for $\mathscr{D}$.

Let r be the rank of $\mathscr{D}$. Then there are, for a suitable open neighborhood U of p in M, r vector fields $\tilde{v}_1, \ldots, \tilde{v}_r$ generating $\mathscr{D}$ on each point $q \in U$. Shrinking U we can take local coordinates $x_1, \ldots, x_n$ on it and write

$$\tilde{v}_i = \sum_{j=1}^{n} \tilde{v}_i^j \frac{\partial}{\partial x_j}$$

The smooth functions $\tilde{v}_i^j$ are naturally arranged in a matrix with r rows and n columns, $r \leq n$, of rank r at each point $q \in U$. Permuting the coordinates, we ensure that the left $r \times r$ minor does not vanish at p; in other words

$$\det\left(\tilde{v}_i^j(p)\right)_{1 \leq i,j \leq r} \neq 0$$

Shrinking Ω if necessary, we can assume that $\det\left(\tilde{v}_i^j\right)_{1 \leq i,j \leq r}$ never vanishes. So there is a $r \times r$ matrix $A = (a_{ij})$ of smooth functions $a_{ij} \in C^\infty(U)$ such that

$$A\left(\tilde{v}_i^j\right)_{1 \leq i,j \leq r} = \left(\tilde{v}_i^j\right)_{1 \leq i,j \leq r} A = I_r$$

Define $\bar{v}_i := \sum_j a_{ij} \tilde{v}_j$. Since A is invertible, $\bar{v}_1, \ldots, \bar{v}_r$ generate $\mathscr{D}$ at each point. We rename the coordinates, setting $y_j := x_{r+j}$. So the local coordinates are from now on named $x_1, \cdots, x_r$, $y_1, \cdots, y_{n-r}$. Then, by definition of A

$$\bar{v}_i = \frac{\partial}{\partial x_i} + \sum_{h=1}^{n-r} v_i^h \frac{\partial}{\partial y_h}$$

Note that this implies that, for all $q \in U$, $\mathscr{D}_q \cap \left\langle \left(\frac{\partial}{\partial y_j}\right)_q \right\rangle = 0$. We compute

$$[\bar{v}_i, \bar{v}_j] = \sum_{s=1}^{n-r} \left(\frac{\partial v_j^s}{\partial x_i} + \sum_{h=1}^{n-r} \frac{\partial v_j^s}{\partial y_h} v_i^h - \frac{\partial v_i^s}{\partial x_j} - \sum_{h=1}^{n-r} \frac{\partial v_i^s}{\partial y_h} v_j^h \right) \frac{\partial}{\partial y_s}.$$

Since $\mathscr{D}$ is involutive and $\mathscr{D}_q \cap \left\langle \left(\frac{\partial}{\partial y_j}\right)_q \right\rangle = 0$, we deduce that $[\bar{v}_i, \bar{v}_j] = 0$.

In other words for all s

$$\frac{\partial v_j^s}{\partial x_i} + \sum_{h=1}^{n-r} \frac{\partial v_j^s}{\partial y_h} v_i^h - \frac{\partial v_i^s}{\partial x_j} - \sum_{h=1}^{n-r} \frac{\partial v_i^s}{\partial y_h} v_j^h = 0$$

Writing $v_i := \left(v_i^1, \ldots, v_i^{n-r}\right)$, we obtain

$$\frac{\partial v_j}{\partial x_i} + \sum_{h=1}^{n-r} \frac{\partial v_j}{\partial y_h} v_i^h = \frac{\partial v_i}{\partial x_j} + \sum_{h=1}^{n-r} \frac{\partial v_i}{\partial y_h} v_j^h$$

that is exactly (4.5).

To apply Lemma 4.3.5, we shrink U to a product $V \times U'$ where V has dimension r, corresponding to the coordinates x_j, and U' has dimension $n-r$, corresponding to the coordinates y_j. Accordingly we write $p = (\bar{x}, \bar{y})$. Then, up to further shrinking V, there is a function $u\colon V \to U'$ such that

$$\begin{cases} u(\bar{x}) = \bar{y} \\ \frac{\partial u}{\partial x_i}(x) = v_i(x, u(x)) \end{cases}$$

Without loss of generality we can assume $x_i(p) = y_j(p) = 0$ for all i, j, so $\bar{x} = 0$, $\bar{y} = 0$. Then we consider the map $\varphi(x,y) = (x, y - u(x))$. Its differential $d\varphi_p$ at p is invertible and a direct straightforward computation shows

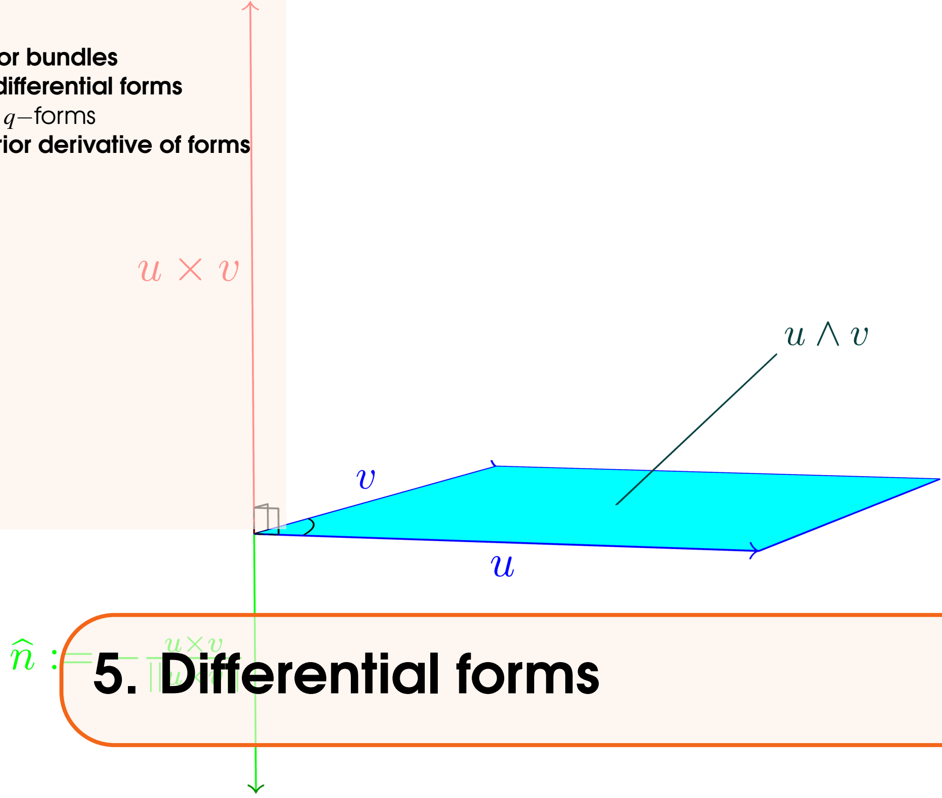
$$d\varphi_p(\bar{v}_i) = \frac{\partial}{\partial x_i}.$$

Then, composing the original chart with φ we obtain a chart in p flat for $\mathscr{D}$. ■

Exercise 4.3.1 Consider the lines $l_{a,b,c}$ in Exercise 2.3.1 and their image $\pi(l_{a,b,c})$ in the torus T.

Prove that for every fixed $a, b \in \mathbb{R}$ there is a distribution $\mathscr{D}_{a,b}$ on T such that the $\pi(l_{a,b,c})$ locally describe integral submanifolds of $\mathscr{D}_{a,b}$.

Recall that, as seen in Exercise 2.3.1, if $\frac{a}{b} \notin \mathbb{Q}$, then $\pi(l_{a,b,c})$ is not a submanifold of T because it is dense in T.



5. Differential forms

5.1 Multilinear algebra

In this section we develop some tools in advanced linear algebra.

Let $V_1, \ldots, V_q$ be finite dimensional vector spaces over a field $\mathbb{K}$. For sake of simplicity we will always assume that $\mathbb{K}$ has characteristic zero; this includes $\mathbb{R}$ and $\mathbb{C}$.

Definition 5.1.1 A map

$$\omega\colon V_1 \times V_2 \times \cdots \times V_q \to \mathbb{K}$$

is **multilinear** or ***q*-linear** or **a tensor of degree *q*** if the following holds: $\forall i \in \{1, \ldots, q\}$ and for every choice, $\forall j \neq i$, of vectors $v_j \in V_j$, the induced map

$$\psi\colon V_i \to \mathbb{K}$$

defined by, $\forall v \in V_i$, $\psi(v) = \omega(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_q)$, is linear.

Example 5.1 The tensors of degree 1 form the dual space V_1^* of V_1.

Example 5.2 The tensors of degree 2 are bilinear maps as, *e.g.*, the standard scalar product on $\mathbb{R}^2$ or $\mathbb{R}^3$ (in which case $V_1 = V_2$).

Indeed, $q-$linearity is the natural generalization of the idea of bilinearity to the case of more than two (but still finitely many) factors.

Example 5.3 If you know the cross product $\times$ in $\mathbb{R}^3$ you may prove that the map

$$(v_1, v_2, v_3) \mapsto (v_1 \times v_2) \cdot v_3$$

defines a tensor of degree 3.

Example 5.4 For every $n \geq 1$ the map $\det\colon (\mathbb{R}^n)^n \to \mathbb{R}$ associating, to each ordered list of n vectors in $\mathbb{R}^n$, the determinant of the matrix whose columns are them, in the same order is a tensor of degree n.

Definition 5.1.2 The space of multilinear maps from $V_1 \times V_2 \times \cdots \times V_q$ to $\mathbb{K}$ is a vector space (see Complement 5.1.2), which is the **tensor product** of V_1^*, V_2^*, ..., V_q^* and is denoted by

$$V_1^* \otimes V_2^* \otimes \cdots \otimes V_q^*$$

R Definition 5.1.2, in the case $q = 1$, gives the vector space V_1^* of all linear maps from V_1 to $\mathbb{K}$: the **dual** space of V_1.

R The expression, $\forall v \in V$ and $\forall \varphi \in V^*$,

$$v(\varphi) := \varphi(v)$$

defines a map $V \to V^{**}$, which is in general (not assuming finite dimensionality) not surjective.

We are assuming V finite dimensional. For each basis $\{e_1, \ldots, e_n\}$ of V, the set of the elements $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of V^* determined by the formula $\varepsilon_i(e_j) = \delta_{ij}$ is easily shown to be a basis of V^*, the **dual basis** of $\{e_1, \ldots, e_n\}$. In particular, if the dimension of V is finite, then V has the same dimension of V^* and therefore of V^{**}.

Moreover, the map $V \to V^{**}$ at the beginning of this remark is obviously injective and therefore an isomorphism. So we may and will use this map to canonically identify V with V^{**}.

Definition 5.1.3 We define then

$$V_1 \otimes V_2 \otimes \cdots \otimes V_q := V_1^{**} \otimes V_2^{**} \otimes \cdots \otimes V_q^{**}$$

There are some very special elements in $V_1^* \otimes V_2^* \otimes \cdots \otimes V_q^*$.

Definition 5.1.4 Choose $\forall 1 \leq i \leq q$, an element $\varphi_i \in V_i^*$.
Then define $\varphi_1 \otimes \cdots \otimes \varphi_q$ by

$$\varphi_1 \otimes \cdots \otimes \varphi_q(v_1, \ldots, v_q) = \varphi_1(v_1) \cdot \varphi_2(v_2) \cdot \cdots \cdot \varphi_q(v_q).$$

These are the **decomposable tensors** in $V_1^* \otimes V_2^* \cdots \otimes V_q^*$.

Note that $\left(\sum_{j_1} a_{1j_1} \varphi_{1j_1}\right) \otimes \cdots \otimes \left(\sum_{j_q} a_{qj_q} \varphi_{qj_q}\right) = \sum a_{1j_1} \cdots a_{qj_q} \varphi_{1j_1} \otimes \cdots \otimes \varphi_{qj_q}$.

We fix bases $\{e_{i1}, \ldots, e_{in_i}\}$ of each space V_i, and we consider the corresponding **dual basis** $\{\varepsilon_{i1}, \ldots, \varepsilon_{in_i}\}$ of V_i^*. They are uniquely determined by the formula $\varepsilon_{ij}(e_{ij'}) = \delta_{jj'}$.

Theorem 5.1.5 The set of decomposable tensors

$$\{\varepsilon_{1i_1} \otimes \varepsilon_{2i_2} \otimes \cdots \otimes \varepsilon_{qi_q}\}$$

form a basis of $V_1^* \otimes V_2^* \cdots \otimes V_q^*$. In particular

$$\dim\left(V_1^* \otimes V_2^* \cdots \otimes V_q^*\right) = (\dim V_1)(\dim V_2) \cdots (\dim V_q).$$

Proof. We skip this proof as it is very similar to the proof of the forthcoming Theorem 5.1.12. ∎

A special case is the complexification of a finitely dimensional real vector space V, as defined in Definition 3.4.1.

Then $V \otimes_{\mathbb{R}} \mathbb{C}$ is the real vector space obtained as in Definition 5.1.3 considering $\mathbb{C}$ as vector space of dimension 2 over $\mathbb{R}$.

It has a natural structure of complex vector space with scalar multiplication by complex numbers defined[1] on the decomposable tensors by

$$\forall \lambda, \mu \in \mathbb{C}\ \forall v \in V \quad \lambda(v \otimes \mu) = v \otimes (\lambda \mu).$$

If $\{e_j\}$ is a basis of V (over $\mathbb{R}$), then $\{e_j \otimes 1\}$ is a basis of $V \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ and $\{e_j \otimes 1\} \cup \{e_j \otimes i\}$ is a basis[2] of $V \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{R}$.

It is natural to consider V embedded in $V \otimes_{\mathbb{R}} \mathbb{C}$ via $v \mapsto v \otimes 1$ writing μv for $v \otimes \mu$. So $\{e_j\}$ is at the same time a basis of V over $\mathbb{R}$ and $V \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ whereas a basis of $V \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{R}$ is the set $\{e_j, ie_j\}$.

The following construction will be useful in the next chapters.

Definition 5.1.6 Consider vector spaces V_1, V_2, W_1, W_2, and linear applications $L_j \colon V_j \to W_j$, $j \in \{1,2\}$. Then there is a unique linear application

$$L_1 \otimes L_2 \colon V_1 \otimes V_2 \to W_1 \otimes W_2$$

such that $\forall v_1 \in V_1$, $\forall v_2 \in V_2$,

$$(L_1 \otimes L_2)(v_1 \otimes v_2) = L_1(v_1) \otimes L_2(v_2). \tag{5.1}$$

Definition 5.1.6 requires the following proof of existence and uniqueness.

Proof. Fix respective bases $\{v_{1j}\}$ of V_1 and $\{v_{2k}\}$ of V_2. By Theorem 5.1.5, $\{v_{1j} \otimes v_{2k}\}$ is a basis of $V_1 \otimes V_2$ so, if $L_1 \otimes L_2$ exists, it is the unique linear application $L \colon V_1 \otimes V_2 \to W_1 \otimes W_2$ such that $L(v_{1j} \otimes v_{2k}) = L_1(v_{1j}) \otimes L_2(v_{2k})$. This shows uniqueness.

The existence follows since (see Complement 5.1.4)

$$\begin{aligned}
L\left(\left(\sum_j a_{1j} v_{1j}\right) \otimes \left(\sum_k a_{2k} v_{2k}\right)\right) &= L\left(\sum_{j,k} a_{1j} a_{2k} v_{1j} \otimes v_{2k}\right) \\
&= \sum_{j,k} a_{1j} a_{2k} L(v_{1j} \otimes v_{2k}) \\
&= \sum_{j,k} a_{1j} a_{2k} L_1(v_{1j}) \otimes L_2(v_{2k}) \\
&= \left(\sum_j a_{1j} L_1(v_{1j})\right) \otimes \left(\sum_k a_{2k} L_2(v_{2k})\right) \\
&= L_1\left(\sum_j a_{1j} v_{1j}\right) \otimes L_2\left(\sum_k a_{2k} v_{2k}\right).
\end{aligned}$$ ∎

The space of the linear applications among two fixed vector spaces can be interpreted as a tensor product as follows.

Proposition 5.1.7 Consider two finitely dimensional vector spaces V and W on the same field. Then there is a canonical isomorphism of vector spaces $W \otimes V^* \to \mathrm{Hom}_{\mathbb{K}}(V,W)$ such that

[1] This extends to the complex numbers the scalar multiplication by real numbers of the real vector space $V \otimes_{\mathbb{R}} \mathbb{C}$. Indeed, for λ real, the equality holds by Definition 5.1.4.

[2] Here $i \in \mathbb{C}$ denotes, as usual, a square root of -1.

every decomposable tensor $w\otimes\varphi$ is mapped on the homomorphism

$$v\mapsto\varphi(v)w.$$

Proof. The proof of the existence and uniqueness of a linear application as stated follows the same strategy of the proof that the Definition 5.1.6 is well posed. Its injectivity is trivial while its surjectivity follows by a dimension count using Theorem 5.1.5.

The reader can easily complete the proof writing the missing details. ■

The most important case for our purposes is the case

$$V_1=\cdots=V_q=:V.$$

In this case we use the shorter form $(V^*)^{\otimes q}$ for $V^*\otimes\cdots\otimes V^*$.

Definition 5.1.8 A tensor $\omega\in(V^*)^{\otimes q}$ is **symmetric** if its value does not depend on the order of the vectors. In other words, if $\forall i\neq j$,

$$\omega(\ldots,v_i,\ldots,v_j,\ldots)=\omega(\ldots,v_j,\ldots,v_i,\ldots).$$

Similarly, $\omega\in(V^*)^{\otimes q}$ is an **alternating tensor** or a **skew tensor** or a **skew form** if, $\forall i\neq j$,

$$\omega(\ldots,v_i,\ldots,v_j,\ldots)=-\omega(\ldots,v_j,\ldots,v_i,\ldots).$$

The symmetric tensors form a vector subspace of $(V^*)^{\otimes q}$ usually denoted $\mathrm{Sym}^q V^*$. The skew tensors form a vector subspace of it usually denoted $\Lambda^q V^*$.

For later convenience we define conventionally $(V^*)^{\otimes 0}=\mathrm{Sym}^0 V^*=\Lambda^0 V^*=\mathbb{K}$.

If $\dim V$ is finite, dualizing twice as in Definition 5.1.3 we obtain $\mathrm{Sym}^q V$ and $\Lambda^q V$.

We are mostly interested in $\Lambda^q V^*$. Note $\Lambda^0 V^*=\mathbb{K}$ and $\Lambda^1 V^*=V^*$. In higher degree an important example of skew tensor is the determinant, seen as tensor of degree $\dim V$ in Example 5.4.

Let's construct some elements in $\Lambda^2 V^*$. For general $\varphi_1,\varphi_2\in V^*$, $\varphi_1\otimes\varphi_2$ is not skew since there is no reason for $\varphi_1(v_1)\varphi_2(v_2)$ to be equal to $-\varphi_2(v_1)\varphi_1(v_2)$. Then we use an averaging procedure.

Definition 5.1.9 $\forall\varphi_1,\varphi_2\in V^*$ we define[a] $\varphi_1\wedge\varphi_2=\frac{1}{2}(\varphi_1\otimes\varphi_2-\varphi_2\otimes\varphi_1)\in\Lambda^2V^*$.

We can equivalently write $\varphi_1\wedge\varphi_2$ in the form

$$\begin{array}{rccc}\varphi_1\wedge\varphi_2\colon & V\times V & \to & \mathbb{K}\\ & (v_1,v_2) & \mapsto & \frac{1}{2}\det\begin{pmatrix}\varphi_1(v_1) & \varphi_1(v_2)\\ \varphi_2(v_1) & \varphi_2(v_2)\end{pmatrix}\end{array}$$

[a] Some author use the alternative definition $\varphi_1\wedge\varphi_2=\varphi_1\otimes\varphi_2-\varphi_2\otimes\varphi_1$. That choice forces a series of small changes in several of the forthcoming definitions to ensure that the Grassmann algebra is a graded algebra. The two resulting algebras are isomorphic.

This is the **wedge product** of φ_1 and φ_2 and may be seen as a map

$$\begin{array}{rccc}\wedge\colon & \Lambda^1V^*\times\Lambda^1V^* & \to & \Lambda^2V^*\\ & (\varphi_1,\varphi_2) & \mapsto & \varphi_1\wedge\varphi_2\end{array}$$

There is a natural extension of this idea to the $\Lambda^q V^*$.

Definition 5.1.10 We define the **wedge product**

$$\begin{array}{cccc} \wedge : & \Lambda^{q_1}V^* \times \Lambda^{q_2}V^* & \to & \Lambda^{q_1+q_2}V^* \\ & (\omega_1, \omega_2) & \mapsto & \omega_1 \wedge \omega_2 \end{array}$$

as follows[a]:

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{q_1+q_2}) =$$
$$= \frac{1}{(q_1+q_2)!} \sum_{\sigma \in \Sigma_{q_1+q_2}} \varepsilon(\sigma)\omega_1(v_{\sigma(1)}, \dots, v_{\sigma(q_1)})\omega_2(v_{\sigma(q_1+1)}, \dots, v_{\sigma(q_1+q_2)})$$

where Σ_k is the group of the permutations of $\{1, \dots, k\}$.

[a] $\varepsilon(\sigma) \in \{\pm 1\}$ is the *sign* of the permutation σ. If σ is the product of l transpositions, $\varepsilon(\sigma) = (-1)^l$. Every permutation σ can be written in many different ways as product of transpositions, and the number l of these transpositions may vary. However $\varepsilon(\sigma)$ is well defined since the parity of l only depends on σ: the reader can find a proof of it in any basic book of group theory.

Note that Definition 5.1.10 makes sense also when $q_1 = 0$ (and/or $q_2 = 0$), in which case $\omega_1 = \lambda \in \mathbb{K}$ and $\omega_1 \wedge \omega_2 = \lambda \omega_2$.

The wedge product has useful properties (see Complement 5.1.12). For example, it is associative, $(k\omega_1) \wedge \omega_2 = k(\omega_1 \wedge \omega_2) = \omega_1 \wedge k\omega_2$, $\omega_1 \wedge \omega_2 = (-1)^{q_1 q_2} \omega_2 \wedge \omega_1$. In particular we can write $k\omega_1 \wedge \cdots \wedge \omega_j$ without ambiguity. When all the ω_i are $1-$forms this has a nice expression.

Proposition 5.1.11 Assume $\varphi_1, \cdots, \varphi_q \in V^*$.
Then

$$\varphi_1 \wedge \cdots \wedge \varphi_q(v_1, \dots, v_q) = \frac{1}{q!} \sum_{\sigma \in \Sigma_q} \varepsilon(\sigma) \prod_{i=1}^{q} \varphi_i(v_{\sigma(i)}) = \frac{1}{q!} \det(\varphi_i(v_j)).$$

where $(\varphi_i(v_j))$ denotes the matrix

$$\begin{pmatrix} \varphi_1(v_1) & \cdots & \varphi_1(v_q) \\ \vdots & \ddots & \vdots \\ \varphi_q(v_1) & \cdots & \varphi_q(v_q) \end{pmatrix}.$$

Proof. The second equality is just the Laplace expansion of the determinant.

We prove the first equality by induction on q. If $q = 1$ the equality becomes the tautology $\varphi_1(v_1) = \varphi_1(v_1)$: there is nothing to prove.

We may then assume the formula true for $q-1$: $\forall w_1, \dots, w_{q-1} \in V$

$$\varphi_1 \wedge \cdots \wedge \varphi_{q-1}(w_1, \dots, w_{q-1}) = \frac{1}{(q-1)!} \sum_{\eta' \in \Sigma_{q-1}} \varepsilon(\eta') \prod_{i=1}^{q-1} \varphi_i(w_{\eta'(i)}). \tag{5.2}$$

We compute the wedge product of $\varphi_1 \wedge \cdots \wedge \varphi_{q-1}$ and φ_q by Definition 5.1.10.

$$(\varphi_1 \wedge \cdots \wedge \varphi_{q-1}) \wedge \varphi_q(v_1, \dots, v_q) =$$
$$= \frac{1}{q!} \sum_{\eta \in \Sigma_q} \varepsilon(\eta) \varphi_1 \wedge \cdots \wedge \varphi_{q-1}(v_{\eta(1)}, \dots, v_{\eta(q-1)}) \varphi_q(v_{\eta(q)})$$

We then apply the inductive assumption (5.2) for $(w_1,\dots,w_{q-1}) = (v_{\eta(1)},\dots,v_{\eta(q-1)})$. Note that, for each i, $w_i = v_{\eta(i)}$ and then, for all $\eta' \in \Sigma_{q-1}$, $w_{\eta'(i)} = v_{\eta(\eta'(i))}$. So

$$\begin{aligned}
&(\varphi_1 \wedge \cdots \wedge \varphi_{q-1}) \wedge \varphi_q(v_1,\dots,v_q) = \\
&= \frac{1}{q!} \sum_{\eta\in\Sigma_q} \varepsilon(\eta) \left(\frac{1}{(q-1)!} \sum_{\eta'\in\Sigma_{q-1}} \varepsilon(\eta') \prod_{i=1}^{q-1} \varphi_i(v_{\eta(\eta'(i))}) \right) \varphi_q(v_{\eta(q)}) = \\
&\qquad = \frac{1}{q!(q-1)!} \sum_{\eta\in\Sigma_q,\ \eta'\in\Sigma_{q-1}} \varepsilon(\eta)\varepsilon(\eta') \left(\prod_{i=1}^{q-1} \varphi_i(v_{\eta\circ\eta'(i)}) \right) \varphi_q(v_{\eta(q)}). \qquad (5.3)
\end{aligned}$$

We consider each permutation $\eta' \in \Sigma_{q-1}$ as a member of Σ_q which fixes q. Then $\eta\circ\eta' \in \Sigma_q$ and (5.3) may be written as

$$\varphi_1 \wedge \cdots \wedge \varphi_q(v_1,\dots,v_q) = \frac{1}{q!(q-1)!} \sum_{\eta\in\Sigma_q,\ \eta'\in\Sigma_{q-1}} \varepsilon(\eta\circ\eta') \left(\prod_{i=1}^{q} \varphi_i(v_{\eta\circ\eta'(i)}) \right).$$

Each summand in the right-hand term do not really depend on η and η', but just on $\sigma := \eta\circ\eta'$. Varying $(\eta,\eta') \in \Sigma_q \times \Sigma_{q-1}$ we obtain each $\sigma \in \Sigma_q$ exactly $(q-1)!$ times, and therefore

$$\begin{aligned}
\varphi_1 \wedge \cdots \wedge \varphi_q(v_1,\dots,v_q) &= \frac{1}{q!(q-1)!} \sum_{\sigma\in\Sigma_q} (q-1)!\varepsilon(\sigma) \left(\prod_{i=1}^{q} \varphi_i(v_{\sigma(i)}) \right) \\
&= \frac{1}{q!} \sum_{\sigma\in\Sigma_q} \varepsilon(\sigma) \left(\prod_{i=1}^{q} \varphi_i(v_{\sigma(i)}) \right) \qquad \blacksquare
\end{aligned}$$

From now on we fix a basis $e_1,\dots,e_n$ of V, and we denote by $\varepsilon_1,\dots,\varepsilon_n$ the dual basis of V^*: then $\varepsilon_i(e_j) = \delta_{ij}$. Since $V^* = \Lambda^1(V^*)$ the ε_i are 1-forms.

From the properties in Complement 5.1.12 follow few very useful *rules*:

- $\varepsilon_i \wedge \varepsilon_j = -\varepsilon_j \wedge \varepsilon_i$;
- $\varepsilon_i \wedge \varepsilon_i = 0$;
- $\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_q} \wedge \varepsilon_j = (-1)^q \varepsilon_j \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_q}$.

Similarly

- $\varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_q} = 0$ when two indices coincide, *i.e.* $\exists i \neq j\ k_i = k_j$;
- if we exchange two indices the form $\varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_q}$ is multiplied by -1.

Consider vectors $v_1,\dots,v_q \in V$, $v_i = \sum_k v_{ik} e_k$. Then $\varepsilon_k(v_i) = v_{ik}$ and

$$\varepsilon_1 \wedge \cdots \wedge \varepsilon_q(v_1,\dots,v_q) = \frac{1}{q!}\det(\varepsilon_i(v_j)) = \frac{1}{q!}\det(v_{ji}).$$

$$\varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_q}(v_1,\dots,v_q) = \frac{1}{q!}\det(\varepsilon_{k_i}(v_j)) = \frac{1}{q!}\det(v_{jk_i})).$$

So the functions $q!\varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_q}$ with increasing indices $(1 \le k_1 < k_2 < \cdots < k_q \le n)$ are the determinants of the minors of the matrix (v_{ij}).

The next theorem shows that all skew forms may be expressed by using determinants.

Theorem 5.1.12 Let $q > 0$. The set

$$\{\varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_q} | 1 \le k_1 \lneqq \cdots \lneqq k_q \le n\}$$

form a basis of $\Lambda^q(V^*)$. In particular

$$\dim \Lambda^q(V^*) = \begin{cases} \binom{n}{q} & if\ q \le n \\ 0 & if\ q > n \end{cases}.$$

Proof. We first show that it is a set of linearly independent forms. Take constants $a_{j_1\cdots j_q} \in \mathbb{R}$ such that

$$\sum_{1\le j_1 \lneqq \cdots \lneqq j_q \le n} a_{j_1\cdots j_q} \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_q} = 0.$$

Let us fix $i_1,\ldots,i_q$ with $1 \le i_1 \lneqq \cdots \lneqq i_q \le n$. From the remark above

$$0 = q!\left(\sum_{1\le j_1 \lneqq \cdots \lneqq j_q \le n} a_{j_1\cdots j_q} \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_q}\right)(e_{i_1},\ldots,e_{i_q}) = a_{i_1\cdots i_q}.$$

To prove that it is a set of generators we need to show that each $\omega \in \Lambda^q(V^*)$ is a linear combination of the forms $\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_q}$.

We define

$$\eta := \omega - q!\left(\sum_{1\le j_1 \lneqq \cdots \lneqq j_q \le n} \omega(e_{j_1},\ldots,e_{j_q}) \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_q}\right)$$

and conclude the proof by showing $\eta = 0$.

By definition of η (still using the formulas of the remark above) $i_1 \lneqq i_2 \lneqq \cdots \lneqq i_q \Rightarrow \eta(e_{i_1},\ldots,e_{i_q}) = 0$. Since η is alternating, it follows that $\eta(e_{i_1},\ldots,e_{i_q}) = 0$ when the e_{i_l} are pairwise distinct. Moreover (Complement 5.1.9) $\eta(e_{i_1},\ldots,e_{i_q}) = 0$ when two of the vectors coincide. So $\eta(e_{i_1},\ldots,e_{i_q})$ vanishes always.

Finally, since η is linear in all factors, $\forall v_1,\ldots,v_q \in V$, $\eta(v_1,\ldots,v_q)$ is a linear combination of the $\eta(e_{i_1},\ldots,e_{i_q})$. It follows $\eta = 0$. ■

Definition 5.1.13 A **graded vector space** $V^\bullet$ is a vector space containing subspaces[a] V^q, $q \in \mathbb{Z}$ such that $V^\bullet = \bigoplus_q V^q$. An element $v \in V^q$ is a **homogeneous element** of **degree** q.

The **Hilbert function** of the graded vector space $V^\bullet$ is the function $HF(V^\bullet)\colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ associating to each integer q the dimension of V^q.

A linear application among two graded vector spaces $L\colon V^\bullet \to W^\bullet$ has **degree** d if $\forall q \in \mathbb{Z}$, $L(V^q) \subset W^{q+d}$.

An **isomorphism of graded vector spaces** is an isomorphism of vector spaces of degree zero.

[a]There are books in literature writing the definition of graded vector space with grading q in $\mathbb{N}$ instead of $\mathbb{Z}$. That definition can be seen as a special case of our definition, the case when $V^q = \{0\}$ for all $q < 0$. In fact we will be in that situation for almost all the graded vector spaces considered in these notes. However it is more usual nowadays to introduce this notion with gradings in $\mathbb{Z}$ because graded vector spaces with negative gradings are important for several applications.

Isomorphic graded vector spaces have the same Hilbert function and, conversely, two graded vector spaces having the same Hilbert function, if their Hilbert function has values in $\mathbb{N}$ (so never ∞), are isomorphic.

Note that every element $v \in V^\bullet$ can be uniquelly decomposed as $v := \sum v_q$ with v_q homogeneous of degree q.

Definition 5.1.14 A **graded algebra** $V^\bullet$ is a graded vector space provided with an internal product $\times\colon V^\bullet \times V^\bullet \to V^\bullet$ giving a structure of algebra on it such that if v and w are homogeneous elements of respective degree p and q then $v \times w$ is homogeneous of degree $p+q$.

A **homomorphism of graded algebras** $L\colon V^\bullet \to W^\bullet$ is a linear application of degree 0 such that $\forall v, w \in V^\bullet$, $L(v \times w) = L(v) \times L(w)$.

An invertible homomorphism of graded algebras is an **isomorphism of graded algebras**.

Definition 5.1.15 The **exterior algebra** or **Grassmann algebra** is the graded algebra $\Lambda^\bullet V^* := \bigoplus_{q \geq 0} \Lambda^q V^*$ considered with the internal product given by the wedge product.

So an element of the exterior algebra is a formal sum of $q-$forms. From Theorem 5.1.12 $\dim \Lambda^\bullet V^* = \sum_{q=0}^{\dim V} \binom{\dim V}{q} = (1+1)^{\dim V} = 2^{\dim V}$.

Linear applications between vector spaces induce naturally linear applications among their spaces of tensors, mapping symmetric tensors to symmetric tensors and skew tensors in skew tensors. Since we are mostly interested in skew tensors, we consider only the latter.

Definition 5.1.16 Let $L\colon V \to W$ be a linear application. It naturally induces linear applications (**pull-backs**)

$$L^*\colon \Lambda^q W^* \to \Lambda^q V^*$$

defined by $(L^*\omega)(v_1, \ldots, v_q) = \omega(L(v_1), \ldots, L(v_q))$, defining a linear application of degree zero

$$L^*\colon \Lambda^\bullet W^* \to \Lambda^\bullet V^*.$$

R To ease the notation we have done an *abuse of notation* attributing the same symbol, L^*, to many different maps. Several similar abuses will follow. This is a standard choice in differential geometry: the student should try to get used to it.

R If $q = 1$, L^* is the usual *dual map* .

R By definition $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$.

By the complements 5.1.13 and 5.1.14 and by Theorem 5.1.12 we can express each L^* (for every q) in terms of the linear application dual to L. The most interesting case is the case when $V = W$ and $q = \dim V$. The next theorem shows that in this case L^* coincides with the multiplication by the determinant of L.

Proposition 5.1.17 Let $L\colon V \to V$ linear, $\omega \in \Lambda^{\dim V} V^*$. Then

$$L^*\omega = (\det L)\omega.$$

Proof. Setting $n := \dim V$, by Theorem 5.1.12 $\dim \Lambda^n V^* = 1$ and therefore the linear application $L^*\colon \Lambda^n V^* \to \Lambda^n V^*$ is the multiplication by a constant $c \in \mathbb{K}$. Since $(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n)(e_1, \ldots, e_n) = \frac{1}{n!}$, then $L^*(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n)(e_1, \ldots, e_n) = \frac{c}{n!}$. It is then enough to show $n! L^*(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n)(e_1, \ldots, e_n) = \det L$.

Indeed, by Definition 5.1.16 and Proposition 5.1.11

$$\begin{aligned} n!L^*(\varepsilon_1\wedge\cdots\wedge\varepsilon_n)(e_1,\dots,e_n) &= n!\varepsilon_1\wedge\cdots\wedge\varepsilon_n(L(e_1),\dots,L(e_n)) \\ &= \det\begin{pmatrix} \varepsilon_1(L(e_1)) & \cdots\varepsilon_1(L(e_n)) \\ \vdots & \ddots\vdots \\ \varepsilon_n(L(e_1)) & \cdots\varepsilon_n(L(e_n)) \end{pmatrix} \\ &= \det L. \end{aligned}$$

■

Complement 5.1.1 — The dual basis. Let $\{e_1,\dots,e_n\}$ be a basis of a vector space V.
Prove that $\forall 1\le j\le n$ there is a unique $\varepsilon_j\in V^*$ such that $\forall 1\le i\le n$, $\varepsilon_j(e_i)=\delta_{ij}$.
Show that $\{\varepsilon_1,\dots,\varepsilon_n\}$ is a basis of V^*.

Complement 5.1.2 Let $V_1,\dots,V_q$ finitely dimensional vector spaces over $\mathbb{K}$.
Show that the following operations give to $V_1^*\otimes\cdots\otimes V_q^*$ a structure of vector space over $\mathbb{K}$.
$+$:

$$\forall\omega_1,\omega_2\in V_1^*\otimes\cdots\otimes V_q^*,\ \forall v_i\in V_i,\ (\omega_1+\omega_2)(v_1,\cdots,v_q)=\omega_1(v_1,\cdots,v_q)+\omega_2(v_1,\cdots,v_q)$$

$\cdot$:

$$\forall\lambda\in\mathbb{K},\forall\omega\in V_1^*\otimes V_2^*\otimes\cdots\otimes V_q^*,\ \forall v_i\in V_i,\ (\lambda\omega)(v_1,\cdots,v_q)=\lambda\omega(v_1,\cdots,v_q)$$

Complement 5.1.3 — Decomposable tensors. Check that the functions $\varphi_1\otimes\cdots\otimes\varphi_q$ in Definition 5.1.4 are tensors in $V_1^*\otimes V_2^*\otimes\cdots\otimes V_q^*$ by showing their multilinearity.

Complement 5.1.4 Let $V_1,\dots,V_q$ be vector spaces over $\mathbb{K}$.
Prove the following equalities.

- Let $i\in\{1,\dots,q\}$. Then for every choice of q elements $\varphi_j\in V_j$, for all $1\le j\le q$, and of a further $\varphi_i'\in V_i$, it holds

$$\varphi_1\otimes\cdots\otimes(\varphi_i+\varphi_i')\otimes\cdots\otimes\varphi_q=\varphi_1\otimes\cdots\otimes\varphi_i\otimes\cdots\otimes\varphi_q+\varphi_1\otimes\cdots\otimes\varphi_i'\otimes\cdots\otimes\varphi_q.$$

- Let $i\in\{1,\dots,q\}$. Then for every choice of q elements $\varphi_j\in V_j$, for all $1\le j\le q$, and of a scalar $\lambda\in\mathbb{K}$, it holds

$$\varphi_1\otimes\cdots\otimes(\lambda\varphi_i)\otimes\cdots\otimes\varphi_q=\lambda(\varphi_1\otimes\cdots\otimes\varphi_i\otimes\cdots\otimes\varphi_q)$$

Let $\{e_{ij}\}$ be respective bases of V_i. Deduce from the previous equalities that for each choice of scalars $\lambda_{ij}\in\mathbb{K}$

$$\left(\sum_{j=1}^{\dim V_1}\lambda_{1j}e_{1j}\right)\otimes\cdots\otimes\left(\sum_{j=1}^{\dim V_q}\lambda_{qj}e_{qj}\right)=\sum_{j_1=1}^{\dim V_1}\cdots\sum_{j_q=1}^{\dim V_q}\left(\left(\prod_{i=1}^{q}\lambda_{ij_i}\right)e_{1j_1}\otimes\cdots\otimes e_{qj_q}\right).$$

Complement 5.1.5 Write the details of the proof of Proposition 5.1.7.

Complement 5.1.6 Prove Theorem 5.1.5.

Complement 5.1.7 Prove that $\mathrm{Sym}^q V^*$ and $\Lambda^q V^*$ are vector subspaces of $(V^*)^{\otimes q}$.

Complement 5.1.8 Show that map det defined in Example 5.4 is a tensor in $((\mathbb{R}^n)^*)^{\otimes n}$ and that it is decomposable if and only if $n = 1$.

Show that $\det \in \Lambda^n(\mathbb{R}^n)^*$.

Complement 5.1.9 Let ω be a skew tensor.

Prove that if $\omega(v_1, \ldots, v_q) \neq 0$, then the v_i are pairwise distinct.

Complement 5.1.10 Let V be a vector space over $\mathbb{K}$.

Prove that, for all $\varphi, \varphi_1, \varphi_2 \in V^*$, $\lambda_1, \lambda_2 \in \mathbb{K}$,

- $\varphi \wedge \varphi = 0$.
- $\varphi_1 \wedge \varphi_2 = -\varphi_2 \wedge \varphi_1$.
- $(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) \wedge \varphi = \lambda_1(\varphi_1 \wedge \varphi) + \lambda_2(\varphi_2 \wedge \varphi)$.
- $\varphi \wedge (\lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1(\varphi \wedge \varphi_1) + \lambda_2(\varphi \wedge \varphi_2)$.

Complement 5.1.11 Let V be a vector space over $\mathbb{K}$.

Prove that, for all $\varphi_1, \varphi_2 \in V^*$, $\lambda_1, \lambda_2 \in \mathbb{K}$, $v, v_1, v_2 \in V$

- $\varphi_1 \wedge \varphi_2(v, v) = 0$.
- $\varphi_1 \wedge \varphi_2(v_1, v_2) = -\varphi_1 \wedge \varphi_2(v_2, v_1)$
- $\varphi_1 \wedge \varphi_2(\lambda_1 v_1 + \lambda_2 v_2, v) = \lambda_1 \varphi_1 \wedge \varphi_2(v_1, v) + \lambda_2 \varphi_1 \wedge \varphi_2(v_2, v)$.
- $\varphi_1 \wedge \varphi_2(v, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \varphi_1 \wedge \varphi_2(v, v_1) + \lambda_2 \varphi_1 \wedge \varphi_2(v, v_2)$.

Complement 5.1.12 Let V be a vector space over $\mathbb{K}$.

Prove that, for all $q_1, q_2, q_3 \in \mathbb{N}$, $\omega_1, \eta_1 \in \Lambda^{q_1} V^*$, $\omega_2 \in \Lambda^{q_2}(V^*)$, $\omega_3 \in \Lambda^{q_3}(V^*)$, $k \in \mathbb{K}$,

- $\omega_1 \wedge \omega_2 \in \Lambda^{q_1+q_2}(V^*)$;
- $(\omega_1 + \eta_1) \wedge \omega_2 = \omega_1 \wedge \omega_2 + \eta_1 \wedge \omega_2$; $\omega_2 \wedge (\omega_1 + \eta_1) = \omega_2 \wedge \omega_1 + \omega_2 \wedge \eta_1$;
- $(k\omega_1) \wedge \omega_2 = k(\omega_1 \wedge \omega_2) = \omega_1 \wedge (k\omega_2)$;
- $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- $\omega_1 \wedge \omega_2 = (-1)^{q_1 q_2} \omega_2 \wedge \omega_1$;

Complement 5.1.13 — Pull-back. Let V, W be vector spaces over the same field and let $L \colon V \to W$ be a linear application.

Prove that all maps $L^* \colon \Lambda^q W^* \to \Lambda^q V^*$ in Definition 5.1.16 are linear.

Complement 5.1.14 — Pull-back and wedge product commute. Let V, W be vector spaces over the same field $\mathbb{K}$ and let $L \colon V \to W$ be a linear application.

Show that $\forall \omega, \eta \in \Lambda^\bullet V^*$, $L^*(\omega \wedge \eta) = L^*\omega \wedge L^*\eta$.

Exercise 5.1.1 Set $V = \mathbb{R}^3$ and consider the map $\omega\colon V^3 \to \mathbb{R}$ defined by $\omega(v_1, v_2, v_3) = (v_1 \times v_2) \cdot v_3$,
Show that $\omega \in (V^*)^{\otimes 3}$.
Show that ω is skew.
Deduce that ω is a generator of $\Lambda^3 V^*$.

Exercise 5.1.2 Show that, if $\forall 1 \le i \le q \ \dim V_i = 1$, all tensors in $V_1^* \otimes \cdots \otimes V_q^*$ are decomposable.

Exercise 5.1.3 Show that, for each decomposable tensor $\omega \in (V^*)^{\otimes q}$ different from 0, the set

$$\{v \in V | \omega(v, v, \dots, v) = 0\}$$

is a union of finitely many hyperplanes of V.

Exercise 5.1.4 Prove that, $\forall q \ge 2$, there is a tensor in $\left(\mathbb{R}^2\right)^{\otimes q}$ not decomposable.

Exercise 5.1.5 Consider, for each bilinear form $\omega \in (\mathbb{K}^n)^* \otimes (\mathbb{K}^n)^*$, the unique square matrix $A \in M_n(\mathbb{K})$ such that $\omega(v, w) = w^T A v$. Show that this gives an isomorphism among $(\mathbb{K}^n)^* \otimes (\mathbb{K}^n)^*$ and $M_n(\mathbb{K})$. Show that this induces two isomorphisms

- among $\mathrm{Sym}^2(\mathbb{K}^n)^*$ and the space of the symmetric $n \times n$ matrices;
- among $\Lambda^2(\mathbb{K}^n)^*$ and the space of the skewsymmetric $n \times n$ matrices.

Exercise 5.1.6 Assume $\dim V \ge 1$. Show that $\mathrm{Sym}^q(V^*) \cap \Lambda^q(V^*) \ne \{0\} \Leftrightarrow q \le 1$.

Exercise 5.1.7 Show that $(V^*)^{\otimes 2} = \mathrm{Sym}^2(V^*) \oplus \Lambda^2 V^*$. Is there a similar relation when $q \ge 3$?

Exercise 5.1.8 Show that there is a canonical (so you are not allowed to use a basis to construct it) isomorphism $V^* \oplus W^* \cong (V \oplus W)^*$.

Exercise 5.1.9 Show that there is a canonical isomorphism $\Lambda^2(V \oplus W)^* \cong \Lambda^2 V^* \oplus \Lambda^2 W^* \oplus (V^* \otimes W^*)$.

Exercise 5.1.10 Let $\varphi_1, \varphi_2 \in (\mathbb{R}^n)^*$. Prove that $2|\varphi_1 \wedge \varphi_2(v_1, v_2)|$ is the area of the parallelogram in $\mathbb{R}^2$ spanned by the vectors $(\varphi_1, \varphi_2)(v_1)$ and $(\varphi_1, \varphi_2)(v_2)$.

Exercise 5.1.11 Show that $\omega \in \Lambda^{2q+1}(V^*) \Rightarrow \omega \wedge \omega = 0$.

Exercise 5.1.12 Show that $\omega \in \Lambda^q(V^*)$, $q > 0$, $\dim V \le 3 \Rightarrow \omega \wedge \omega = 0$.

Exercise 5.1.13 Find an alternating form $\omega \in \Lambda^q(V^*)$, $q > 0$ with $\omega \wedge \omega \neq 0$.

Exercise 5.1.14 Let $v_1, \dots, v_m$ and $w_1, \dots, w_m$ two sets of linearly independent vectors in the same vector space V. Show that

$$\langle v_1, \dots, v_m \rangle = \langle w_1, \dots, w_m \rangle$$

if and only if there is $\lambda \in \mathbb{K} \setminus \{0\}$ such that

$$v_1 \wedge \cdots \wedge v_m = \lambda w_1 \wedge \cdots \wedge w_m.$$

Exercise 5.1.15 Assume $V = \mathbb{R}^3$. Compute explicitly the wedge product of two general 1-forms. Compare the result with the usual definition of cross product on $\mathbb{R}^3$.

5.2 Operations on vector bundles

All standard constructions in linear algebra have a *relative version* in the category of vector bundles. We give here some of them.

Direct sum. Consider two vector bundles E and E' of respective ranks r and r' on the same base B.

The fibre product $E \times_B E'$ has a natural structure of vector bundle of rank $r + r'$ over B given by the induced map $\pi \oplus \pi' \colon E \times_B E' \to B$ defined by the commutative diagram below.

$$\begin{tikzcd} E \times_B E' \arrow[r] \arrow[d] \arrow[dr, "\pi\oplus\pi'"] & E \arrow[d, "\pi"] \\ E' \arrow[r, "\pi'"] & B \end{tikzcd}$$

We note that by definition there is a canonical isomorphism, for each $p \in B$

$$E_p \oplus E'_p \cong (E \times_B E')_p.$$

Indeed, we define the direct sum $E \oplus E'$ of the vector bundles E and E' to be the vector bundle whose total space is $E \oplus E' := E \times_B E'$ and whose map is $\pi \oplus \pi'$.

Let us have a look to the cocycles of these bundles. We take trivializations of E and E' relative to the same open cover $\{U_\alpha\}$ of B (see Complement 3.1.2); they induce a trivialization of $E \oplus E'$ in a natural way.

Consider the corresponding cocycles $\{g_{\alpha\beta}\}$ for E and $\{g'_{\alpha\beta}\}$ for E'. Then the cocycle of $E \oplus E'$ is $\{g_{\alpha\beta} \oplus g'_{\alpha\beta}\}$ where

$$g_{\alpha\beta} \oplus g'_{\alpha\beta} := \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g'_{\alpha\beta} \end{pmatrix} \in \mathrm{GL}(r + r', \mathbb{K}). \tag{5.4}$$

Then $E \oplus E'$ is a vector bundle of rank equal to the sum of the ranks of E and of E'.

By Proposition 3.2.2 we could use (5.4) as definition of the direct sum $(E \oplus E', \pi \oplus \pi')$ (up to isomorphisms). Indeed, we will do the next constructions with the last method, by producing cocycles, since it is more convenient in those cases.

Tensor product. For every two vector bundles E and E' of respective ranks r and r' on the same base B, we define the vector bundle $E \otimes E'$ as the vector bundle of rank rr' on B given as follows: given cocycles $\{g_{\alpha\beta}\}$ of E and $\{g'_{\alpha\beta}\}$ of E' relative to the same cover of B, we define $E \otimes E'$ through the cocycle $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$ (see Definition 5.1.6).

Then, for every $p \in B$, the fibre $(E \otimes E')_p$ is canonically isomorphic to $E_p \otimes E'_p$.

Similarly we define, for every vector bundle E of rank r and for all $q \in \mathbb{N}$, the vector bundles $E^{\otimes q}$ and $(E^*)^{\otimes q}$ of rank rq.

Dual. Let E be a vector bundle on a base B with cocycle $\{g_{\alpha\beta}\}$. Then we define E^* to be the bundle with cocycle $\{{}^t(g_{\alpha\beta}^{-1})\}$ (here ^{t} stands for "transpose"). Then we have canonical isomorphisms among each fibre E_p^* and the dual of E_p.

Exterior powers. We define $\Lambda^q E^*$ as the subbundle of $(E^*)^{\otimes q}$ given, as subset, by all elements that are skew as $q-$linear application on the corresponding fibre E_p of E.

Complexification.[3] If E is a real vector bundle with cocycle $g_{\alpha\beta}$, then, since every matrix with real coefficients is also a matrix with complex coefficients, the same cocycle $g_{\alpha\beta}$ gives also a complex vector bundle[4] $E_{\mathbb{C}}$. Indeed, for every $p \in B$, the fibre $(E_{\mathbb{C}})_p$ is canonically isomorphic to the complex vector space $E_p \otimes_{\mathbb{R}} \mathbb{C}$.

Hom. For each pair of vector bundles E and E' over the same B, we define the vector bundle $\mathrm{Hom}(E,E')$ as $E' \otimes E^*$. The canonical isomorphisms $\mathrm{Hom}(E,E')_p \cong \mathrm{Hom}(E_p,E'_p)$ follows by Proposition 5.1.7.

Complement 5.2.1 Show that $E \oplus E'$ is well defined up to isomorphisms by showing that if we choose different cocycles for E and E' we obtain an isomorphic vector bundle.

Exercise 5.2.1 Show that, if E is any line bundle, then $E \otimes E^*$ is trivial.

Exercise 5.2.2 — The Picard group. Prove that the tensor product $\otimes$ defines a structure of abelian group on the set of line bundles over a fixed base B modulo isomorphisms.

Exercise 5.2.3 Show that the hyperplane bundle of $\mathbb{P}^1_{\mathbb{K}}$ is dual to the tautological bundle.

Exercise 5.2.4 Let H be a hyperplane of $\mathbb{P}^n_{\mathbb{C}}$, the locus

$$\left\{\sum_0^n a_i x_i = 0\right\}$$

where the a_i are complex numbers not all equal to zero.

Show that there is a holomorphic section of the hyperplane bundle vanishing exactly along H.

Exercise 5.2.5 Note that every complex vector bundle is also a real vector bundle.

Prove that $E_{\mathbb{C}}$, as real vector bundle, is isomorphic to $E \oplus E$.

5.3 The algebra of the differential forms

We define the differential forms as sections of suitable vector bundles.

[3] Note that this definition is equivalent to Definition 3.4.2

[4] In this case, even if E is a smooth manifold, we are not claiming that $E_{\mathbb{C}}$ has any structure of complex manifold.

Indeed we have defined, for every real manifold M, the tangent bundle TM, which induces $\forall 1 \le q \le \dim M$, by the theory of the vector bundles, a bundle $\Lambda^q T^*M := \Lambda^q (TM)^*$. Conventionally we set $\Lambda^0 T^*M$ to be the trivial bundle of rank 1. The bundle $\Lambda^1 T^*M$ is the **cotangent bundle**. The bundle $\Lambda^{\dim M} T^*M$ is the **real canonical**[5] **bundle**.

Definition 5.3.1 A (smooth)[a] **differential q-form** or **differential form of degree** q on a manifold M is smooth section ω of the vector bundle $\Lambda^q T^*M \to M$.

The form ω is **smooth** if it is smooth as a map among manifolds. The smooth q-forms form the vector space $\Omega^q(M)$.

Conventionally,

$$\Omega^q(M) = \{0\} \text{ for } q < 0 \qquad \Omega^0(M) = C^\infty(M) \qquad \Omega^\bullet(M) = \oplus_{q \in \mathbb{Z}} \Omega^q(M)$$

[a] We will only consider smooth sections in this notes, so we will often drop the word smooth for sake of brevity.

Note that, since for all $q > \dim M$ the vector bundle $\Lambda^q T^*M$ has rank zero, then $\Omega^q(M) = \{0\}$ and we can equivalently write

$$\Omega^\bullet(M) = \oplus_{q=0}^{\dim M} \Omega^q(M)$$

The smooth q-forms act naturally on $\mathfrak{X}(M)^q$; that is we can see every q-form ω as a map

$$\omega \colon \mathfrak{X}(M)^q \to C^\infty(M)$$

as follows. For every choice of q smooth vector fields $v_1, \dots, v_q$, $\omega(v_1, \dots, v_q)$ is the function defined by

$$\forall p \quad \omega(v_1, \dots, v_q)(p) := \omega_p(v_1(p), \dots, v_q(p)).$$

We have then a natural map

$$\Omega^q(U) \times (\mathfrak{X}(U))^q \to C^\infty(U).$$

For every q-form ω, charts (U, φ) for M may be used to represent locally the form, that is to write the restriction of ω to U $\omega_{|U}$, as follows. Let $x_1, \dots, x_n$ be the local coordinates induced by the chart. $\forall p \in U$ we have an induced basis $\left\{ \left(\frac{\partial}{\partial x_i} \right)_p \right\}$ of T_pM.

Definition 5.3.2 We denote by $\{(dx_1)_p, \dots, (dx_n)_p\}$ the basis of $(T_pM)^*$ dual to the basis $\left\{ \left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_n} \right)_p \right\}$ of T_pM.

In other words $(dx_i)_p \left(\frac{\partial}{\partial x_j} \right)_p = \delta_{ij}$.

This let us write locally in a simple way the $1-$ forms, since for every $\omega \in \Omega^1(M)$ locally there are unique smooth functions $\omega_1, \dots, \omega_n$ such that, for all p,

$$\omega_p = \sum_1^n \omega_i(p)(dx_i)_p$$

[5] To the knowledge of the author the word "canonical bundle" in literature is usually reserved to the complex case. People refers to the canonical bundle for the bundle of the holomorphic $n-$forms, the bundle of the $(n,0)$-forms discussed at the end of this section.

We find it natural to use the same word for the analogous real case, but we want to inform the student that this notation is not common in the literature.

Note $\omega_i = \omega\left(\frac{\partial}{\partial x_i}\right)$. We will write

$$\omega = \sum_i \omega_i dx_i$$

To write similarly the $q-$forms we introduce the following notation.

Notation 5.1. *A* **multiindex** $I = (i_1, \dots, i_q)$ *of positive integers, is an ordered sequence such that* $\forall j,\ i_j \in \mathbb{N}$.

We will say that I has **length** *q, and we will denote by* dx_I *the element* $dx_{i_1} \wedge \cdots \wedge dx_{i_q} \in \Omega^q(U)$. *Similarly* $(dx_I)_p = (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_q})_p$ *for all p.*

If $q = 0$, $I = \emptyset$, *in which case* $dx_\emptyset := 1$.

Locally, by Theorem 5.1.12, for every smooth differential $q-$form ω, there are smooth functions $\omega_I = \omega_{i_1 \cdots i_q} : U \to \mathbb{R}$ where I runs along all multiindices $I = (i_1, \dots, i_q)$ with $1 \lneqq i_1 \leq \cdots \lneqq i_q \leq n$, such that

$$\omega_p = \sum_{1 \lneqq i_1 \leq \cdots \lneqq i_q \leq n} \omega_{i_1 \cdots i_q}(p)(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_q})_p = \sum_I \omega_I(p)(dx_I)_p$$

In fact $\omega_{i_1 \cdots i_q} = q!\omega\left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_q}}\right)$. We will write

$$\omega_{|U} = \sum_{1 \lneqq i_1 \leq \cdots \lneqq i_q \leq n} \omega_{i_1 \cdots i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q} = \sum_I \omega_I dx_I.$$

Also the action of ω on $\mathfrak{X}(M)^q$ can be described in local coordinates. If $\omega_{|U} = \sum \omega_I dx_I$, then

$$\omega(v_1, \dots, v_q)(p) = \sum \omega_I(p)(dx_I)_p(v_1(p), \dots, v_q(p)).$$

The graded vector space $\Omega^\bullet(M) := \oplus_q \Omega^q(M)$ is called **algebra of the differential forms** analogous to the exterior algebra $\Lambda^\bullet V^*$ of a vector space. The internal product defining its algebra structure is the wedge product, defined intrinsecally using Definition 5.1.10 of the wedge product of alternating forms as follows.

Definition 5.3.3 Let ω_1 be a differential q_1-form on M, ω_2 be a differential q_2-form on M. Then we define $\omega_1 \wedge \omega_2$ as the $(q_1 + q_2)$-form such that $\forall p \in U$, $(\omega_1 \wedge \omega_2)_p = (\omega_1)_p \wedge (\omega_2)_p$.

The wedge product of smooth forms is smooth, since sums of products of smooth functions are smooth. So we get bilinear maps

$$\wedge\colon \Omega^{q_1}(M) \times \Omega^{q_2}(M) \to \Omega^{q_1+q_2}(M)$$

which inherits all the properties of the wedge product of alternating forms. Namely (see Complement 5.1.12)

- Let $\omega_1, \eta_1 \in \Omega^{q_1}(M)$, $\omega_2 \in \Omega^{q_2}(M)$; then $(\omega_1 + \eta_1) \wedge \omega_2 = \omega_1 \wedge \omega_2 + \eta_1 \wedge \omega_2$; $\omega_2 \wedge (\omega_1 + \eta_1) = \omega_2 \wedge \omega_1 + \omega_2 \wedge \eta_1$.
- Let $\omega_1 \in \Omega^{q_1}(M)$, $\omega_2 \in \Omega^{q_2}(M)$, $f \in C^\infty(M)$. Then $(f\omega_1) \wedge \omega_2 = f(\omega_1 \wedge \omega_2) = \omega_1 \wedge (f\omega_2)$.
- Let $\omega_1 \in \Omega^{q_1}(M)$, $\omega_2 \in \Omega^{q_2}(M)$, $\omega_3 \in \Omega^{q_3}(M)$, then $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- Let $\omega_1 \in \Omega^{q_1}(M)$, $\omega_2 \in \Omega^{q_2}(M)$, then $\omega_1 \wedge \omega_2 = (-1)^{q_1 q_2} \omega_2 \wedge \omega_1$.

Note that this applies also to $0-$forms. Namely, if $f \in \Omega^0(M) = C^\infty(M)$, then

$$f \wedge \omega = f\omega$$

Then $\Omega^\bullet(M)$, with the three given operations (multiplication by scalar, sum, wedge product), is a graded $\mathbb{R}-$algebra.

Example 5.5 $\omega := x_1 dx_2$ is a 1-form; $\omega \in \Omega^1(M)$, $\deg \omega = 1$.
$\tau := x_2 dx_1 \wedge dx_2 + dx_3 \wedge dx_1$ is a 2-form; $\tau \in \Omega^2(M)$, $\deg \tau = 2$.
$\omega + \tau$ is a form, $\omega + \tau \in \Omega^\bullet(M)$ but $\omega + \tau$ is not a q-form. Indeed, $\omega \notin \bigcup_q \Omega^q(M)$.
In contrast $\omega \wedge \tau$ is a 3-form:

$$\begin{aligned}
\omega \wedge \tau &= x_1 dx_2 \wedge (x_2 dx_1 \wedge dx_2 + dx_3 \wedge dx_1)\\
&= x_1 dx_2 \wedge (x_2 dx_1) \wedge dx_2 + x_1 dx_2 \wedge dx_3 \wedge dx_1\\
&= x_1 x_2 dx_2 \wedge dx_1 \wedge dx_2 - x_1 dx_2 \wedge dx_1 \wedge dx_3\\
&= -x_1 x_2 dx_1 \wedge dx_2 \wedge dx_2 + x_1 dx_1 \wedge dx_2 \wedge dx_3\\
&= x_1 dx_1 \wedge dx_2 \wedge dx_3.
\end{aligned}$$

Now let us consider the real canonical bundle. It is a line bundle.

Definition 5.3.4 Let M be a manifold of dimension n. A **volume form** on M is a form $\omega \in \Omega^n(M)$ such that $\forall p \in M$, $\omega_p \neq 0$.

By Exercise 3.2.2 the real canonical bundle of M is trivial if and only if there is a volume form on M.

The real canonical bundle of S^1 is the dual of its tangent bundle, which is trivial by Exercise 3.3.4. So its real canonical bundle is trivial as well and we may conclude that there is a volume form on S^1. Note however that this argument does not allow us to conclude anything about the triviality of the real canonical bundle of any other sphere (including the odd dimensional ones).

5.3.1 The holomorphic $q-$forms

Let us consider a complex manifold M of dimension n. Then we have:

- the **real cotangent bundle** $\Lambda^1_{\mathbb{R}} T^*M$, which is the cotangent bundle of M as real manifold;
- its complexification: the **complexified real cotangent bundle** $\Lambda^1_{\mathbb{C}} T^*M$;
- the **holomorphic cotangent bundle** $\Lambda^{1,0} T^*M$: this is the cotangent bundle as complex manifold, and it is naturally embedded as subbundle of the complexified real cotangent bundle;
- the **antiholomorphic cotangent bundle** $\Lambda^{0,1} T^*M = \overline{\Lambda^{1,0} T^*M}$: this is the conjugated of the holomorphic cotangent bundle in the complexified real cotangent bundle.

If $z_j = x_j + iy_j$ are local coordinates then locally

- the **real cotangent bundle** is generated by the dx_j, dy_j (on the real numbers);
- the **complexified real cotangent bundle** is generated by the dx_j, dy_j (on the complex numbers);
- the **holomorphic cotangent bundle** is generated by the $dz_j = (dx_j + idy_j)$;
- the **antiholomorphic cotangent bundle** is generated by the $d\bar{z}_j = (dx_j - idy_j)$.

It follows immediately

$$\Lambda^1_{\mathbb{C}} T^*M = \Lambda^{1,0} T^*M \oplus \Lambda^{0,1} T^*M$$

Please note

$$dz_j\left(\frac{\partial}{\partial z_k}\right) = \delta_{jk}, \quad dz_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = 0, \quad d\bar{z}_j\left(\frac{\partial}{\partial z_k}\right) = 0, \quad d\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = \delta_{jk}.$$

Something similar happens for higher forms. We have

- the **real higher cotangent bundles** $\Lambda^q_{\mathbb{R}} T^*M$, the bundle of $q-$forms as real manifold;

- their complexification: the **complexified real higher cotangent bundles** $\Lambda^q_{\mathbb{C}} T^*M$;
- the **holomorphic higher cotangent bundles** $\Lambda^{q,0}T^*M$: this is the holomorphic analog of the bundle of $q-$forms, and it is naturally embedded in the complexified real higher cotangent bundle $\Lambda^q_{\mathbb{C}} T^*M$ as subbundle generated by the $dz_{i_1} \wedge \cdots \wedge dz_{i_q}$;
- the **(p,q)-cotangent bundles** $\Lambda^{p,q}T^*M$: this is the subbundle of $\Lambda^{p+q}_{\mathbb{C}} T^*M$ locally generated by the $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$.

The reader can easily prove the following properties, and namely

- the (p,q)-cotangent bundle are interesting only for p,q both not bigger than $\dim M$ since
$$\forall P \in M \text{ if } \max(p,q) \geq 1 + \dim M \text{ then } (\Lambda^{p,q}T^*M)_P = \{0\};$$
- the (p,q)-cotangent bundles split the complexified real higher cotangent bundles as direct sum:
$$\Lambda^k_{\mathbb{C}} T^*M = \oplus_{p+q=k} \Lambda^{p,q}T^*M;$$
- the complex conjugation on $\Lambda^k_{\mathbb{C}} T^*M$ acts on them exchanging p and q:
$$\Lambda^{p,q}T^*M = \overline{\Lambda^{q,p}T^*M}.$$

It is natural then to write $\Omega^q(M)$ for the $q-$forms as real manifolds, $\Omega^{p,0}(M)$ for the $p-$forms as complex manifolds, so holomorphic sections of the complexified real higher cotangent bundle $\Lambda^{p,0}T^*M$, and finally $\Omega^{p,q}(M)$ for the holomorphic sections of $\Lambda^{p,q}T^*M$, the **holomorphic (p,q)-forms**. The usual notation for the bigger space of the smooth sections of $\Lambda^{p,q}T^*M$ is $A^{p,q}(M)$.

For example, both $z_1 dz_1 \wedge d\bar{z}_2$ and $\bar{z}_1 dz_1 \wedge d\bar{z}_2$ belong to $A^{1,1}(\mathbb{C}^2)$ but the latter does not belong to $\Omega^{1,1}(\mathbb{C}^2)$.

Exercise 5.3.1 Show that $\Omega^q(U) = \{0\} \Leftrightarrow q > n$ or $q < 0$.

Exercise 5.3.2 Check that $(x_2 dx_1 \wedge dx_2)\left(x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}\right) = \frac{x_1 x_2^2}{2}$.

Exercise 5.3.3 Compute

- $(x_2 dx_1 \wedge dx_2)\left(x_1 \frac{\partial}{\partial x_1}, x_1 \frac{\partial}{\partial x_1}\right)$
- $(x_2 dx_1 \wedge dx_1)\left(x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}\right)$
- $(x_2 dx_1 \wedge dx_2)\left(x_2 \frac{\partial}{\partial x_2}, x_1 \frac{\partial}{\partial x_1}\right)$
- $(x_2 dx_2 \wedge dx_1)\left(x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}\right)$.
- $(x_2 dx_1 \wedge dx_2 + x_2 dx_2 \wedge dx_1)\left(x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}\right)$.
- $(x_2 dx_1 \wedge dx_2)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)$.

Exercise 5.3.4 Let M be a complex manifold of dimension n and consider a holomorphic $n-$form $\omega \in \Omega^{n,0}(M)$.

Consider the form $\omega \wedge \overline{\omega} \in A^{n,n}(M)$.

1. Show that $i^n \omega \wedge \overline{\omega} \in \Omega^{2n}(M)$.

2. Prove that, if $\omega_p \neq 0$ for all $p \in M$, then $\omega \wedge \overline{\omega}$ is a volume form.

Exercise 5.3.5 Show[a] that the real canonical bundle of S^n is trivial for all n.

[a] Use the form

$$\sum_{i=0}^{n+1} (-1)^i x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_n$$

Exercise 5.3.6 Consider $\mathbb{P}^1_{\mathbb{C}}$ with homogeneous coordinates $(z_0 : z_1)$ with the complex structure given by the charts $\{(U_i, \varphi_i)\}_{i\in\{0,1\}}$ with

$$U_i = \{z_i \neq 0\}, \qquad \varphi_0((z_0 : z_1)) = \frac{z_0}{z_1}, \qquad \varphi_1((z_0 : z_1)) = \frac{z_1}{z_0}.$$

Show[a] that every holomorphic $1-$form on $\mathbb{P}^1_{\mathbb{C}}$ vanishes identically[b].

[a] Hint: Consider the local coordinates z resp. z' on U_0 resp. U_1 given by φ_0 resp. φ_1. The restriction of a holomorphic $1-$form on $\mathbb{P}^1_{\mathbb{C}}$ on U_0 resp. U_1 is of the form $f(z)dz$ resp. $f'(z')dz'$with f, f' holomorphic. Write a relation among f and f' and deduce that f' has a pole, a contradiction.

[b] So this bundle is not trivial!

5.4 Pull-back and exterior derivative of forms

Let $F\colon M \to N$ be a smooth function between two manifolds. For every point $p \in M$ the differential $dF_p\colon T_pM \to T_{F(p)}N$ induce by Definition 5.1.16, $\forall q \in \mathbb{N}$, linear applications $dF_p^*\colon \Lambda^q T^*_{F(p)}N \to \Lambda^q T^*_pM$. *Gluing* them we get the pull-back map

$$F^*\colon \Omega^\bullet(N) \to \Omega^\bullet(M);$$

as follows.

If $q = 0$, $f \in \Omega^0(N) = C^\infty(N)$, then $F^*f := f \circ F \in \Omega^0(M)$.

If $q > 0$, for a form $\omega \in \Omega^q(N)$, its pull-back $F^*\omega$ is defined by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}).$$

It is clear by the definition that $F^*\omega$ is a section of the vector bundle $\Lambda^q T^*M$. We claim its smoothness without proving it for the time being. We will discuss the smoothness of $F^*\omega$ later in this section.

R From the analogous properties of the alternating forms all F^* are linear and

$$F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2,$$

and therefore F^* is a morphism of $\mathbb{R}-$algebras. Moreover

$$(F \circ G)^* = G^* \circ F^*.$$

In particular, if F is a diffeomorphism, then F^* is invertible with inverse $(F^{-1})^*$.

To define the exterior derivative we start by defining a linear application $d\colon \Omega^0(M) \to \Omega^1(M)$.

Definition 5.4.1 Let $f \in \Omega^0(M)$ be a smooth function. Fix a point $p \in M$.

The **exterior derivative of f at p** is the linear map

$$(df)_p \colon T_pM \to \mathbb{R}$$

defined by $df_p(v) = v(f)$. Note that $\forall p \in U$, $(df)_p \in (T_pM)^*$.

The **exterior derivative of f** is the $1-$form df giving $(df)_p$ at each p.

In local coordinates, since $df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

and therefore df is smooth: $df \in \Omega^1(M)$.

This defines a map

$$d \colon \Omega^0(M) \to \Omega^1(M)$$

the **exterior derivative**, characterized by the formula

$$\forall v \in \mathfrak{X}(M), \ \ df(v) = v(f).$$

R For any $f \in \Omega^0(M)$, we have denoted by df not only the exterior derivative $df \in \Omega^1(M)$ but also the differential of f, the map $df \colon TM \to T\mathbb{R}$ given by, writing locally $v = \sum_i v_i \left(\frac{\partial}{\partial x_i}\right)_p$,

$$df(v) = df_p\left(\sum_i v_i \left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_i \frac{\partial f}{\partial x_i}(p) v_i \left(\frac{d}{dt}\right)_{f(p)} = v(f)\left(\frac{d}{dt}\right)_{f(p)}$$

So we can see this abuse of notation as a "forgetting $\frac{d}{dt}$"

One more apparent abuse of notation is that we denoted by $dx_i \in \Omega^1(U)$ both the exterior derivative of the coordinate function x_i and the $1-$form giving at each point the element $(dx_i)_p$ of the basis of $(T_pM)^*$ dual to $\left\{\left(\frac{\partial}{\partial x_i}\right)_p\right\}$. This is not an abuse of notation, since these two $1-$forms coincide. In fact

$$dx_i\left(\sum_j v_j \frac{\partial}{\partial x_j}\right) = \left(\sum_j v_j \frac{\partial}{\partial x_j}\right)(x_i) = v_i$$

We investigate the relation among pull-back and exterior derivative for forms of degree zero.

Lemma 5.4.2 Let $F \colon M \to N$ be a smooth function and consider any $f \in C^\infty(N)$. Then $F^*(df) = d(F^*f)$.

Proof. Both $F^*(df)$ and $d(F^*f)$ are $1-$forms on M and therefore it is enough to prove that

$$\forall p \in M, \ (F^*(df))_p = d(F^*f)_p$$

as elements of the dual vector space $(T_pM)^*$. In other words we need to prove that

$$\forall p \in M, \ \forall v \in T_pM, \ (F^*(df))_p(v) = d(F^*f)_p(v).$$

Indeed

$$(F^*(df))_p(v) = dF_p^*((df)_{F(p)})(v) = (df)_{F(p)}(dF_p(v)) = ((df)_{F(p)} \circ dF_p)(v) = \\ = d(f \circ F)_p(v) = d(F^* f)_p(v).$$

■

Now we can write the pull-back of a form explicitly.

Proposition 5.4.3 Let $F\colon M \to N$ be a smooth function. Fix a point $p \in M$, and choose a chart (U, φ) for M in p with coordinates $x_1, \ldots, x_n$, and a chart (V, ψ) for N in $F(p)$ with coordinates $y_1, \ldots, y_m$ such that $F(U) \subset V$. Assume

$$\omega_{|V} = \sum_{1 \le i_1 < \ldots < i_q \le n} \omega_{i_1 \cdots i_q} dy_{i_1} \wedge \cdots \wedge dy_{i_q} \in \Omega^q(V).$$

Then

$$(F^*\omega)_{|U} = \sum_{1 \le i_1 < \ldots < i_q \le n} (\omega_{i_1 \cdots i_q} \circ F) dF_{i_1} \wedge \cdots \wedge dF_{i_q},$$

where $F_k := y_k \circ F$.

Proof.

$$\begin{aligned} F^*\omega = F^* &\left(\sum_{1 \le i_1 < \ldots < i_q \le n} \omega_{i_1 \cdots i_q} dy_{i_1} \wedge \cdots \wedge dy_{i_q} \right) \\ &= \sum_{1 \le i_1 < \ldots < i_q \le n} F^* \left(\omega_{i_1 \cdots i_q} dy_{i_1} \wedge \cdots \wedge dy_{i_q} \right) \\ &= \sum_{1 \le i_1 < \ldots < i_q \le n} \left(\omega_{i_1 \cdots i_q} \circ F \right) F^* dy_{i_1} \wedge \cdots \wedge F^* dy_{i_q}. \end{aligned}$$

We have then only to check $F^* dy_k = dF_k$. which follows since $F^* dy_k = dy_k \circ dF = d(y_k \circ F) = dF_k$. ■

Definition 5.4.4 There is one case which is rather important, it is the case when F is an embedding. In this case we will write $\omega_{|M}$ for $F^*\omega$.

Obviously if $p \in M$, $\omega_p = 0 \Rightarrow (F^*\omega)_p = 0$. It is rather important to notice that the converse is not true: it may be that $\omega_p \neq 0$ but still $(F^*\omega)_p = 0$; the reader will find important examples among the exercises of this section.

Now we can write the pull-back of a form explicitly. For example, if F is the function $F(x_1, x_2) = (x_1 x_2, x_1^2 + x_2^2)$, then

$$F^*(y_1 dy_2) = (x_1 x_2) d(x_1^2 + x_2^2) = 2x_1^2 x_2 dx_1 + 2x_1 x_2^2 dx_2.$$

We can finally prove the smoothness of a pull-back.

Corollary 5.4.5 If $F\colon M \to N$ is smooth and $\omega \in \Omega^q(N)$ then $F^*\omega$ is smooth, so it defines an element of $\Omega^q(M)$.

Proof. Since

$$F^*\omega = \sum_{1\le i_1<\ldots<i_q\le n} (\omega_{i_1\cdots i_q}\circ F)dF_{i_1}\wedge\cdots\wedge dF_{i_q},$$

and $dF_{i_j} = \sum \frac{\partial F_{i_j}}{\partial x_k}dx_k$, then we obtain that $F^*\omega = \sum g_I dx_I$ where all g_I are sums of products of the smooth functions $(\omega_{i_1\cdots i_q}\circ F)$ and of the partial derivatives $\frac{\partial F_i}{\partial x_k}$. ■

To extend the exterior derivative to $q-$forms, we first consider the local case. In other words instead of manifolds M,N we consider open subset U,V of $\mathbb{R}^n_\pm$ and $\mathbb{R}^n_\pm$.

We extend the exterior derivative $d\colon \Omega^0(U)\to\Omega^1(U)$ to an operator $d\colon \Omega^\bullet(U)\to\Omega^\bullet(U)$ of degree 1 of the graded algebra of the differential forms. We will do that by defining all restrictions

$$d_{|\Omega^q(U)}\colon \Omega^q(U)\to\Omega^{q+1}(U).$$

Theorem 5.4.6 There is a unique linear operator $d\colon \Omega^\bullet(U)\to\Omega^\bullet(U)$ of degree 1 such that
- i) $\forall f\in\Omega^0(U), \forall v\in\mathfrak{X}(U), df(v)=v(f)$;
- ii) $\forall q_1,q_2\ge 0, \forall\omega_1\in\Omega^{q_1}(U), \forall\omega_2\in\Omega^{q_2}(U), d(\omega_1\wedge\omega_2)=d\omega_1\wedge\omega_2+(-1)^{q_1}\omega_1\wedge d\omega_2$;
- iii) $d\circ d=0$.

If $\omega=\sum_I\omega_I dx_I$, then $d\omega=\sum_I d\omega_I\wedge dx_I=\sum_I\sum_{i=1}^n\frac{\partial\omega_I}{\partial x_i}dx_i\wedge dx_I$.

Proof. The existence is easy: we just need to consider the formal expression given in the statement, $d\omega=\sum d\omega_I\wedge dx_I$, and check that it has the required properties. We only check iii), leaving the other simpler checks to the reader.

By the linearity of d it is enough if we prove the statement for $\omega = f dx_{i_1}\wedge\cdots\wedge dx_{i_q}$. Then using the equality $dx_i\wedge dx_j=-dx_j\wedge dx_i$, since by Schwarz' Theorem $\frac{\partial^2 f}{\partial x_j\partial x_i}=\frac{\partial^2 f}{\partial x_i\partial x_j}$,

$$\begin{aligned}
d(d(fdx_I)) &= d\left(\sum_{i=1}^n\frac{\partial f}{\partial x_i}dx_i\wedge dx_I\right)\\
&= \sum_{i,j=1}^n\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i\wedge dx_I\\
&= \sum_{i\neq j}\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i\wedge dx_I\\
&= \sum_{i<j}\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i\wedge dx_I+\sum_{i>j}\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i\wedge dx_I\\
&= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i+\frac{\partial^2 f}{\partial x_i\partial x_j}dx_i\wedge dx_j\right)\wedge dx_I\\
&= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i-\frac{\partial^2 f}{\partial x_i\partial x_j}dx_j\wedge dx_i\right)\wedge dx_I=0.
\end{aligned}$$

We prove the uniqueness by showing that every linear operator with the properties i), ii), and iii) coincides with it.

By linearity $d\omega=\sum_I d(\omega_I dx_I)$, so by the properties i) and ii) (for $q_1=0$) it follows $d\omega=\sum_I(d\omega_I\wedge dx_I+\omega_I d(dx_I))$, and we conclude the proof by showing that for every multiindex $I=(i_1,\ldots,i_q)$

$$d(dx_{i_1}\wedge\cdots\wedge dx_{i_q})=0. \tag{5.5}$$

We prove (5.5) by induction on q. If $q = 1$, since by the property i) dx_i is the differential of the coordinate function x_i, $d(dx_i) = (d \circ d)x_i$ vanishes by the property iii).

Finally, we may assume (5.5) true for r-forms, $r < q$. Then

$$\begin{aligned} d(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}) &= d(dx_{i_1} \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_q})) \\ &= d(dx_{i_1}) \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_q}) - dx_{i_1} \wedge d(dx_{i_2} \wedge \cdots \wedge dx_{i_q})) \\ &= 0 - 0 = 0. \end{aligned}$$ ■

Note that $d\colon \Omega^\bullet(U) \to \Omega^\bullet(U)$ is NOT a ring homomorphism, as in general $d(\omega_1 \wedge \omega_2) \neq d\omega_1 \wedge d\omega_2$.

We can now complete the discussion of the special case of maps among open subset of $\mathbb{R}^n_\pm$ and $\mathbb{R}^m_\pm$ by proving that Lemma 5.4.2 extend to forms of higher degree.

Proposition 5.4.7 If $F\colon U \to V$ is a smooth function between open subsets $U \subset \mathbb{R}^n_\pm$, $V \subset \mathbb{R}^m_\pm$, and $\omega \in \Omega^q(V)$. Then $F^* d\omega = dF^*\omega$.

Proof. We write

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} \omega_{i_1 \cdots i_q} dy_{i_1} \wedge \cdots \wedge dy_{i_q},$$

which yields

$$F^*\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} (\omega_{i_1 \cdots i_q} \circ F) dF_{i_1} \wedge \cdots \wedge dF_{i_q}.$$

By Theorem 5.4.6 and Lemma 5.4.2

$$\begin{aligned} dF^*\omega = &\sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} d(\omega_{i_1 \cdots i_q} \circ F) \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_q} = \\ &= \sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} F^* d\omega_{i_1 \cdots i_q} \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_q}. \end{aligned}$$

On the other hand

$$d\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} d\omega_{i_1 \cdots i_q} \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_q},$$

and therefore

$$\begin{aligned} F^* d\omega &= \sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} F^* d\omega_{i_1 \cdots i_q} \wedge F^* dy_{i_1} \wedge \cdots \wedge F^* dy_{i_q} \\ &= \sum_{1 \le i_1 < i_2 < \cdots < i_q \le n} F^* d\omega_{i_1 \cdots i_q} \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_q}. \end{aligned}$$ ■

Now we generalize the results of this section to manifolds.

Note that if (U, φ) is a chart, with coordinates $x_1, \ldots x_n$ then (Exercise 5.4.1) $\varphi^* du_i = dx_i$ and therefore $\varphi^* \sum \omega_I du_I = \sum (\omega_I \circ \varphi) dx_I$.

Theorem 5.4.8 There is a unique operator, called **exterior derivative** or **differential**

$$d\colon \Omega^\bullet(M) \to \Omega^\bullet(M),$$

of degree 1 such that

i) $\forall f \in \Omega^0(M), \forall v \in \mathfrak{X}(M), df(v) = v(f)$.

ii) $\forall q_1, q_2 \geq 0, \forall \omega_1 \in \Omega^{q_1}(M), \forall \omega_2 \in \Omega^{q_2}(M)$,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{q_1} \omega_1 \wedge d\omega_2;$$

iii) $d \circ d = 0$.

If (U, φ) is a chart with coordinates $x_1, \ldots, x_n$ and on U

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} \omega_{i_1 \cdots i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q},$$

then

$$\begin{aligned} d\omega &= \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} d\omega_{i_1 \cdots i_q} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \cdots i_q}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}. \\ &= \varphi^* d((\varphi^{-1})^* \omega) \end{aligned}$$

Proof. We first prove the uniqueness part of the statement.

Assume that there are two exterior derivatives d and d'. Then, to prove the uniqueness part of the statement, we need to show that for each ω, $d\omega = d'\omega$. This is a local statement: it is enough if we prove that $d\omega$ and $d\omega'$ coincide in each chart U. Chosen a chart, we write ω in local coordinates as $\sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} \omega_{i_1 \cdots i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q}$. Repeating word-by-word the proof of the analogous statement in Theorem 5.4.6 we obtain that, in U, $d\omega = d'\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \cdots i_q}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}$.

Now we prove the existence part of the statement. We can use the expression

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \cdots i_q}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

to define $d\omega$ in a chart. This gives a well defined *global* form if and only if two such forms coincide in the intersection of the respective domain of definitions.

In other words we need to show that the local expression given for $d\omega$ is independent on the choice of the chart. Then let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two charts in p and set $d_\alpha \omega = \varphi_\alpha^* d((\varphi_\alpha^{-1})^* \omega)$, $d_\beta \omega = \varphi_\beta^* d((\varphi_\beta^{-1})^* \omega)$. We need to show that $d_\alpha \omega = d_\beta \omega$ on $U_\alpha \cap U_\beta$; this[6] follows easily by the uniqueness part of the statement (that we have already shown) applied to the manifold $U_\alpha \cap U_\beta$.

The reader can easily check that d, so defined, has the properties i)-iii). ■

We can now conclude the section generalizing Proposition 5.4.7.

[6] Here is a different proof. We need to show $\varphi_\alpha^* d((\varphi_\alpha^{-1})^* \omega) = \varphi_\beta^* d((\varphi_\beta^{-1})^* \omega)$, which may be rewritten, setting $\eta := (\varphi_\alpha^{-1})^* \omega \in \Omega^1(D_\alpha)$ as

$$\varphi_\alpha^* d\eta = \varphi_\beta^* d(\varphi_\beta^{-1})^* \varphi_\alpha^* \eta$$

which is equivalent to

$$\varphi_{\alpha\beta}^* d\eta = d\varphi_{\alpha\beta}^* \eta$$

that follows from Proposition 5.4.7.

Corollary 5.4.9 Let $F\colon M \to N$ be a smooth function, $\omega \in \Omega^\bullet(N)$.
Then $F^* d\omega = dF^*\omega$.

Proof. This is a local statement: it is enough if we prove the statement in a neighbourhood of every point $p \in M$. As a general principle, local statements on manifolds holds if and only if they hold for affine spaces, and we have proved that in Proposition 5.4.7.

More precisely, we choose charts (U,φ) in M and (V,ψ) in N such that $p \in U$, $F(U) \subset V$ and observe that by Proposition 5.4.7

$$(\psi \circ F \circ \varphi^{-1})^* d((\psi^{-1})^*\omega) = d(\psi \circ F \circ \varphi^{-1})^*((\psi^{-1})^*\omega).$$

Then

$$\begin{aligned}
F^* d\omega &= F^*\psi^* d((\psi^{-1})^*\omega) \\
&= \varphi^*(\varphi^{-1})^* F^*\psi^* d((\psi^{-1})^*\omega) \\
&= \varphi^*(\psi \circ F \circ \varphi^{-1})^* d((\psi^{-1})^*\omega) \\
&= \varphi^* d(\psi \circ F \circ \varphi^{-1})^*((\psi^{-1})^*\omega) \\
&= \varphi^* d(\varphi^{-1})^* F^*\psi^*(\psi^{-1})^*\omega \\
&= \varphi^* d(\varphi^{-1})^* F^*\omega \\
&= dF^*\omega.
\end{aligned}$$

■

Complement 5.4.1 Assume that ω, τ are homogeneous forms (possibly of different degree). Then prove

$$\tau \wedge \omega = (-1)^{(\deg\tau)\cdot(\deg\omega)}\omega \wedge \tau.$$

Complement 5.4.2 Prove that $(F \circ G)^* = G^* \circ F^*$.

Complement 5.4.3 Prove that $F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2$.

Exercise 5.4.1 Let (U,φ) be a chart for a manifold M, and let $x_1, \dots x_n$ be the corresponding local coordinates.
Prove $\varphi^* du_i = dx_i$.

Exercise 5.4.2 Let M,N be diffeomorphic manifolds.
Show that $\Omega^\bullet(N)$ is isomorphic to $\Omega^\bullet(M)$ as graded $\mathbb{R}-$algebra..

Exercise 5.4.3 Let M be a manifold, $\omega \in \Omega^q(M)$. Consider an open subset $U \subset M$ as manifold embedded in M, and choose a point $p \in M$.
Show that $\omega_p = 0 \Leftrightarrow (\omega_{|U})_p = 0$.

Exercise 5.4.4 Assume that M is a manifold without boundary, $f \in C^\infty(M)$, $y \in \mathrm{Reg}(f)$, and consider $X := f^{-1}(y)$ with the differentiable structure such that the inclusion $i\colon X \hookrightarrow M$ is an embedding (as in Theorem 2.4.7). Consider the $1-$form $df \in \Omega^1(M)$.

Show that $df_{|X} = 0$.

Exercise 5.4.5 Assume that X is a manifold embedded in a manifold M.

For every $q-$form $\omega \in \Omega^q(M)$, consider the sets

$$Z_M(\omega) := \{p \in M | \omega_p = 0\}, \quad Z_X(\omega) := \left\{p \in X | (\omega_{|X})_p = 0\right\}.$$

Show that $Z_M(\omega) \cap X \subset Z_X(\omega)$.

Consider the $1-$form $dx_1 \in \Omega^1(\mathbb{R}^2)$. Show that $Z_{\mathbb{R}^2}(dx_1) \cap S^1 \neq Z_{S^1}(dx_1)$.

Exercise 5.4.6 Consider the following two open subsets of S^1: $U_i = \{p \in S^1 | x_i \neq 0\}$ for $i = 1, 2$. Consider the $1-$form ω on S^1 defined by

$$\omega_p = \begin{cases} \left(\left(-\frac{dx_2}{x_1}\right)_{|U_1}\right)_p & \text{if } p \in U_1 \\ \left(\left(\frac{dx_1}{x_2}\right)_{|U_2}\right)_p & \text{if } p \in U_2 \end{cases}$$

Show that this gives a well defined 1-form $\omega \in \Omega^1(S^1)$ which is a volume form on S^1.

Exercise 5.4.7 Consider a function $f \in C^\infty(\mathbb{R}^n)$, $y \in \operatorname{Reg}(f)$, $M = f^{-1}(y)$. Prove[a] that the real canonical bundle of M is trivial.

[a]Hint: consider the open subsets $M_i := \{p \in M | \frac{\partial f}{\partial x_i}(p) \neq 0\}]$. Try to define $\omega_i \in \Omega^{n-1}(M)$ so that $\forall p \in M_i$,

$$\omega_p = (-1)^i \frac{(dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n)_p}{\frac{\partial f}{dx_i}(p)}$$

Exercise 5.4.8 Compute explicit formulas for the differential of a general 0-form, 1-form resp. 2-form on $\mathbb{R}^3$ and relate the results with the usual definition of gradient, curl and divergence. What's the differential of a $3-$form?

Exercise 5.4.9 Let $U, V \subset \mathbb{R}^n$ be open subsets, $F : U \to V$ be a smooth map.

Show that

$$F^*(dx_1 \wedge \cdots \wedge dx_n) = \det(J(F)) dx_1 \wedge \cdots \wedge dx_n$$

where $J(F)$ is the Jacobi matrix of F.

Exercise 5.4.10 Let U, V be open sets of $\mathbb{R}^n$ and assume that they are diffeomorphic.

Prove that $F^* : \Omega^\bullet(V) \to \Omega^\bullet(U)$ is an isomorphism.

6. Orientability and Integrability

6.1 Orientability

The orientability of a real manifold is an interesting geometrical property which has no analogs in the complex case. Indeed, as we see in the next definition, to consider it we need to be able to distinguish "positive" and "negative" numbers.

Definition 6.1.1 Let V be a finite dimensional real vector space. Then we will say that two bases of V are **orientation equivalent** if the determinant of the corresponding base change matrix is positive. This equivalence relation partitions the bases of V in two equivalence classes, the two **orientations** of V.

Consequently we will say that a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ **preserves the orientation** if $\det A > 0$. In other words, a base change matrix preserves the orientation if and only if it maps bases to bases in the same orientation class. Analogously A **reverses the orientation** if $\det A < 0$.

Let Ω, Ω' be two open subsets of $\mathbb{R}^n_\pm$ and let $F\colon \Omega \to \Omega'$ be a smooth function. F **preserves the orientation** if $\forall p \in \Omega$, the Jacobi matrix of F in p preserves the orientation. F **reverses the orientation** if $\forall p \in \Omega$, the Jacobi matrix of F in p reverses the orientation.

Note that, if Ω is connected and F is a diffeomorphism, then F either preserves or reverses the orientation.

Proposition 6.1.2 Let $\varphi\colon U \to V$ be a smooth map among open subsets of $\mathbb{R}^n_\pm$. Then

$$\varphi^* dx_1 \wedge \cdots \wedge dx_n = \det(J(\varphi))\, dx_1 \wedge \cdots \wedge dx_n.$$

Proof. It is enough to check the formula as equality on $\Lambda^n T^*_p \mathbb{R}^n$ on each point p of U, and this follows immediately by the definition of pull-back and Proposition 5.1.17. ■

Consider now a real manifold M, so we have a fixed differentiable structure, corresponding to several (pairwise *compatible*) atlases. Those whose transition functions preserve the orientation define an orientation on M as follows.

Definition 6.1.3 Let M be a real manifold of positive dimension. An atlas for M is **oriented** if all its transition functions preserve the orientation. In other words, two different local coordinates at the same point p induce bases of T_pM (taking the corresponding partial derivatives) in the same orientation class.

M is **orientable** if it admits an oriented atlas.

Two oriented atlases are **orientedly compatible** or **orientedly equivalent** if their union is oriented. This defines an equivalence relation on the set of atlases of the differentiable structure of M. An equivalence class for this equivalence relation is an **orientation** on M. A manifold with a chosen orientation is an **oriented** manifold.

If $\dim M = 0$ (then if M is a point) we set conventionally that an orientation on M is the choice of a sign: either $+$ or $-$.

In the framework of oriented manifolds, the following functions are crucial.

Definition 6.1.4 Let M, N be oriented manifolds of the same dimension, and let $F\colon M \to N$ be a smooth map.

We say that F **preserves**, resp. **reverses the orientation** if, $\forall p \in M$, given local coordinates $x_1,\ldots,x_n$ around p induced by a chart of an atlas of the orientation of M and local coordinates $y_1,\ldots,y_n$ around $F(p)$ induced by a chart of an atlas of the orientation of N, then the Jacobi matrix $\left(\frac{\partial F_i}{\partial x_j}(p)\right)$ preserves, resp. reverses the orientation.

Proposition 6.1.5 Each connected orientable manifold admits exactly two orientations.

To give an orientation of a manifold with more connected components is equivalent to choose an orientation on each component. It follows that a manifold with k connected components is orientable if and only if all its connected components are orientable, and the number of possible orientations is 2^k.

Proof. The case $\dim M = 0$ is obvious. Assume $\dim M \geq 1$.

Consider the linear application $L\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$L(x_1,x_2,\ldots,x_n) = (x_1,\ldots,x_{n-1},-x_n).$$

L is a linear isomorphism and a diffeomorphism. Moreover $L(\mathbb{R}^n_+) = \mathbb{R}^n_-$, $L(\mathbb{R}^n_-) = \mathbb{R}^n_+$ and both $L_{|\mathbb{R}^n_+}\colon \mathbb{R}^n_+ \to \mathbb{R}^n_-$ and $L_{|\mathbb{R}^n_-}\colon \mathbb{R}^n_- \to \mathbb{R}^n_+$ are diffeomorphisms.

Assume now M orientable. Let $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ be an oriented atlas for M, and consider the atlas $\{(U_\alpha, L\circ\varphi_\alpha)\}_{\alpha\in I}$. This is an oriented atlas, since $(L\circ\varphi_\alpha)\circ(L\circ\varphi_\beta)^{-1} = L\circ\varphi_\alpha\circ\varphi_\beta^{-1}\circ L^{-1} = L\circ\varphi_{\alpha\beta}\circ L$ preserves the orientation by

$$\begin{aligned}\det J\left(L\circ\varphi_{\alpha\beta}\circ L\right) &= \det J\left(L\right)\cdot\det J\left(\varphi_{\alpha\beta}\right)\cdot\det J\left(L\right) = \\ &= (-1)\cdot\det J\left(\varphi_{\alpha\beta}\right)\cdot(-1) = \det J\left(\varphi_{\alpha\beta}\right) > 0.\end{aligned}$$

The new atlas is not orientedly compatible with the first one, since $\forall\alpha$, $L\circ\varphi_\alpha\circ\varphi_\alpha^{-1}$ reverses the orientation. Therefore every orientable manifolds has at least two orientations, and it remains only to show that every further orientable atlas $\{(V_\beta,\psi_\beta)\}_{\beta\in J}$ for M is compatible with one ot these two.

For every point $p\in M$ we choose $\alpha\in I$, $\beta\in J$ with $p\in U_\alpha\cap V_\beta$. we define

$$\nu(p) := \frac{|\det J(\varphi_\alpha\circ\psi_\beta^{-1})_{\psi_\beta(p)}|}{\det J(\varphi_\alpha\circ\psi_\beta^{-1})_{\psi_\beta(p)}} \in \{\pm 1\} \subset \mathbb{R}.$$

Since both atlases $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ and $\{(V_\beta,\psi_\beta)\}_{\beta\in J}$ are oriented, $\nu(p)$ do not depend on the choice of α and β. Moreover ν is smooth, therefore continous. But M is connected, $\{\pm 1\}$ is discrete, so ν is constant. We have then two cases: either $\nu\equiv 1$ or $\nu\equiv -1$.

If $\nu \equiv 1$, a straightforward computation shows that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in J}$ are compatible. Else, $\nu \equiv -1$, and similarly $\{(U_\alpha, L\circ\varphi_\alpha)\}_{\alpha\in I}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in J}$ are compatible. ■

Notation 6.1. *If M is an oriented manifold, we will denote by $\overline{M}$ the same manifold with the* **opposite** *orientation, obtained by changing the orientation of each component.*

There is no natural way to extend the definition of orientability of the category of complex manifolds, since we can't decide if a complex number is "positive" or "negative" in a reasonable way. On the other hand, we know that every complex manifold of dimension n has a natural differentiable structure of real manifold without boundary of dimension $2n$, sometimes denoted as *the underlying real manifold*. It is then natural to ask, for every complex manifold, if its underlying real manifold is orientable or not. This natural question has a surprisingly simple answer.

Assume first for sake of simplicity that M is a complex manifold of dimension 1, with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$. Then the $\varphi_{\alpha\beta}\colon \mathbb{C}\to\mathbb{C}$ are holomorphic functions in one variable. The underlying real manifold has atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$, and the transition functions $\psi_{\alpha\beta}$ are obtained by the $\varphi_{\alpha\beta}$ removing the complex structures from its domain and from its codomain: in other words $\psi_{\alpha\beta} = (a_{\alpha\beta}, b_{\alpha\beta})$ is exactly the map $\varphi_{\alpha\beta}$ where we are considering its domain and its codomain as open subsets of $\mathbb{R}^2$ instead of $\mathbb{C}$.

By the Cauchy-Riemann relations the Jacobi matrix of $\psi_{\alpha\beta}$ is

$$\begin{pmatrix} \frac{\partial a_{\alpha\beta}}{\partial x} & -\frac{\partial b_{\alpha\beta}}{\partial x} \\ \frac{\partial b_{\alpha\beta}}{\partial x} & \frac{\partial a_{\alpha\beta}}{\partial x} \end{pmatrix}$$

whose determinant is $\frac{\partial a_{\alpha\beta}}{\partial x}^2 + \frac{\partial b_{\alpha\beta}}{\partial x}^2 > 0$. Therefore the real atlas induced by the complex atlas is already an oriented atlas.

A similar result holds in higher dimension,

Proposition 6.1.6 Let U,V be open subsets of $\mathbb{C}^n$ and let $F\colon U\to V$ be a holomorphic map. Choose a point p in U and let $B\in\mathbb{C}$ be the determinant of the $n\times n$ Jacobi matrix of F at p. Let $A\in\mathbb{R}$ be the determinant of the real Jacobi map of F, the $2n\times 2n$ real matrix representing the differential of F, seen as smooth map among real affine spaces, at p. Then

$$A = B\cdot\overline{B}.$$

In particular $A\geq 0$.

Proof. Let $z_1,\dots,z_n$ and $z'_1,\dots,z'_n$ be the complex coordinates respectively of U and V and write $z_j = x_j + iy_j$ and $z'_j = x'_j + iy'_j$, so that $(x_1,y_1,\dots,x_n,y_n)$ and $(x'_1,y'_1,\dots,x'_n,y'_n)$ are real coordinates of respectively of U and V.

By Proposition 6.1.2, denoting by $A\in\mathbb{R}$ the determinant of the Jacobi matrix of the corresponding transition functions then

$$F^*\left(dx'_1\wedge dy'_1\wedge\cdots\wedge dx'_n\wedge dy'_n\right) = A\,dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n.$$

Note that $dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n\in\Omega^{2n}(U)$. Recall that $\Omega^{2n}(U)\subset A^{n,n}(U)$. In this bigger space

$$dz_j\wedge d\bar{z}_j = (dx_j+idy_j)\wedge(dx_j-idy_j) = dx_j\wedge(-idy_j)+(idy_j)\wedge dx_j = -2i\,dx_j\wedge dy_j$$

so

$$dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n = (-2i)^n \, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

similarly

$$dz'_1 \wedge d\bar{z}'_1 \wedge \cdots \wedge dz'_n \wedge d\bar{z}'_n = (-2i)^n \, dx'_1 \wedge dy'_1 \wedge \cdots \wedge dx'_n \wedge dy'_n,$$

and then

$$F^* \left(dz'_1 \wedge d\bar{z}'_1 \wedge \cdots \wedge dz'_n \wedge d\bar{z}'_n \right) = A dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

Reorder both terms putting first the holomorphic differentials and then the antiholomorphic differentials. It changes both sides of the equation by the same power of (-1) and then

$$F^* \left(dz'_1 \wedge \cdots \wedge dz'_n \wedge d\bar{z}'_1 \wedge \cdots \wedge d\bar{z}'_n \right) = A dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n. \tag{6.1}$$

By the complex analogous of Proposition 6.1.2,

$$F^* \left(dz'_1 \wedge \cdots \wedge dz'_n \right) = B dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n. \tag{6.2}$$

Since by definition $d\bar{z}_i$, $d\bar{z}'_i$ are complex conjugated of dz_i, dz'_i

$$F^* \left(d\bar{z}'_1 \wedge \cdots \wedge d\bar{z}'_n \right) = \overline{B} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n. \tag{6.3}$$

Finally by (6.1, 6.2, 6.3) it follows

$$A = B\overline{B} \geq 0.$$

■

This shows that the underlying real manifold of any complex manifold is naturally oriented by considering any complex atlas as real atlas as above.

Theorem 6.1.7 The real atlas obtained by a complex atlas is oriented.

Two equivalent complex atlases induce orientedly equivalent real atlas, so determining a natural orientation on the underlying real manifold of any complex manifold.

Proof. By Proposition 6.1.6 a transition function of a complex atlas preserves the orientation as a map among open subsets of $\mathbb{R}^{2n}$: in fact the determinant A of its real Jacobi matrix at any point p equals $B\overline{B}$ for some $B \in \mathbb{C}$, $B \neq 0$, and then A is positive,

This implies first that the real atlas obtained by a complex atlas is oriented, and then that if two complex atlases are equivalent (*i.e.* the union is a complex atlas) the induced real atlases are orientedly equivalent (*i.e.* the union is a complex atlas). ■

R Whenever we have a complex atlas, we can construct a second complex atlas by "twisting" the first atlas by the conjugation map $c(z) = \bar{z}$ as follows.
Note that the function c is not holomorphic. However if $\{(U_\alpha, \varphi_\alpha)\}$ is a complex atlas, setting $\psi_\alpha := c \circ \varphi_\alpha$ the atlas $\{(U_\alpha, \psi_\alpha)\}$ is again a complex atlas. Indeed the *new* transition functions

$$\psi_{\alpha\beta} = c \circ \varphi_{\alpha\beta} \circ c$$

are holomorphic if and only if the *old* transition functions $\varphi_{\alpha\beta}$ are holomorphic.
Note that the induced orientations on the real underlying differentiable manifolds are the same if and only if the (complex) dimension is even.

Now we introduce a very powerful tool in the theory of real manifolds, the partitions of unity.

Definition 6.1.8 Let X be a topological space. A family $\mathfrak{S} := \{S_\alpha\}_{\alpha \in I} \subset \mathscr{P}(X)$ of subsets of X is **locally finite** if $\forall p \in X$ there exists an open set $U \ni p$ such that $U \cap S_\alpha \neq \emptyset$ only for finitely many $\alpha \in I$.

The definition is posed for every $\mathfrak{S} \subset \mathscr{P}(X)$, but we will only use it for families of open sets $\mathfrak{U} \subset \mathscr{T}(X)$ (here $\mathscr{T}(X)$ is the topology of X).

Definition 6.1.9 Let $\mathfrak{U} := \{U_\alpha\}_{\alpha \in I}$ be an open covering of a manifold M. A **partition of unity subordinate to** $\mathfrak{U}$ is a family of smooth functions $\rho_i \colon M \to [0,1]$, i varying in a countable set of indices J, such that

a) $\forall i$, the support $\operatorname{supp}(\rho_i) := \overline{\{p \in M | \rho_i(p) \neq 0\}}$ is compact;
b) $\forall i \in J$, $\exists \alpha(i) \in I$ such that $\operatorname{supp}(\rho_i) \subset U_{\alpha(i)}$;
c) $\{\operatorname{supp}(\rho_i)\}_{i \in J} \subset \mathscr{P}(M)$ is locally finite;
d) $\forall p \in M$, $\sum_{i \in J} \rho_i(p) = 1$.

Note that the sum at the point d) is meaningful because, by c), it reduces to a finite sum on a suitable small neighbourhood of every point.

We will use the next result without proving it. We only mention that the proof uses the fact that M has a countable basis of open subsets.

Theorem 6.1.10 Let $\mathfrak{U} := \{U_\alpha\}$ be an open covering of a real manifold M. Then there exists a partition of unity subordinate to $\mathfrak{U}$.

We will also need the following

Lemma 6.1.11 Let M be a manifold and let $U \subset M$ be an open subset.

Consider a form $\omega \in \Omega^q(U)$ and assume that its support $\operatorname{supp}\omega = \overline{\{p \in U | \omega_p \neq 0\}}$ is compact. Then there is a form $\widetilde{\omega} \in \Omega^q(M)$ such that

$$\begin{cases} \widetilde{\omega}_p = \omega_p & \forall p \in U \\ \widetilde{\omega}_p = 0 & \forall p \in M \setminus U \end{cases} \tag{6.4}$$

Proof. The expression (6.4) defines obviously a section of the bundle $\Lambda^q T^*M$; we only need to prove that $\widetilde{\omega}$ is smooth.

By definition on all points of U $\widetilde{\omega}$ equals ω: $\widetilde{\omega}_{|U} = \omega$.

Set $K := \operatorname{supp}\omega$. Its complement $V := M \setminus K$ is an open subset of M where $\widetilde{\omega}$ vanishes.

We have then found two open subsets U, V of M such that $U \cup V = M$ and $\widetilde{\omega}$ restricted to both is smooth. Since smoothness is a local property, then $\widetilde{\omega}$ is smooth. ■

Proposition 6.1.12 Let M be a manifold of dimension $n > 0$. Then M is orientable if and only if there exists a volume form on M, *i.e.* if and only if its real canonical bundle is trivial (compare Exercise 3.2.2).

Proof. ($\Rightarrow$) Choose an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$.

Take a partition of unity $\{\rho_i\}_{i \in \mathbb{N}}$ subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$. For every $i \in \mathbb{N}$ choose $\alpha(i)$ with $\operatorname{supp}(\rho_i) \subset U_{\alpha(i)}$ and define ω_i by

$$\omega_i(p) = \begin{cases} \rho_i \varphi^*_{\alpha(i)}(du_1 \wedge \cdots \wedge du_n) & \text{if } p \in U_{\alpha(i)} \\ 0 & \text{else.} \end{cases}$$

By Lemma 6.4, $\omega_i \in \Omega^n(M)$.

Then we can consider the form $\omega = \sum_i \omega_i \in \Omega^n(M)$. Indeed, since the support of each ω_i is $\operatorname{supp}\omega_i := \overline{\{p \in M | (\omega_i)_p \neq 0\}} = \operatorname{supp}\rho_i$, then the family $\{\operatorname{supp}\omega_i\}$ is locally finite, and therefore $\sum_i \omega_i$ is locally a finite sum.

We show that, $\forall p \in M$, $\omega_p \neq 0$.

First of all choose i with $\rho_i(p) \neq 0$. Let $x_{1\alpha}, \dots x_{n\alpha}$ be the coordinates induced by a chart $(U_\alpha, \varphi_\alpha)$ with $\operatorname{supp}\omega_i = \operatorname{supp}\rho_i \subset U_\alpha$. Then $\omega_i = \rho_i dx_{1\alpha} \wedge \cdots \wedge x_{n\alpha}$.

For every $j \neq i$, $\omega_j = \rho_j dx_{1\beta} \wedge \cdots \wedge x_{n\beta}$ for the coordinates $x_{1\beta}, \dots, x_{n\beta}$ induced by a chart (U_β, φ_β). Since $dx_{i\alpha} = \varphi_\alpha^* du_i$, $dx_{i\beta} = \varphi_\beta^* du_i$,

$$\begin{aligned}(dx_{1\beta} \wedge \cdots \wedge dx_{n\beta})_p &= (\varphi_\beta^* du_1 \wedge \cdots \wedge du_n)_p = \\ &= (\varphi_\alpha^* \varphi_{\beta\alpha}^* du_1 \wedge \cdots \wedge du_n)_p = \det J(\varphi_{\beta\alpha})_{\varphi_\alpha(p)} (dx_{1\alpha} \wedge \cdots \wedge dx_{n\alpha})_p\end{aligned}$$

and therefore, since our atlas is supposed oriented, $\forall j$, $\exists \lambda_j \geq 0$ such that $(\omega_j)_p = \lambda_j (dx_{1\alpha} \wedge \cdots \wedge x_{n\alpha})_p$. Since $\lambda_i(p) = \rho_i(p) > 0$, $\omega_p \neq 0$.

($\Leftarrow$) Take an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ for M such that all U_α are connected. We construct a further atlas for M which is oriented, by using the same open sets: an atlas of the form $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$.

Fix $\alpha \in I$, and let $x_{1\alpha}, \dots, x_{n\alpha}$ be the local coordinates induced by the chart $(U_\alpha, \varphi_\alpha)$. Then we may write $\omega_{|U_\alpha} = f_\alpha dx_{1\alpha} \wedge \cdots \wedge dx_{n\alpha}$ with $f_\alpha \in C^\infty(U_\alpha)$.

By assumption f_α never vanishes. Since U_α is assumed connected, then the function f_α is either strictly positive or strictly negative. In the former case we take $\psi_\alpha = \varphi_\alpha$; in the latter case we take $\psi_\alpha = L \circ \varphi_\alpha$ for the map L introduced in the proof of Proposition 6.1.5.

We show that the atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ is oriented. Denoting by $y_{1\alpha}, \dots, y_{n\alpha}$ the local coordinates of the chart (U_α, ψ_α), we write $\omega_{|U_\alpha} = g_\alpha dy_{1\alpha} \wedge \cdots \wedge dy_{n\alpha}$ with $g_\alpha \in \mathscr{C}^\infty(U_\alpha)$, obtaining $g_\alpha(p) > 0$ for all p. Indeed, if we had $f_\alpha > 0$, then $g_\alpha = f_\alpha$. Else $f_\alpha < 0$, and then $dy_{n\alpha} = -dx_{n\alpha}$ whence for $i < n$ $dy_{i\alpha} = dx_{i\alpha}$. In particular $f_\alpha dx_{1\alpha} \wedge \cdots \wedge dx_{n\alpha} = -f_\alpha dy_{1\alpha} \wedge \cdots \wedge dy_{n\alpha}$ and therefore $g_\alpha = -f_\alpha$.

Arguing as before $(dy_{1\beta} \wedge \cdots \wedge dy_{n\beta})_p = \det J(\psi_{\beta\alpha})_{\psi_\alpha(p)} (dy_{1\alpha} \wedge \cdots \wedge dy_{n\alpha})_p$, and therefore $\det J(\psi_{\beta\alpha})_{\psi_\alpha(p)} = \frac{g_\alpha(p)}{g_\beta(p)} > 0$. ∎

The proof of Proposition 6.1.12 shows a bit more than the statement. Fix a volume form ω, a point p in M, and a chart in p, if $x_1, \dots, x_n$ are the corresponding local coordinates, then clearly $\omega_p = \lambda(dx_1 \wedge \cdots \wedge dx_n)_p$, for some $\lambda \neq 0$. The proof of Proposition 6.1.12 shows that, if we choose the chart in an oriented atlas, the sign of λ_p does not depend neither from the chart nor from the point, but only from the choice of the orientation. Then we can give the following definition.

Definition 6.1.13 Let M be an oriented manifold and let ω be a volume form on M.

We will say that M is **positively oriented** with respect to ω if for every choice of a chart, in the given local coordinates $\omega = \lambda dx_1 \wedge \cdots \wedge dx_n$ with $\forall p\ \lambda(p) > 0$.

Similarly we will say that M is **negatively oriented** with respect to ω if for every choice of a chart, in the given local coordinates $\omega = \lambda dx_1 \wedge \cdots \wedge dx_n$ with $\forall p\ \lambda(p) < 0$.

R By Proposition 6.1.6 every local biholomorphism preserves the orientation.

The proof of Proposition 6.1.12 shows that, for each volume form $\omega \in \Omega^n(M)$, one of the two orientations of M is positively oriented with respect to ω, the other one is negatively oriented with respect to ω (and positively oriented with respect to $-\omega$).

For every oriented manifold M we can define the induced orientations on M° and ∂M.

Definition 6.1.14 Assume that M is oriented, and take an atlas for the chosen orientation. Then the atlas induced (by restriction) on M° is oriented too, giving what we call "the **induced orientation** on M°".

Definition 6.1.15 Let M be an oriented manifold. We define an orientation on ∂M, the one induced by M, as follows.

- If $\dim M = 0$, then $\partial M = \emptyset$, and there is nothing to do.
- If $\dim M = 1$, then ∂M is discrete, so to orient it we need to associate a sign to each point of it. We choose the *opposite* sign with respect to the one induced by the codomain of any chart in this point, in the following sense. If $p \in \partial M$, we pick an oriented chart (U, φ) in M with $p \in U$: if $\varphi(U) \subset \mathbb{R}^1_-$ we choose the $+$, if $\varphi(U) \subset \mathbb{R}^1_+$ we choose the $-$. This do not depend on the choice of the chart, see Exercise 6.1.6.
- if $\dim M \geq 2$ is even, we choose an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that $\forall \alpha \in I$, $\varphi_\alpha(U_\alpha)$ is open either in $\mathbb{R}^n$ or in $\mathbb{R}^n_+$ (see Exercise 6.1.7). Then we take on ∂M the orientation of the atlas $\{(U_\alpha \cap \partial M, (\varphi_\alpha)_{|U_\alpha \cap \partial M})\}_{\alpha \in I}$; we ask the student to check that it is oriented in Complement 6.1.4.
- if $\dim M \geq 3$ is odd, we take the orientation *opposite* to the one of the atlas $\{(U_\alpha \cap \partial M, (\varphi_\alpha)_{|U_\alpha \cap \partial M})\}_{\alpha \in I}$ induced by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of M.

The defintion of orientability has a natural *relative* version on bundles that, roughly speaking, correspond to give an orientation to each fibre in a continous way. We do it only for vector bundles, as this is the only case we need.

Definition 6.1.16 We will say that a vector bundle E is **orientable** if it admits a cocycle $\{g_{\alpha\beta}\}$ such that, $\forall \alpha, \beta, p$, $\det g_{\alpha\beta}(p) > 0$.

If such a cocycle exists, a trivialization $\{\Phi_\alpha\}$ associated to it induce an orientation on each fibre $E_p = \pi^{-1}(p)$. Indeed Φ_α maps E_p diffeomorphically onto $\{p\} \times \mathbb{R}^r$, so inducing an orientation on E_p from the natural orientation of $\mathbb{R}^r$; the positivity of the determinant of $g_{\alpha\beta}(p)$ ensures that the given orientation of E_p does not depend on the choice of α. Different trivializations may induce different orientations on the E_p.

Definition 6.1.17 An **orientation** on a vector bundle E is the choice of an orientation of every fibre E_p induced as above by a cocycle $\{\Phi_\alpha\}$ such that all $\det g_{\alpha\beta}(p)$ are positive.

We will say that an orientable bundle is **oriented** if an orientation is chosen.

If B is connected, then every orientable vector bundle admits exactly two orientations.

We will later need to orient the direct sum of two orientable vector bundles, so we conclue this section with the following natural

Definition 6.1.18 Let E, F be two oriented vector bundles on the same base B.

The **induced** orientation on the vector bundle $E \oplus F$ is the one such that for all $p \in B$, if $\{e_1, \ldots, e_r\}$ is an oriented basis of E_p and $\{f_1, \ldots, f_s\}$ is an oriented basis of F then $\{e_1, \ldots, e_r, f_1, \ldots, f_s\}$ is an oriented basis of $(E \oplus F)_p$.

Complement 6.1.1 Let M be an orientable manifold, $U \subset M$ an open subset. Show that U is orientable.

Complement 6.1.2 — The cylinder. The cylinder C is the quotient of $[0,1] \times [0,1] \subset \mathbb{R}^2$ by the equivalence relation $\forall y \in [0,1]$, $(0,y) \sim (1,y)$. We denote by $\pi\colon [0,1] \times [0,1] \to C$ the projection map.

We give an atlas for C with 4 charts: $\{(U_i,\varphi_i)\}_{i\in\{1,2,3,4\}}$ where

$$U_1=\pi\left(\left(\left[0,\tfrac{2}{3}\right)\cup\left(\tfrac{5}{6},1\right]\right)\times\left[0,\tfrac{2}{3}\right)\right) \quad \varphi_1(\pi(x,y))=\begin{cases}(x,y) & \text{if } x<\frac{2}{3}\\ (x-1,y) & \text{if } x>\frac{5}{6}\end{cases}$$
$$U_2=\pi\left(\left(\left[0,\tfrac{1}{6}\right)\cup\left(\tfrac{1}{3},1\right]\right)\times\left[0,\tfrac{2}{3}\right)\right) \quad \varphi_2(\pi(x,y))=\begin{cases}(x,y) & \text{if } x<\frac{1}{6}\\ (x-1,y) & \text{if } x>\frac{1}{3}\end{cases}$$
$$U_3=\pi\left(\left(\left[0,\tfrac{2}{3}\right)\cup\left(\tfrac{5}{6},1\right]\right)\times\left(\tfrac{1}{3},1\right]\right) \quad \varphi_3(\pi(x,y))=\begin{cases}(x,1-y) & \text{if } x<\frac{2}{3}\\ (x-1,1-y) & \text{if } x>\frac{5}{6}\end{cases}$$
$$U_4=\pi\left(\left(\left[0,\tfrac{1}{6}\right)\cup\left(\tfrac{1}{3},1\right]\right)\times\left(\tfrac{1}{3},1\right]\right) \quad \varphi_4(\pi(x,y))=\begin{cases}(x,1-y) & \text{if } x<\frac{1}{6}\\ (x-1,1-y) & \text{if } x>\frac{1}{3}.\end{cases}$$

i) Compute all transition functions. Notice that the atlas is not oriented, but all transition functions either preserve or reverse the orientation
ii) Prove that the cylinder is orientable by producing an oriented atlas $\{(U_i,\psi_i)\}_{i\in\{1,2,3,4\}}$.

Complement 6.1.3 Prove that the sign of the Jacobi matrix in Definition 6.1.4 do not depend on the choice of the local coordinates.

Complement 6.1.4 Show that the atlas given for X in definition 6.1.15 is oriented.

Complement 6.1.5 Show that an orientable vector bundle on an orientable manifold is an orientable manifold.

Exercise 6.1.1 Let M be a manifold and assume that there exist two charts (U_1,φ_1) and (U_2,φ_2) such that U_1 and U_2 are connected, $U_1\cap U_2\neq\emptyset$ and the transition function φ_{12} neither preserves nor reverses the orientation. Show that then M is not orientable.

Exercise 6.1.2 — The Moebius band. The Moebius band M is the quotient of the square $[0,1]\times[0,1]\subset\mathbb{R}^2$ by the equivalence relation $\forall y\in[0,1]$, $(0,y)\sim(1,1-y)$. We denote by $\pi\colon [0,1]\times[0,1]\to M$ also the projection on this quotient.

We give an atlas for M: $\{(U_i,\varphi_i)\}_{i\in\{1,2,3,4\}}$ where

$$U_1=\pi\left(\left(\left[0,\tfrac{2}{3}\right)\times\left[0,\tfrac{2}{3}\right)\right)\cup\left(\left(\tfrac{5}{6},1\right]\times\left(\tfrac{1}{3},1\right]\right)\right) \quad \varphi_1(\pi(x,y))=\begin{cases}(x,y) & \text{if } x<\frac{2}{3}\\ (x-1,1-y) & \text{if } x>\frac{5}{6}\end{cases}$$
$$U_2=\pi\left(\left(\left[0,\tfrac{1}{6}\right)\times\left(\tfrac{1}{3},1\right]\right)\cup\left(\left(\tfrac{1}{3},1\right]\times\left[0,\tfrac{2}{3}\right)\right)\right) \quad \varphi_2(\pi(x,y))=\begin{cases}(x,1-y) & \text{if } x<\frac{1}{6}\\ (x-1,y) & \text{if } x>\frac{1}{3}\end{cases}$$
$$U_3=\pi\left(\left(\left[0,\tfrac{2}{3}\right)\times\left(\tfrac{1}{3},1\right]\right)\cup\left(\left(\tfrac{5}{6},1\right]\times\left[0,\tfrac{2}{3}\right)\right)\right) \quad \varphi_3(\pi(x,y))=\begin{cases}(x,1-y) & \text{if } x<\frac{2}{3}\\ (x-1,y) & \text{if } x>\frac{5}{6}\end{cases}$$
$$U_4=\pi\left(\left(\left[0,\tfrac{1}{6}\right)\times\left[0,\tfrac{2}{3}\right)\right)\cup\left(\left(\tfrac{1}{3},1\right]\times\left(\tfrac{1}{3},1\right]\right)\right) \quad \varphi_4(\pi(x,y))=\begin{cases}(x,y) & \text{if } x<\frac{1}{6}\\ (x-1,1-y) & \text{if } x>\frac{1}{3}.\end{cases}$$

- Show that the Moebius band is not an orientable manifold.
- Consider the open set $M^\circ:=\pi([0,1]\times(0,1))$. Show that M° and every manifold which contains an open set diffeomorphic to M° is not orientable.

Exercise 6.1.3 Show that the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ is not orientable, and deduce that there is no complex structure on $\mathbb{P}^2_{\mathbb{R}}$; in other words, no complex manifold has $\mathbb{P}^2_{\mathbb{R}}$ as underlying real manifold.

Exercise 6.1.4 Let M be a complex manifold with complex atlas $\{U_\alpha, \varphi_\alpha\}$. Let $conj\colon \mathbb{C}^n \to \mathbb{C}^n$ be the conjugation map $conj(z_1,\dots,z_n) = (\bar{z}_1,\dots,\bar{z}_n)$. Then $\{U_\alpha, conj\circ\varphi_\alpha\}$ is a complex atlas, yielding then a possibly different complex structure on the same manifold. Set M' for the new complex manifold obtained.

Show that M and M' are diffeomorphic as real manifold, through a diffeomorphism that preserves the orientation if the complex dimension of M is even and reverses the orientation if the complex dimension of M is odd.

Deduce that the underlying real manifold of M' is opposite to the one induced by M ($\overline{M}$) if and only if the complex dimension of M is odd.

Exercise 6.1.5 — Interpretation of the relation among a volume form and the induced orientation of the manifold. Let M be an oriented manifold, (U,ϕ) a chart in a corresponding oriented atlas, and let as usual $x_1,\dots,x_n$ be the induced local coordinates on U. Let ω be a volume form on M.

1) Show that M is positively oriented with respect to ω if and only if $\forall p\in U$,

$$\omega_p\left(\left(\frac{\partial}{\partial x_1}\right)_p,\dots,\left(\frac{\partial}{\partial x_n}\right)_p\right) > 0.$$

1) Show that M is negatively oriented with respect to ω if and only if $\forall p\in U$,

$$\omega_p\left(\left(\frac{\partial}{\partial x_1}\right)_p,\dots,\left(\frac{\partial}{\partial x_n}\right)_p\right) < 0.$$

Exercise 6.1.6 Recall that, for every chart $(U_\alpha,\varphi_\alpha)$ of a manifold of dimension n, $\varphi_\alpha(U_\alpha)$ is an open subsets of one ot the following: $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\mathbb{R}^n_-$.

Let M be a 1-dimensional oriented manifold, $p\in\partial M$. Show that

- either for every chart $(U_\alpha,\varphi_\alpha)$ with $p\in U_\alpha$, $\varphi_\alpha(U_\alpha)$ is an open subsets of $\mathbb{R}^1_+$,
- or for every chart $(U_\alpha,\varphi_\alpha)$ with $p\in U_\alpha$, $\varphi_\alpha(U_\alpha)$ is an open subsets of $\mathbb{R}^1_-$.

Exercise 6.1.7 Let M be an oriented manifold of dimension at least 2.

Show that there is an atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ for the chosen orientation such that $\forall\alpha\in I$, $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Show that the previous statement fails if we suppose $\dim M = 1$.

Exercise 6.1.8 Consider the identity map of an orientable manifold, taking two different orientations for the domain and for the codomain: $\mathrm{Id}_M\colon M\to\overline{M}$.

Show that, with this choice of the orientations, Id_M reverses the orientation.

Exercise 6.1.9 Assume that a map $F\colon M\to N$ preserves the orientation.

Prove that the map F considered as a map $F\colon\overline{M}\to N$ or as a map $F\colon M\to\overline{N}$, reverses

the orientation.
What can be said on the map $F: \overline{M} \to \overline{N}$?
What if we assume instead that F reverses the orientation?

Exercise 6.1.10 Show that, if M is an orientable manifold, then its tangent bundle $TM \to M$ is an orientable vector bundle.

Exercise 6.1.11 Show that, if M is an orientable manifold, then its cotangent bundle $T^*M \to M$ is an orientable vector bundle.

Exercise 6.1.12 Show that, if M is an orientable bundle, then the vector bundle

$$\Lambda^{\dim M} T^*M \to M$$

is an orientable vector bundle.

Exercise 6.1.13 Let M be an oriented manifold, and let S be a manifold embedded in M. Show that $TM_{|S}$ is an orientable bundle.

Exercise 6.1.14 Let S, M be oriented manifolds, and assume that S is embedded in M. Show that $\mathscr{N}_{S|M}$ is an orientable bundle.

6.2 Integrating forms

We know how to integrate smooth functions on open subsets of $\mathbb{R}^n_\pm$; the classical Riemann's integration theory is enough for this class of functions.

Every idea on $\mathbb{R}^n$ which is sufficiently independent from the choice of the coordinates may be lifted to the larger category of the real manifolds. Unfortunately, the integration does not have this property.

Actually the area of an open subsets $U \subset \mathbb{R}^n$, which is the integral on U of the constant function 1, depends on the choice of the coordinates: if you "double" all coordinates the area is multiplied by 2^n.

The action of coordinate changes on integrals is precisely described by the following famous result.

Theorem 6.2.1 Let U and V be two open subsets of $\mathbb{R}^n_\pm$ and let $\varphi: V \to U$ be a diffeomorphism. Let $f: U \to \mathbb{R}$ be a smooth function with compact support. Then

$$\int_U f = \int_V (f \circ \varphi) |\det J(\varphi)|.$$

Here and in the following we assume that f has compact support to avoid convergence problems.

Theorem 6.2.1 shows that the action of a coordinate change on an integral depends only on the determinant of the Jacobi matrix of the coordinate change. Exercise 5.4.9 suggests then to consider n-forms where $n = \dim M$.

We first restrict our attention to the forms with compact support.

Definition 6.2.2 The **space of q-forms with compact support** $\Omega^q_c(M)$ is the vector subspace of $\Omega^q(M)$

$$\Omega^q_c(M) := \{\omega \in \Omega^q(M) \,|\, \operatorname{supp}\omega \text{ is compact}\}$$

First of all, we define the integral of a form in $\omega \in \Omega^n_c(U)$ on an open subset $U \subset \mathbb{R}^n_{\pm}$. Then ω may be uniquely written as $\omega = f du_1 \wedge \cdots \wedge du_n$ for a smooth function $f \in \Omega^0_c(U) = C^\infty_c(U)$.

Definition 6.2.3 If $\omega = f du_1 \wedge \cdots \wedge du_n \in \Omega^n_c(U)$ then we define

$$\int_U \omega := \int_U f. \tag{6.5}$$

Is this definition independent from the choice of the coordinates? Not completely.

Proposition 6.2.4 Let U, V be two open subsets of $\mathbb{R}^n_{\pm}$, $\omega \in \Omega^n_c(U)$, and let $\varphi \colon V \to U$ be a diffeomorphism.

If φ preserves the orientation, then

$$\int_U \omega = \int_V \varphi^*\omega.$$

If φ reverses the orientation, then

$$\int_U \omega = -\int_V \varphi^*\omega.$$

Proof. Assume that φ preserves the orientation; in other words, assume that $\det(J(\varphi))$ is always positive.

Write $\omega = f du_1 \wedge \cdots \wedge du_n$. Then, by Theorem 6.2.1 and Exercise 5.4.9,

$$\begin{aligned}
\int_U \omega = \int_U f &= \int_V (f \circ \varphi) |\det(J(\varphi))| \\
&= \int_V (\varphi^* f) \det(J(\varphi)) \\
&= \int_V (\varphi^* f) \det(J(\varphi)) \, du_1 \wedge \cdots du_n \\
&= \int_V (\varphi^* f) \varphi^*(du_1 \wedge \cdots \wedge du_n) \\
&= \int_V \varphi^*\omega.
\end{aligned}$$

■

If follows that, to have a definition of integral which is independent from the coordinates, we have to ensure that all transition functions preserve the orientation: we have to fix an orientation.

This allows us to define an integration theory on $\Omega^n_c(M)$ only if M is an oriented manifold. We start by considering forms whose support is contained in a chart.

Definition 6.2.5 Let M be an oriented manifold of dimension n, and let $\omega \in \Omega^n_c(M)$. Assume that there exists (U, φ) in the oriented atlas of M such that $\operatorname{supp}\omega \subset U$. Then we define

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega \tag{6.6}$$

Proposition 6.2.4 ensures that Definition 6.2.5 is well posed, showing that the right-hand term of (6.6) is independent from the choice of the chart.

More precisely, if $\operatorname{supp}\omega \subset U_\alpha \cap U_\beta$, since $\varphi_{\alpha\beta}$ preserves the orientation, then

$$\int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*\omega = \int_{\varphi_\alpha(U_\alpha\cap U_\beta)} (\varphi_\alpha^{-1})^*\omega =$$
$$= \int_{\varphi_\beta(U_\alpha\cap U_\beta)} \varphi_{\alpha\beta}^*(\varphi_\alpha^{-1})^*\omega = \int_{\varphi_\beta(U_\alpha\cap U_\beta)} (\varphi_\beta^{-1})^*\omega = \int_{\varphi_\beta(U_\beta)} (\varphi_\beta^{-1})^*\omega.$$

To extend Definition 6.2.5 to any $\omega \in \Omega_c^n(M)$ we need to use the partitions of unity.

Definition 6.2.6 Let M be an oriented manifold and choose one of the corresponding oriented atlases $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$. Choose a partition of unity subordinate to the cover $\mathfrak{U} := \{U_\alpha\}_{\alpha\in I}$. For every $i \in \mathbb{N}$ choose $\alpha(i)$ with $\operatorname{supp}\rho_i \subset U_{\alpha(i)}$ and define $\omega_i := \rho_i\omega$.

Then we define

$$\int_M \omega := \sum_{i\in\mathbb{N}} \int_M \omega_i.$$

Apparently the right-hand term is an infinite sum. One can prove that since $\{\operatorname{supp}\omega_i\}$ is locally finite and $\operatorname{supp}\omega$ is compact, then there are only finitely many indices such that ω_i is not identically 0. So all but finitely many addenda of the right-hand term are zero: it is a finite sum.

Anyway, at a first glance this is still not a good definition, since the formula defining $\int_M \omega$ appears to be dependent on the chosen atlas and on the chosen partition of unity. This problem is solved by the next proposition.

Proposition 6.2.7 Definition 6.2.6 do not depend neither on the choice of the partition nor on the choice of the atlas, but only on the orientation of M.

More precisely, if $\overline{M}$ is the same manifold taken with the opposite orientation, then

$$\int_M \omega = -\int_{\overline{M}} \omega$$

Proof. A partition of unity may be subordinate to many different atlases. Obviously if we change atlas (for the same orientation) without changing the partition of unity, the ω_i do not change, and therefore Definition 6.2.6 do not depend on the choice of the atlas.

Consider now the general case of two different partitions of unity $\{\rho_i\}_{i\in\mathbb{N}}$ and $\{\sigma_j\}_{j\in\mathbb{N}}$, subordinate to two different atlases $\{(U_\alpha,\varphi_\alpha)\}$ and $\{(U_\beta,\varphi_\beta)\}$.

First of all, we notice that $\{(U_\alpha,\varphi_\alpha)\} \cup \{(U_\beta,\varphi_\beta)\}$ is an atlas orientedly compatible with both, and such that both partitions of unity are subordinate to it. So we can assume $\{(U_\alpha,\varphi_\alpha)\} = \{(U_\beta,\varphi_\beta)\}$.

Second, we note that also the family of functions $\{\rho_i\sigma_j\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is a partition of unity! Indeed $\mathbb{N}\times\mathbb{N}$ is countable and all other properties follow from the analogous properties of $\{\rho_i\}$ and $\{\sigma_j\}$.

We define $\omega_{ij} := \rho_i\sigma_j\omega$. If we prove, $\forall i \in \mathbb{N}$,

$$\int_M \rho_i\omega = \sum_j \int_M \omega_{ij}.$$

then $\sum_i \int_M \rho_i\omega = \sum_{i,j}\int_M \omega_{ij}$, and similarly it equals $\sum_j \int_M \sigma_j\omega$ concluding our proof.

This is simple to prove: take a chart (U,φ) containing $\operatorname{supp}\omega_i$, and compute

$$\sum_j \int_M \omega_{ij} = \sum_j \int_M \sigma_j \rho_i \omega = \sum_j \int_{\varphi(U)} (\sigma_j \circ \varphi^{-1})(\varphi^{-1})^* \rho_i \omega =$$
$$= \int_{\varphi(U)} \left(\sum_j \sigma_j \circ \varphi^{-1} \right) (\varphi^{-1})^* \rho_i \omega = \int_{\varphi(U)} (\varphi^{-1})^* \rho_i \omega = \int_M \rho_i \omega.$$

Finally, if $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha, L \circ \varphi_\alpha)\}$ (where $L(u_1, \ldots, u_{n-1}, u_n) = (u_1, \ldots, u_{n-1}, -u_n)$) is an atlas for $\overline{M}$. Computing the integrals using these atlases and the same partition of unity, by Proposition 6.2.4 follows $\int_M \omega = -\int_{\overline{M}} \omega$. ■

Definition 6.2.8 If $\dim M = 0$, then $M = \{p\}$ is a point, and its orientation is a sign, $\varepsilon(p) \in \{\pm\}$. The objects to integrate are the functions $f\colon \{p\} \to \mathbb{R}$, which are naturally identified with $\mathbb{R}$ by $f \mapsto f(p)$. Then we define $\int_M f := \varepsilon(p) f(p)$.

R Arguing as in 6.2.7, it is not difficult to show (see Complement 6.2.1) that if $f\colon M \to N$ is a diffeomorphism which preserves the orientation for all $\omega \in \Omega^n_c(N)$ then $\int_M f^*\omega = \int_N \omega$ and similarly, if f reverses the orientation, then $\int_M f^*\omega = -\int_N \omega$.

In the definition of partition of unity we have requested the supports of the ρ_i to be compact, which was somewhere convenient. Anyway, we notice that in the proof of Proposition 6.2.7, we haven't used the compactness of the supports of the σ_j. It follows that, when computing the integral of a form, we can also use a "partition of unity" with support not compact.

This is important for solving Complement 6.2.4 and then Complement 6.2.5, which are very important to compute integrals explicitly. Indeed, nobody computes integrals using directly the definition, since partitions of unity produce functions usually very hard to integrate. Anyway, most manifolds contains a chart whose complement is a union of one or more embedded manifolds of smaller dimension. Then by the above mentioned complements, the integral of a form does not change when we restrict to such a chart, and then we can reduce the computation to a single "classical" integral.

We conclude this section with the following famous theorem.

Theorem 6.2.9 — Stokes' Theorem. Let M be an oriented manifold of dimension n, $\omega \in \Omega^{n-1}_c(M)$. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Consider an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ for M, a partition of unity $\{\rho_i\}_{i \in \mathbb{N}}$ subordinate to $\{U_\alpha\}_{\alpha \in I}$, and define $\omega_i := \rho_i \omega$. Then $\omega = \sum_i \omega_i$ and therefore

$$\int_{\partial M} \omega = \int_{\partial M} \sum_i \omega_i = \sum_i \int_{\partial M} \omega_i.$$

On the other hand

$$\int_M d\omega = \int_M d(\sum_i \omega_i) = \sum_i \int_M d\omega_i.$$

Therefore if the theorem holds for each ω_i, then it holds for ω. We may then assume that $\operatorname{supp}\omega \subset U$ for an (oriented) chart (U,φ).

We assume for simplicity $\varphi\colon U \to \mathbb{R}^n_+$, the proof for the case in which the codomain of φ is $\mathbb{R}^n_-$ being almost identical.

We write

$$(\varphi^{-1})^*\omega = \sum_{i=1}^{n} a_i(u_1,\dots,u_n)du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n,$$

for the a_i some smooth functions whose compact support is contained in the open set $\varphi(U)$ of $\mathbb{R}^n_+$. We extend these functions to functions $a_i \in C^\infty(\mathbb{R}^n_+)$ setting them zero out of $\varphi(U)$; this extends $(\varphi^{-1})^*\omega$ to a form in $\Omega^{n-1}_c\mathbb{R}^n_+$. Since $\int_M d\omega = \int_{\varphi(U)}(\varphi^{-1})^*d\omega = \int_{\varphi(U)} d(\varphi^{-1})^*\omega$, we get

$$\begin{aligned}
\int_M d\omega &= \int_{\mathbb{R}^n_+} d\left(\sum_{i=1}^{n} a_i du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n\right) = \\
&= \int_{\mathbb{R}^n_+} \left(\sum_{i=1}^{n} da_i \wedge du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n\right) = \\
&= \int_{\mathbb{R}^n_+} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial a_i}{\partial u_j} du_j \wedge du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n\right) \\
&= \int_{\mathbb{R}^n_+} \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial u_i} du_i \wedge du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n\right) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n_+} \frac{\partial a_i}{\partial u_i} du_1 \wedge \cdots \wedge du_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n_+} \frac{\partial a_i}{\partial u_i} du_1 \cdots du_n.
\end{aligned}$$

We start the computation of $\int_{\mathbb{R}^n_+} \frac{\partial a_i}{\partial u_i} du_1 \cdots du_n$. by integrating with respect to the variable u_i.

We need to distinguish two cases, since we are integrating on $\mathbb{R}^n_+$, so all variables vary from $-\infty$ to ∞ but the last one, u_n, which varies from 0 to $+\infty$.

$$\begin{aligned}
\int_M d\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n_+} \frac{\partial a_i}{\partial u_i} du_1 \cdots du_n \\
&= (-1)^{n-1} \int_{\mathbb{R}^n_+} \frac{\partial a_n}{\partial u_n} du_1 \cdots du_n + \sum_{i=1}^{n-1} (-1)^{i-1} \int_{\mathbb{R}^n_+} \frac{\partial a_i}{\partial u_i} du_1 \cdots du_n \\
&= (-1)^{n-1} \int du_1 \cdots du_{n-1} \int_0^{+\infty} \frac{\partial a_n}{\partial u_n} du_n + \sum_{i=1}^{n-1} (-1)^{i-1} \int \cdots \int_{-\infty}^{\infty} \frac{\partial a_i}{\partial u_i} du_i
\end{aligned}$$

Finally we note that, since all the a_i have compact support, $\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial u_i} du_i = 0$ and $\int_0^{+\infty} \frac{\partial a_n}{\partial u_n} du_n = -a_n(u_1,\dots,u_{n-1},0)$. Therefore

$$\int_M d\omega = (-1)^n \int a_n(u_1,\dots,u_{n-1},0) du_1 \cdots du_{n-1}. \tag{6.7}$$

To compute $\int_{\partial M} \omega = \int_{\partial\mathbb{R}^n_+} (\varphi^{-1})^*\omega$ we recall that the orientation of $\partial\mathbb{R}^n_+$ coincides with the standard orientation of $\mathbb{R}^{n-1}$ if and only if n is even. Then

$$\int_{\partial M} \omega = (-1)^n \int_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} a_i du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n.$$

The restriction of every form du_i, $i < n$ to $\mathbb{R}^{n-1}$ is the namesake form du_i. In contrast, the restriction of the form du_n to $\mathbb{R}^{n-1}$, is the zero form! Therefore all summands vanish but the last one (for $i = n$) and

$$\int_{\partial X} \omega = (-1)^n \int_{\mathbb{R}^{n-1}} a_n(u_1, \dots, u_{n-1}, 0) du_1 \cdots du_{n-1}. \tag{6.8}$$

The statement follows by comparing (6.7) and (6.8). ∎

A first easy consequence is very important for the next chapter.

Corollary 6.2.10 Let M be oriented of dimension n and $\partial M = \emptyset$, $\omega \in \Omega_c^{n-1}(M)$. Then $\int_M d\omega = 0$

Complement 6.2.1 If $F\colon M \to N$ is a diffeomorphism which preserves the orientation, and $\omega \in \Omega_c^n(N)$ then

$$\int_M F^*\omega = \int_N \omega.$$

If $F\colon M \to N$ is a diffeomorphism which reverses the orientation, and $\omega \in \Omega_c^n(Y)$ then

$$\int_M F^*\omega = -\int_N \omega.$$

Complement 6.2.2 Let M_1, M_2 oriented manifolds, assume $\partial M_1 = \emptyset$ and consider the manifold $M_1 \times M_2$ with the orientation induced by the orientations of the M_i. Let $\pi_i\colon M_1 \times M_2 \to M_i$ be the natural projections and consider two forms $\omega_i \in \Omega_c^{\dim M_i}(M_i)$.

Prove that

$$\int_{M_1 \times M_2} (\pi_1^*\omega_1 \wedge \pi_2^*\omega_2) = \left(\int_{M_1} \omega_1\right)\left(\int_{M_2} \omega_2\right)$$

Complement 6.2.3 Show that the function $\int_M\colon \Omega_c^{\dim M}(M) \to \mathbb{R}$ is linear.

Complement 6.2.4 Let M be an oriented manifold of dimension n, N a manifold of strictly smaller dimension.

Let $i\colon N \hookrightarrow M$ an embedding with closed image. Consider the open subset $M' := M \setminus i(N) \subset M$ with the orientation induced by M.

Prove that, if $\omega \in \Omega_c^n(M)$, and $\omega_{|M'} \in \Omega_c^n(M')$, then

$$\int_M \omega = \int_{M'} \omega.$$

Complement 6.2.5 Let M, n, N, i, M' as in the previous exercise. We assume $\omega \in \Omega_c^n(M)$ (but we do not do any assumption on $\operatorname{supp} \omega_{|M'}$. Extend Definition 6.2.6 to a definition of $\int_{M'} \omega$, and show that it is a good definition.

Exercise 6.2.1 Let M be a compact complex manifold of dimension n, and let $\omega \in \Omega^{n-1,0}(M)$ be a holomorphic form of degree $n-1$. Recall (see Exercise 5.3.4) that $i^n d\omega \wedge d\overline{\omega}$ is a real form, an element of $\Omega^{2n}(M)$.

1. Prove that $\int_M i^n d\omega \wedge d\overline{\omega} = 0$.
2. Prove that $d\omega = 0$. In other words all holomorphic forms of degree $n-1$ are closed.

6.3 Integrating functions

We can define an integration theory on orientable manifolds for smooth functions by choosing a volume form ω on M, whose existence is guaranteed by Proposition 6.1.12, as follows.

Definition 6.3.1 Consider a manifold M and a volume form ω on it.

Then, for every $f \in C_c^\infty(M)$, we define $\int_M f := \int_M f\omega$ where in the right-hand term M is taken with the positive orientation with respect to ω.

Then the choice of a volume form allows us to integrate functions. Please note that we write for simplicity $\int_M f$ but this strongly depends on the choice of ω. If we change the volume form, $\int_M f$ changes!

If M is compact, we can then define its volume.

Definition 6.3.2 Let M be a compact manifold of dimension n, $\omega \in \Omega^n(M)$ be a volume form. Then, we define the **volume** of M as

$$V(M) := \int_M 1 = \int_M \omega.$$

The main example of volume form is the form $du_1 \wedge \cdots \wedge du_n$ on $\mathbb{R}^n$.

Consider a function $f \in \Omega^0(\mathbb{R}^n)$, $y \in \mathrm{Reg}(f)$, and set $M := f^{-1}(y) \subset \mathbb{R}^n$.

To ease the notation we write $du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n$ for the form $du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n$. Then by Exercise 5.4.7 the expression

$$\eta_p := (-1)^{n+i} \frac{(du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n)_p}{\frac{\partial f}{du_i}(p)} \tag{6.9}$$

gives a well-defined volume form η on the whole M.

Indeed there is at least one index i for which $\frac{\partial f}{du_i}(p)$ does not vanish, and then also the numerator does not vanish as alternating form even when restricted to $T_pM = \ker df_p$.

Moreover, for two different choices of the index i such that $\frac{\partial f}{du_i}(p) \neq 0$, the right-hand term of (6.9) gives the same alternating form on $\ker df_p = T_pM$. Let us insist on the fact that this is true only on M. The right-hand term of (6.9) gives, for different i, forms on $\mathbb{R}^n$ that differs in any point $p \in \mathbb{R}^n$, including the points of M.

They are different alternating forms on $T_p\mathbb{R}^n$ whose restriction to T_pM coincide.

Notice however that

$$(-1)^{n+i} \frac{(du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n)_p}{\frac{\partial f}{du_i}(p)} \wedge df_p = (du_1 \wedge \cdots \wedge du_n)_p.$$

In particular, if $v_1,\dots,v_{n-1}$ are vectors in T_pM, $v_n \in T_p\mathbb{R}^n$, then

$$\begin{aligned}
&du_1\wedge\cdots\wedge du_n(v_1,\dots,v_{n-1},v_n) =\\
&= \left((-1)^{n+i}\frac{(du_1\wedge\cdots\wedge\widehat{du_i}\wedge\cdots\wedge du_n)_p}{\frac{\partial f}{du_i}(p)}\wedge df_p\right)(v_1,\dots,v_{n-1},v_n) =\\
&= \sum_{\sigma\in\mathfrak{S}_n}\frac{\varepsilon(\sigma)(-1)^{n+i}}{n!}\frac{(du_1\wedge\cdots\wedge\widehat{du_i}\wedge\cdots\wedge du_n)_p}{\frac{\partial f}{du_i}(p)}(v_{\sigma(1)},\dots,v_{\sigma(n-1)})df_p(v_{\sigma(n)}) =\\
&= \sum_{\sigma\in\mathfrak{S}_{n-1}}\frac{\varepsilon(\sigma)(-1)^{n+i}}{n!}\frac{(du_1\wedge\cdots\wedge\widehat{du_i}\wedge\cdots\wedge du_n)_p}{\frac{\partial f}{du_i}(p)}(v_{\sigma(1)},\dots,v_{\sigma(n-1)})df_p(v_n) =\\
&= \frac{df_p(v_n)}{n!}\sum_{\sigma\in\mathfrak{S}_{n-1}}\varepsilon(\sigma)\omega_p(v_{\sigma(1)},\dots,v_{\sigma(n-1)}) =\\
&\qquad\qquad = \frac{(n-1)!df_p(v_n)}{n!}\omega_p(v_1,\dots,v_{n-1}) = \frac{df_p(v_n)}{n}\omega_p(v_1,\dots,v_{n-1}),
\end{aligned}$$

so, for $v_n \notin T_pM$ (which means $df_p(v_n)\neq 0$)

$$\eta_p(v_1,\dots,v_{n-1}) = \frac{n du_1\wedge\cdots\wedge du_n}{df_p(v_n)}(v_1,\dots,v_{n-1},v_n). \tag{6.10}$$

We notice that the induced volume form depends not only on M but also on the choice of f. Indeed, replacing, $\forall\lambda\neq 0$, f by λf and y by λy we get the same M but the induced volume form changes, being divided by λ. This is not convenient, so we multiply the form by the norm of the gradient ∇f.

Definition 6.3.3 Consider a function $f\in\Omega^0(\mathbb{R}^n)$, define $\nabla_p f = \sum\frac{\partial f}{\partial u_i}(p)\left(\frac{\partial}{\partial u_i}\right)_p$ and set therefore $||\nabla_p f|| = \sqrt{\sum\left(\frac{\partial f}{\partial u_i}(p)\right)^2}$.

Pick $y\in\mathrm{Reg}(f)$ and set $M := f^{-1}(y)\subset\mathbb{R}^n$. The **induced volume form** ω on M is defined by the following equality, holding $\forall p\in M$, $\forall v_i\in T_pM$:

$$\omega_p(v_1,\dots,v_{n-1}) = n\frac{1}{||\nabla_p f||}du_1\wedge\cdots\wedge du_n(v_1,\dots,v_{n-1},\nabla_p f).$$

Note that $df_p(\nabla_p f) = ||\nabla_p f||^2$, so by (6.10) $\omega = ||\nabla f||\eta$.

Note that we need $y\in\mathrm{Reg}(f)$ to ensure that we are not dividing by zero. Clearly Definition 6.3.3 of the induced volume form does not change if we substitute f with λf, $\lambda>0$, whereas if we substitute f by $-f$, ω is substituted by $-\omega$ changing then the orientation induced on M. Correspondingly the linear application $\int_M\colon \Omega_c^{n-1}\to C_c^\infty(M)$ depends on the choice of the orientation of M. In contrast, the induced integral $\int_M\colon C_c^\infty(M)\to\mathbb{R}$ does not depend on the choice of f.

Let us see an example.

Example 6.1 Let $f\in C^\infty(\mathbb{R}^2)$, $y\in\mathrm{Reg}(f)$, and assume that $M := f^{-1}(y)$ is compact: a compact closed regular plane curve. Let $\omega\in\Omega^1(M)$ be the volume form induced by f as in Definition 6.3.3.

Consider a **regular parametrization** γ of M, a surjective immersion $\gamma\colon [0,1]\to M$ such that $\gamma_{|[0,1)}$ is injective and $\gamma(0)=\gamma(1)$. Set $\gamma'(t_0) := d\gamma_{t_0}\left(\frac{d}{dt}\right)_p$.

Then, $\forall v\in T_{\gamma(t_0)}M$, $v=\lambda\gamma'(t_0)$. Let us compute $\omega_p(v)$. Set $\gamma'(t_0) =: (\gamma_1,\gamma_2)$. Then, up to rescaling the function f defining M we can assume $\nabla f = (-\gamma_2,\gamma_1)$.

So

$$\omega_p(v) = \omega_p(\lambda\gamma'(t_0)) = \lambda\,\omega_p(\gamma'(t_0)) =$$
$$= \lambda\frac{2}{||\gamma'(t_0)||}du_1\wedge du_2(\gamma'(t_0),\nabla f) = \frac{\lambda}{||\gamma'(t_0)||}\det\begin{pmatrix}\gamma_1 & \gamma_2\\ -\gamma_2 & \gamma_1\end{pmatrix} = \lambda||\gamma'(t_0)||.$$

We conclude this section by showing some classical applications of the Stokes' theorem.

Example 6.2 — Fundamental Theorem of Calculus. Take $M = [a,b] \subset \mathbb{R}$, with the natural orientation, $\omega = f \in C^\infty([a,b])$.

The boundary is $\partial M = \{a,b\}$ oriented by taking the $+$ in b and the $-$ in a. Therefore $\int_{\partial M} f = f(b) - f(a)$. By $df = f'(t)dt$ Stokes' theorem in this case is just the fundamental theorem of the calculus

$$\int_{[a,b]} f'(t)dt = f(b) - f(a),$$

Similarly suppose that M is an arc, which means that M is the image of an embedding $i\colon [a,b] \to \mathbb{R}^n$. Consider M with the orientation making i an orientation preserving diffeomorphism. Take a function $f \in C^\infty(\mathbb{R}^n)$.

Then, as in the previous case

$$\int_M df = f(i(b)) - f(i(a)).$$

More generally, if $\dim M = 1$, then $\int_M df$ is the sum (with suitable signs) of the values of f on the boundary points (if any) of M.

Example 6.3 — Green formula. Let $A \subset \mathbb{R}^2$ be an *open subset with regular boundary*, which means that $\overline{A}$ is a manifold with boundary embedded in $\mathbb{R}^2$ whose interior is A. We consider a $1-$form $\omega \in \Omega^1_c(\overline{A})$, $\omega = P(x,y)dx + Q(x,y)dy$, so $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$.

Then Stokes' theorem in this case gives

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_\gamma Pdx + Qdy,$$

where γ is ∂A positively (counterclockwise) oriented.

Complement 6.3.1 Consider a smooth function $f \in C^\infty(\mathbb{R}^3)$, $y \in \mathrm{Reg}(f)$, $M := f^{-1}(y)$. Construct the volume form ω induced on M in Definition 6.3.3.

Consider a parametrization of an open subset of M, which means consider an open subset $U \subset \mathbb{R}^2$ and an embedding $P\colon U \hookrightarrow M$. Show that $P^*\omega = \pm\sqrt{\det G}\,du_1\wedge du_2$.

Complement 6.3.2 Let $f \in C^\infty(\mathbb{R}^n)$, $y \in \mathrm{Reg}(f)$, $\lambda \in \mathbb{R}\setminus\{0\}$, $g := \lambda f$. Then $\lambda y \in \mathrm{Reg}(g)$ and $M = f^{-1}(y) = g^{-1}(\lambda y)$

Definition 6.3.3 induces two different volume forms ω_f and ω_g on M, respectively induced by f and g.

Show that $\omega_f = \omega_g \Leftrightarrow \lambda > 0$.

Show that the two corresponding linear applications $\int_M\colon C^\infty_c(M) \to \mathbb{R}$ coincide, regardless the positivity of λ.

Complement 6.3.3 — The divergence theorem. Prove

$$\int_A div(f) = \int_{\partial A} f \cdot \hat{n},$$

for an open set $A \subset \mathbb{R}^3$ with regular boundary ∂A. Here $f\colon \overline{A} \to \mathbb{R}^3$ is a smooth function, $\hat{n}$ is one of the two vectors of norm 1 orthogonal to the surface (which one?), and the **divergence** of f is the function $div(f) := \sum_{i=1}^3 \frac{\partial f_i}{\partial x_i}$.

Complement 6.3.4 — Stokes' theorem on the curl. If $S \subset \mathbb{R}^3$ is an oriented embedded surface and $\Gamma = \partial S$ is its boundary with the induced orientation. Consider a $1-$form $\omega := f_1 dx_1 + f_2 dx_2 + f_3 dx_3$.

Prove

$$\int_S curl(f) \cdot \hat{n} = \int_\Gamma \omega.$$

where $curl(f)$ is the function with values in $\mathbb{R}^3$

$$curl(f) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right).$$

Exercise 6.3.1 Consider a parametrized plane curve $\gamma\colon [0,1] \to \mathbb{R}^2$, and assume that γ is an embedding in a submanifold M of $\mathbb{R}^2$ as in Example 6.1. Endowe $\Gamma := \gamma([0,1])$ with the volume form pull-back of the volume form of M.

Prove that the volume of Γ (say **the length**) equals $\int_0^1 |\gamma'|$.

Exercise 6.3.2 Find a form $\omega \in \Omega^2(\mathbb{R}^2)$ whose restriction to S^1 is the volume form induced by $f = x_1^2 + x_2^2$ as in Definition 6.3.3.

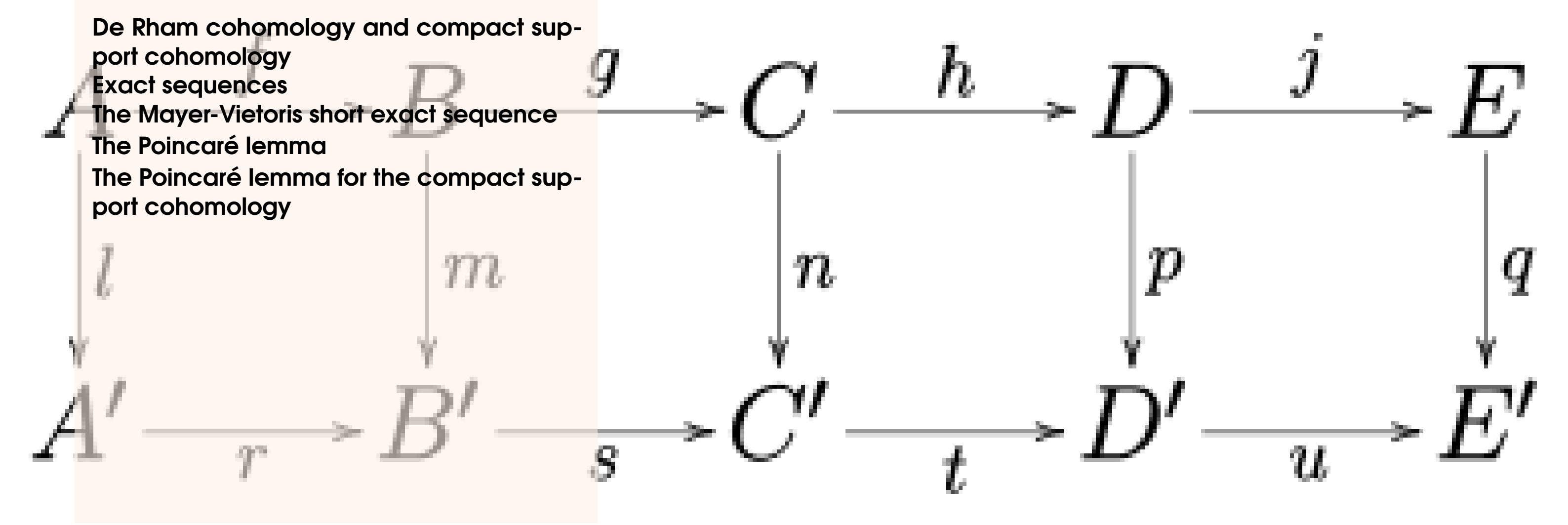
Exercise 6.3.3 Consider $S^1 = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$. Prove that the volume of S^1 is 2π.

Exercise 6.3.4 Let $\Pi \subset \mathbb{R}^2$ be a polygon[a] of vertices $P_1, \dots, P_r$, ordered counterclockwise. Set $(x_i, y_i) := P_i$, $x_0 := x_r$, $x_{r+1} := x_1$.

Prove that the area of Π equals

$$\frac{1}{2} \sum_{i=1}^{r} y_i (x_{i+1} - x_{i-1}).$$

[a] Warning: a polygon is NOT a manifold with boundary embedded in the plane because of the corners at the vertices.



7. De Rham cohomology

We can now define the De Rham cohomology of a real manifold.

Unless we do explicitly state something different, all manifolds of this chapter are real manifolds. When we consider the De Rham cohomology of a complex manifold M, we are considering M just as oriented real manifold, with the orientation given in Theorem 6.1.7.

7.1 De Rham cohomology and compact support cohomology

Definition 7.1.1 A **differential complex** is a pair $(V^\bullet, d)$ where $V^\bullet = \bigoplus_{q\in\mathbb{Z}} V^q$ is a graded vector space and $d\colon V^\bullet \to V^\bullet$ is an operator of degree 1 such that $d\circ d = 0$.

If $(V^\bullet, d)$ is a differential complex $\operatorname{Im} d \subset \ker d$ and we can define its **cohomology**

$$H_d^\bullet(V^\bullet) := \frac{\ker d}{\operatorname{Im} d}.$$

For every $\omega \in \ker d$ we denote by $[\omega]$ its class in $H_d^\bullet(V^\bullet)$.

$H_d^\bullet(V^\bullet)$ has a natural structure of graded vector space $H_d^\bullet(V^\bullet) = \bigoplus_{q\in\mathbb{Z}} H_d^q(V^\bullet)$, obtained by defining $H_d^q(V^\bullet) := \{[\omega] \in H_d^\bullet(V^\bullet) | \omega \in V^q\}$.

In particular

$$H_d^q(V^\bullet) = \frac{\ker d_{|V^q}}{dV^{q-1}}.$$

Let M be a real manifold. The graded algebra $\Omega^\bullet(M) := \bigoplus_{q\in\mathbb{Z}} \Omega^q(M)$ with the operator d defined in Theorem 5.4.8. is a differential complex. Its algebra structure passes to its cohomology by the following computation.

Proposition 7.1.2 Let $(V^\bullet, d)$ be a differential complex.

Let $\times\colon V^\bullet \times V^\bullet \to V^\bullet$ be a product inducing a graded algebra structure on $V^\bullet$ with the property that for all $q_1, q_2 \in \mathbb{Z}$, for all $\omega_1 \in V^{q_1}$, $\omega_2 \in V^{q_2}$, there exists $\lambda, \mu \in \mathbb{K}\setminus\{0\}$ with

$$d(\omega_1 \times \omega_2) = \lambda\, \omega_1 \times \omega_2 + \mu\, \omega_1 \times d\omega_2.$$

Then $[\omega_1]\times[\omega_2] := [\omega_1 \times \omega_2]$ is a good definition of a graded algebra structure on $H_d^\bullet(V^\bullet)$.

Proof. We prove that $[\omega_1] \times [\omega_2] := [\omega_1 \times \omega_2]$ is a good definition, leaving the remaining checks to the reader. By the bilinearity of the product we may assume without loss of generality $\omega_1 \in V^{q_1}$, $\omega_2 \in V^{q_2}$.

First of all we need that, if ω_1 and ω_2 belong to $\ker d$, also $\omega_1 \times \omega_2$ belongs to $\ker d$. Indeed $d(\omega_1 \times \omega_2) = \lambda d\omega_1 \times \omega_2 + \mu \omega_1 \times d\omega_2 = 0 + 0 = 0$.

Then we need to show that the cohomology class of $\omega_1 \times \omega_2$ only depends on the cohomology classes of the ω_i. Indeed, if $[\omega_i] = [\omega_i']$, then $\exists \eta_i$ with $d\eta_i = \omega_i - \omega_i'$. It follows, since $d\omega_1' = d\omega_2 = 0$,

$$\begin{aligned}
\omega_1 \times \omega_2 &= (\omega_1' + d\eta_1) \times (\omega_2' + d\eta_2) = \\
&= \omega_1' \times \omega_2' + \omega_1' \times d\eta_2 + d\eta_1 \times \omega_2 = \\
&= \omega_1' \times \omega_2' + \frac{1}{\lambda} d(\omega_1' \times \eta_2) + \frac{1}{\mu} d(\eta_1 \times \omega_2) = \\
&= \omega_1' \times \omega_2' + d\left(\frac{\omega_1' \times \eta_2}{\lambda} + \frac{\eta_1 \times \omega_2}{\mu}\right)
\end{aligned}$$

so $[\omega_1 \times \omega_2] = [\omega_1' \times \omega_2']$. ■

Let's then have a better look to the differential complexes $(\Omega^\bullet(M), d)$.

Definition 7.1.3 A differential form $\omega \in \Omega^\bullet(M)$ is **closed** if $d\omega = 0$, *i.e.* if $\omega \in \ker d$.

A differential form ω is **exact** if there is a differential form η such that $\omega = d\eta$, *i.e.* if $\omega \in \operatorname{Im} d$.

By Theorem 5.4.8 *every exact form is closed*, and then $(\Omega^\bullet(M), d)$ is a differential complex.

Definition 7.1.4 For every manifold M, the differential complex $(\Omega^\bullet(M), d)$ is the **De Rham complex** of M.

Its cohomology is the **De Rham cohomology algebra** (sometimes denoted just by **De Rham cohomology** for short) of M, the graded algebra

$$H^\bullet_{DR}(M) = \frac{\{\text{closed forms}\}}{\{\text{exact forms}\}} = \bigoplus H^q_{DR}(M),$$

where

$$H^q_{DR}(M) = \frac{\{\text{closed } q\text{-forms}\}}{\{\text{exact } q\text{-forms}\}}$$

is the q^{th} **De Rham cohomology group** of M. The algebra structure on $H^\bullet_{DR}(M)$ is defined, by Proposition 7.1.2 by the **wedge product of De Rham cohomology classes**

$$[\omega_1] \wedge [\omega_2] = [\omega_1 \wedge \omega_2].$$

Notation 7.1. *We will denote by $h^q_{DR}(M) \in \mathbb{N} \cup \{\infty\}$ the dimension of $H^q_{DR}(M)$.*

Let M be a manifold such that all De Rham cohomology groups are finitely dimensional. Then the **Hilbert function** *of M is the Hilbert function of $H^\bullet_{DR}(M)$, the function $\mathbb{Z} \to \mathbb{N}$ mapping each q to $h^q_{DR}(M)$.*

The **Euler number** *of M is $e(M) := \sum(-1)^q h^q_{DR}(M)$.*

Note that $H^q_{DR}(M)$ is defined for all $q \in \mathbb{Z}$, but it is different from $\{0\}$ only for $0 \le q \le n$.

The forthcoming Exercise 7.1.2 shows that $h^0_{DR}(M)$ only depends on the topology of M; more precisely it counts the connected components of M. Some similar interpretations hold true also for other cohomology groups; we will discuss some of them later.

There is a class of maps among differential complexes that is very useful.

Definition 7.1.5 A **chain map** is a linear application

$$L\colon V^\bullet \to W^\bullet$$

among two differential complexes $(V^\bullet, d_V)$, $(W^\bullet, d_W)$ that *commutes with the differentials*, which means

$$L \circ d_V = d_W \circ L.$$

Their more interesting property is that chain maps induce maps among the respective cohomologies.

Proposition 7.1.6 Let $(V^\bullet, d_V)$ and $(W^\bullet, d_W)$ be differential complexes, and let $L\colon V^\bullet \to W^\bullet$ be a chain map.

Then there is a linear application

$$H^\bullet(L)\colon H^\bullet_{d_V}(V^\bullet) \to H^\bullet_{d_W}(W^\bullet)$$

defined by $H^\bullet(L)[\omega] = [L\omega]$.

If L has degree d, then $H^\bullet(L)$ has degree d.

If both $(V^\bullet, d_V)$ and $(W^\bullet, d_W)$ have algebra structures fulfilling the assumptions of Proposition 7.1.2 and L is a morphism of algebras then, considering $H^\bullet_{d_V}(V^\bullet)$ and $H^\bullet_{d_W}(W^\bullet)$ with the induced algebra structures, $H^\bullet(L)$ is a morhpism of algebras too.

Proof. The only nontrivial thing to prove is that $H^\bullet(L)[\omega] = [L\omega]$ is a good definition.

First of all, for all $\omega \in \ker d_V$, $d_W L\omega = L d_V \omega = L0 = 0$, so $L\omega \in \ker d_W$ has a class $[L\omega] \in H^\bullet_{d_W}(W^\bullet)$.

Then, if $[\omega] = [\omega']$ then $\exists \eta$ such that $\omega - \omega' = d_V \eta$ and therefore $L\omega - L\omega' = L(\omega - \omega') = L d_V \eta = d_W L\eta$. It follows $[L\omega] - [L\omega'] = [d_W L\eta] = 0$ and therefore $[L\omega] = [L\omega']$. ■

It follows

Corollary 7.1.7 Let $F\colon M \to N$ be a smooth map. Then there is a graded algebra homomorphism $F^*\colon H^\bullet_{DR}(N) \to H^\bullet_{DR}(M)$ of degree zero such that, for each closed form $\omega \in \Omega^q_{DR}(N)$, $F^*[\omega] = [F^*\omega]$.

Proof. By Proposition 5.4.7 the pull-back F^* defines a chain map of degree zero $F^*\colon \Omega^\bullet(N) \to \Omega^\bullet(M)$ that is moreover, as we have already remarked, an algebra homomorphism.

Then the result follows by Proposition 7.1.6. ■

By abuse of notation we have given the same name to the map $F^*\colon \Omega^\bullet(N) \to \Omega^\bullet(M)$ and to the induced map $F^*\colon H^\bullet_{DR}(N) \to H^\bullet_{DR}(M)$.

Since the latter is induced by the former by the formula $F^*[\omega] = [F^*(\omega)]$, most of the properties of the first map pass in a natural way to the second one.

For example the formula

$$(F \circ G)^* = G^* \circ F^*$$

holds also in cohomology. It follows (Complement 7.1.2) that diffeomorphic manifolds have isomorphic De Rham cohomology algebras.

Since the wedge product of forms with compact support has compact support and the exterior derivative of a form with compact support has also compact support, the subset

$$\Omega^\bullet_c(M) := \{\omega \in \Omega^\bullet(M) \,|\, \operatorname{supp}\omega \text{ is compact}\}$$

is a graded subalgebra of $\Omega^\bullet(M)$ and a differential (sub)complex whose degree 1 operator is given by the restriction of d.

Definition 7.1.8 The **compact support cohomology algebra** or **compact support cohomology ring** of M is the graded algebra

$$H_c^\bullet(M) := H_d^\bullet(\Omega_c^\bullet(M)) = \frac{\ker d_{|\Omega_c^\bullet(M)}}{\operatorname{Im} d_{|\Omega_c^\bullet(M)}} = \frac{\{\text{closed forms with compact support}\}}{\{\text{differentials of forms with compact support}\}}$$

whose grading is given by the decomposition $H_c^\bullet(M) = \bigoplus_q H_c^q(M)$ as direct sum of

$$H_c^q(M) := \frac{\ker d_{|\Omega_c^q(M)}}{d\left(\Omega_c^{q-1}(M)\right)} = \frac{\{\text{closed } q\text{-forms with compact support}\}}{\{\text{differentials of } (q-1)\text{-forms with compact support}\}}.$$

The graded piece $H_c^q(M)$ is the **q^{th}-cohomology group with compact support**. The product of the algebra structure is defined as

$$[\omega_1] \wedge [\omega_2] := [\omega_1 \wedge \omega_2].$$

As in the case of the De Rham cohomology, $H_c^q(M)$ is defined for all $q \in \mathbb{Z}$, but it equals $\{0\}$ unless $0 \le q \le n$.

Note that, if M is compact, $\Omega^\bullet(M) = \Omega_c^\bullet(M)$ and therefore $H_{DR}^\bullet(M) = H_c^\bullet(M)$.

Notation 7.2. *We will denote by $h_c^q(M) \in \mathbb{N} \cup \{\infty\}$ the dimension of $H_c^q(M)$.*

We would like to generalize Corollary 7.1.7 to the cohomology with compact support, but we cannot because in general[1] $F^*(\Omega_c^\bullet(N)) \not\subset \Omega_c^\bullet(M)$. But there is an important class of functions for which it works.

Definition 7.1.9 Let M, N be manifolds and let $F : M \to N$ be a function.
F is **proper** if, $\forall K \subset N$ compact, then $F^{-1}(K) \subset M$ is compact too.

If F is a smooth proper map then obviously $F^*(\Omega_c^\bullet(N)) \subset \Omega_c^\bullet(M)$ and then

Corollary 7.1.10 Let M, N be manifolds and let $F : M \to N$ be a smooth proper map.
Then there is a graded algebra homomorphism $F^* : H_c^\bullet(N) \to H_c^\bullet(M)$ such that, for each closed form $\omega \in \Omega_c^q(N)$, $F^*[\omega] = [F^*\omega]$.

Since diffeomorphisms are proper maps, it follows that diffeomorphic manifolds have isomorphic compact support cohomology algebras.

Complement 7.1.1 Show that, for every differential complex $(V^\bullet, d)$, $H_d^q(V^\bullet) = \frac{\ker d_{|V^q}}{dV^{q-1}}$.

Complement 7.1.2 Show that the De Rham cohomologies of diffeomorphicmanifolds are isomorphic as graded algebras.

Exercise 7.1.1 Show that $f \in \Omega^0(M) = C^\infty(M)$ is closed if and only if it is locally constant, *i.e.* $\forall p \in M$ there exists an open neighbourhood U of p such that $f_{|U}$ is constant.

[1] As example take the first projection $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Exercise 7.1.2 Show that $h^0_{DR}(M)$ equals the number of connected components of M. Find a similar description for $h^0_c(M)$.

Exercise 7.1.3 Compute $h^1_{DR}(\mathbb{R})$.

Exercise 7.1.4 Show that $H^\bullet_{DR}(\mathbb{R})$ is isomorphic as graded algebra to $\mathbb{R}[t]/(t)$. Show that $H^\bullet_c(\mathbb{R})$ is isomorphic as graded algebra to $t\mathbb{R}[t]/(t^2)$.

Hint: We need to determine when a $1-$ *form with compact support has a primitive whose support is compact. Look for a criterium in terms of integrals.*

Exercise 7.1.5 Compute the De Rham cohomology ring and the compact support cohomology ring of the intervals $[0,1)$ and $[0,1]$.

Exercise 7.1.6 Show that the restrictions to S^1 of the forms xdy and $xdy-ydx$ of $\mathbb{R}^2$ are closed but not exact.

Exercise 7.1.7 Show that if M is oriented and $\partial M=\emptyset$, then there is a well defined linear map $\int_M : H^n_c(M)\to\mathbb{R}$ associating to each class $[\omega]$ the number $\int_M \omega$.

Show moreover that the map $\int_M$ is surjective.

7.2 Exact sequences

Definition 7.2.1 An **exact sequence** is a (finite or not finite) sequence of linear applications

$$\cdots\to V^{q-1}\to V^q\to V^{q+1}\to\cdots$$

such that the image of each map coincides with the kernel of the next one.

One can see an exact sequence as a graded vector space $V^\bullet := \bigoplus_q V^q$. Then the linear applications build naturally an operator d on $V^\bullet$ of degree 1, and the exact sequence condition means that $(V^\bullet,d)$ is a differential complex with trivial cohomology $\{0\}$.

Definition 7.2.2 A **short exact sequence** is an exact sequence of the form

$$0\to A\xrightarrow{f} B\xrightarrow{g} C\to 0$$

where 0 stands for the $0-$dimensional vector space $\{0\}$.

In other words we have an injective map $f: A\to B$, a surjective map $g: B\to C$ such that $\operatorname{Im} f=\ker g$.

A special role is played by the following exact sequences.

Definition 7.2.3 A **short exact sequence of complexes** is a short exact sequences of chain maps of degree zero among differential complexes

$$0\to A^\bullet\xrightarrow{f} B^\bullet\xrightarrow{g} C^\bullet\to 0.$$

These are commutative diagrams

$$\begin{tikzcd}
 & \vdots \arrow[d, "d"'] & \vdots \arrow[d, "d"'] & \vdots \arrow[d, "d"] & \\
0 \arrow[r] & A^{q-1} \arrow[r, "f"] \arrow[d, "d"'] & B^{q-1} \arrow[r, "g"] \arrow[d, "d"'] & C^{q-1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{q} \arrow[r, "f"] \arrow[d, "d"'] & B^{q} \arrow[r, "g"] \arrow[d, "d"'] & C^{q} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{q+1} \arrow[r, "f"] \arrow[d, "d"'] & B^{q+1} \arrow[r, "g"] \arrow[d, "d"'] & C^{q+1} \arrow[r] \arrow[d, "d"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{7.1}$$

whose rows are exact, and whose columns are differential complexes. Therefore in the diagram (7.1)

- all maps f, g and d are linear;
- $d \circ d = 0$;
- all f are injective;
- all g are surjective;
- $\operatorname{Im} f = \ker g$;
- $d \circ f = f \circ d$ and $d \circ g = g \circ d$.

The key result is the following

Theorem 7.2.4 Assume that there is a short exact sequence of complexes

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0.$$

Then there is a long exact sequence of cohomology groups

$$\cdots \to H^{q-1}(C^\bullet) \xrightarrow{d_*} H^q(A^\bullet) \xrightarrow{f_*} H^q(B^\bullet) \xrightarrow{g_*} H^q(C^\bullet) \xrightarrow{d_*} H^{q+1}(A^\bullet) \to \cdots \tag{7.2}$$

Proof. We have to define the maps f_*, g_* and d_* in (7.2) and then prove that (7.2) is an exact sequence by showing that the image of each map equals the kernel of the next one.

Since f, g are chain maps of degree zero, by Proposition 7.1.6 they induce linear applications f_*, g_* of degree zero among the respective cohomologies

$$f_*\colon H^\bullet(A^\bullet) \to H^\bullet(B^\bullet), \qquad g_*\colon H^\bullet(B^\bullet) \to H^\bullet(C^\bullet),$$

so that

$$\forall a \in A^\bullet \text{ with } da = 0 \;\; f_*([a]) = [f(a)], \qquad \forall b \in B^\bullet \text{ with } db = 0 \;\; g_*([b]) = [g(b)].$$

To describe the linear application d_* of degree 1 we define each of its graded pieces $d_*\colon H^{q-1}(C^\bullet) \to H^q(A^\bullet)$ as follows.

By the surjectivity of g, for every $c \in C^{q-1}$ we can pick an element $b \in B^{q-1}$ such that $g(b) = c$. When c is a representative of a cohomology class, then $d(c) = 0$ and $g(d(b)) = d(g(b)) = d(c) = 0$. Then $d(b) \in \ker g = \operatorname{Im} f$ and therefore there is an element $a \in A^q$ such that $f(a) = d(b)$. The following diagram summarizes how we constructed a and b.

$$\begin{tikzcd}
 & \vdots \arrow[dd, "d"] & & \vdots \arrow[dd, "d"] & & \vdots \arrow[dd, "d"] & & \\
 & & & & b \arrow[rr, mapsto] \arrow[dd, mapsto] \arrow[dl, dash] & & c \arrow[dd, mapsto] \arrow[dl, dash] & \\
0 \arrow[r] & A^{q-1} \arrow[rr, "f"] \arrow[dd, "d"] & & B^{q-1} \arrow[rr, "g"] \arrow[dd, "d"] & & C^{q-1} \arrow[r] \arrow[dd, "d"] & & 0 \\
 & & a \arrow[rr, mapsto] \arrow[dl, dash] & & db \arrow[rr, mapsto] \arrow[dl, dash] & & 0 \arrow[dl, dash] & \\
0 \arrow[r] & A^{q} \arrow[rr, "f"] \arrow[d, "d"] & & B^{q} \arrow[rr, "g"] \arrow[d, "d"] & & C^{q} \arrow[r] \arrow[d, "d"] & & 0 \\
 & \vdots & & \vdots & & \vdots & &
\end{tikzcd}$$

We define then

$$d_*([c]) = [a].$$

We need to prove that the definition is well done, *i.e.* that

1. $d(a) = 0$ (so that we can consider its cohomology class $[a]$);
2. the cohomology class $[a]$ do not depend on the choices we have done:
   of $a \in f^{-1}(d(b))$;
   of $b \in g^{-1}(c)$;
   of c in its cohomology class.

The proof of point 1) is easy. Indeed $f(d(a)) = d(f(a)) = d(d(b)) = 0$, so $d(a) \in \ker f$. Since f is injective, then $d(a) = 0$.

Point 2) are really three different checks, one for each choice we have done, the choice of a, the choice of b and finally the choice of c.

The first check, "the choice of a", is obvious: since f is injective, $f^{-1}(d(b))$ has cardinality 1 and then we had no choice there!

For the second check, let's consider a different b' with $g(b') = c$, and set a' for the unique element in A^q with $f(a') = db'$. Then $g(b-b') = g(b) - g(b') = c - c = 0$, so $b - b' \in \ker g = \operatorname{Im} f$, so there exists $\overline{a} \in A^{q-1}$ such that $f(\overline{a}) = b - b'$. Then

$$f(d(\overline{a})) = d(f(\overline{a})) = d(b-b') = db - db' = f(a) - f(a') = f(a-a').$$

By the injectivity of f it follows $a - a' = d\overline{a}$ and then $[a] = [a']$. The following diagram describes the argument used.

$$\begin{tikzcd}
 & \vdots \arrow[dd, "d"] & & \vdots \arrow[dd, "d"] & & \vdots \arrow[dd, "d"] & & \\
 & & \overline{a} \arrow[rr, mapsto] \arrow[dl, dash] \arrow[dd, mapsto] & & b-b' \arrow[rr, mapsto] \arrow[dl, dash] \arrow[dd, mapsto] & & 0 \arrow[dl, dash] & \\
0 \arrow[r] & A^{q-1} \arrow[rr, "f"] \arrow[dd, "d"] & & B^{q-1} \arrow[rr, "g"] \arrow[dd, "d"] & & C^{q-1} \arrow[rr] \arrow[dd, "d"] & & 0 \\
 & & d\overline{a}=a-a' \arrow[rr, mapsto] \arrow[dl, dash] & & db-db' \arrow[dl, dash] & & & \\
0 \arrow[r] & A^{q} \arrow[rr, "f"] \arrow[d, "d"] & & B^{q} \arrow[rr, "g"] \arrow[d, "d"] & & C^{q} \arrow[rr] \arrow[d, "d"] & & 0 \\
 & \vdots & & \vdots & & \vdots & &
\end{tikzcd}$$

The last check, the independence of $[a]$ by the choice of c in its cohomology class, can be done by a similar *diagram chasing* argument. We leave it to the reader, as the rest of the proof.

More precisely, to complete the proof the reader should show that d_* is linear (this is standard undergraduate linear algebra) and prove

- $\operatorname{Im} f_* \subset \ker g_*$;
- $\operatorname{Im} f_* \supset \ker g_*$;
- $\operatorname{Im} g_* \subset \ker d_*$;
- $\operatorname{Im} g_* \supset \ker d_*$;
- $\operatorname{Im} d_* \subset \ker f_*$;
- $\operatorname{Im} d_* \supset \ker f_*$.

all statements that can be proved by *diagram chasing* as above (do it, it is fun!) ■

Complement 7.2.1 Run all details of the Proof of Theorem 7.2.4.

Exercise 7.2.1 Let $(V^\bullet, d)$ be a differential complex of finitely dimensional vector spaces such that $\{q | \dim V^q > 0\}$ is finite. Then

$$\sum(-1)^q \dim V^q = \sum(-1)^q \dim H^q(V^\bullet).$$

In particular, if $0 \to V^a \to V^{a+1} \to \cdots \to V^b \to 0$ is an exact sequence of finitely dimensional vector spaces, then $\sum(-1)^q \dim V^q = 0$.

Exercise 7.2.2 — The dual exact sequence. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of finitely dimensional vector spaces. Prove that

$$C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$$

is also an exact sequence.

7.3 The Mayer-Vietoris short exact sequence

At the moment we have considered, for each manifold M, two differential complexes, the De Rham complex $(\Omega^\bullet(M), d)$ and its subcomplex $(\Omega^\bullet_c(M), d)$ with respective cohomologies $H^\bullet_{DR}(M)$ and $H^\bullet_c(M)$.

To apply Theorem 7.2.4 to our cohomology theories we need to construct suitable short exact sequences of complexes.

Definition 7.3.1 Let M be manifold, $U \subset M$ be an open subset with the induced differentiable structure. Consider the restriction map

$$\rho^M_U : \Omega^\bullet(M) \to \Omega^\bullet(U)$$

defined by $\rho^M_U(\omega) := \omega_{|U}$

Since the restriction $\omega_{|U}$ is the pull-back for the inclusion map $U \hookrightarrow M$, and pull-back and differential commute, ρ^M_U is a chain map.

Theorem 7.3.2 Let $\{U, V\}$ be an open covering of a manifold M.

Then there is a short exact sequence of chain maps

$$0 \to \Omega^\bullet(M) \xrightarrow{f} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{g} \Omega^\bullet(U \cap V) \to 0 \tag{7.3}$$

where $f(\omega) = (\rho^M_U \omega, \rho^M_V \omega)$, and $g(\omega_U, \omega_V) = \rho^V_{U\cap V}\omega_V - \rho^U_{U\cap V}\omega_U$.

Proof. The only nontrivial check is the surjectivity of g.

By Theorem 6.1.10 and its proof (that we have not seen) there is a *partition of unity* made by two smooth functions $f_U, f_V : M \to [0,1]$ such that

- $f_U + f_V = 1$;
- $\operatorname{supp} f_U \subset U$;
- $\operatorname{supp} f_V \subset V$.

For every $\tau \in \Omega^q(U \cap V)$ consider the form $f_U \tau \in \Omega^q(U \cap V)$.

We extend it to a form on V by setting $\forall p \in V \setminus U$, $(f_U \tau)_p = 0$. We obtain a smooth form (that we keep calling $f_U \tau$), $f_U \tau \in \Omega^q(V)$: indeed the smoothness is obvious on $U \cap V$, whereas for every $p \in V \setminus (U \cap V) = V \setminus U$ there is a neighbourhood of p, namely $V \setminus \operatorname{supp} f_U$, where $f_U \tau = 0$, and therefore $f_U \tau$ is smooth at these points too.

Similarly for f_V: we have constructed two forms $f_U \tau \in \Omega^q(V)$, $f_V \tau \in \Omega^q(U)$ such that

$$(f_U \tau)_{|U\cap V} + (f_V \tau)_{|U\cap V} = \tau.$$

We conclude by

$$g(-f_V \tau, f_U \tau) = (f_U \tau)_{|U\cap V} + (f_V \tau)_{|U\cap V} = \tau. \qquad \blacksquare$$

Corollary 7.3.3 Let $\{U,V\}$ be an open covering ot a manifold M.
Then there is an exact sequence

$$\begin{array}{ccccccc} & & & \cdots & \to & H^{q-1}_{DR}(U\cap V) & \to \\ \to & H^q_{DR}(M) & \to & H^q_{DR}(U)\oplus H^q_{DR}(V) & \to & H^q_{DR}(U\cap V) & \to \\ \to & H^{q+1}_{DR}(M) & \to & \cdots & & & \end{array}$$

Proof. It follows immediately applying Theorem 7.2.4 to the exact sequence (7.3). ■

The same construction does not work for forms with compact support because the support of the restriction of a form with compact support to an open subset may be not compact.

Still, a different construction gives a similar result.

Definition 7.3.4 Let M be a manifold and let $U \subset M$ be an open subset.
Consider the inclusion $U \hookrightarrow M$.
Then we define

$$j^U_M \colon \Omega^\bullet_c(U) \to \Omega^\bullet_c(M)$$

so that, $\forall \omega \in \Omega^\bullet_c(U)$, $j^U_M\omega \in \Omega^\bullet_c(M)$ is the form that coincides with ω on the points of U, and vanishes elsewhere.

Note that $j^U_M\omega$ is smooth by Lemma 6.1.11 because $\operatorname{supp}\omega$ is compact.

Note moreover that j^U_M is a chain map.

Theorem 7.3.5 Let $\{U,V\}$ be an open covering ot a manifold M.
Then there is a short exact sequence of chain maps

$$0 \to \Omega^\bullet_c(U\cap V) \xrightarrow{f} \Omega^\bullet_c(U)\oplus\Omega^\bullet_c(V) \xrightarrow{g} \Omega^\bullet_c(M) \to 0$$

where $f(\omega) = (-j^{U\cap V}_U\omega, j^{U\cap V}_V\omega)$, and $g(\omega_U,\omega_V) = j^U_M\omega_U + j^V_M\omega_V$.

Proof. The proof follows the same lines of the proof of Theorem 7.3.2. Do it! ■

Corollary 7.3.6 Let $\{U,V\}$ be an open covering of a manifold M.
Then there is an exact sequence

$$\begin{array}{ccccccc} & & & \cdots & \to & H^{q-1}_c(M) & \to \\ \to & H^q_c(U\cap V) & \to & H^q_c(U)\oplus H^q_c(V) & \to & H^q_c(M) & \to \\ \to & H^{q+1}_c(U\cap V) & \to & \cdots & & & \end{array}$$

Proof. This follows immediately by Theorem 7.2.4 and Theorem 7.3.5. ■

Complement 7.3.1 Prove Theorem 7.3.5.

Exercise 7.3.1 Use Corollary 7.3.3 to compute $H^1_{DR}(S^1)$.

Exercise 7.3.2 Use Corollary 7.3.6 to compute $H^1_c(S^1)$.

Exercise 7.3.3 Let M be a manifold and let $U,V \subset M$ be open subsets such that $U \cup V = M$ and all De Rham cohomology groups of U, V and $U \cap V$ are finitely dimensional.

Prove that then all De Rham cohomology groups of M are finitely dimensional and moreover

$$e(M) + e(U \cap V) = e(U) + e(V).$$

7.4 The Poincaré lemma

Let M be a manifold, let $\pi\colon M \times \mathbb{R} \to M$ be the projection on the first factor, fix $c \in \mathbb{R}$, and let $s\colon M \to M \times \mathbb{R}$ be corresponding constant section, so $\forall p \in M$, $s(p) = (p,c)$. Notice $\pi \circ s = Id_M$: s is a section of a trivial bundle.

Lemma 7.4.1 There exist a linear operator

$$K\colon \Omega^\bullet(M \times \mathbb{R}) \to \Omega^\bullet(M \times \mathbb{R})$$

of degree -1 such that

$$Id_{\Omega^q(M\times\mathbb{R})} - \pi^* \circ s^* = (-1)^q (K \circ d - d \circ K) \tag{7.4}$$

The operator K above is called **integration along the fibres**.

Proof. The ordinary derivative $\frac{d}{dt}$ defines a vector field on $\mathbb{R}$. Consider the inclusions $\mathbb{R} \hookrightarrow M \times \mathbb{R}$ given by the fibres of π, namely $\forall p \in M$, $t \mapsto (p,t)$. Their differentials map the vector field $\frac{d}{dt}$ on vector fields on each $\{p\} \times \mathbb{R}$, giving then, $\forall (p,t) \in M \times \mathbb{R}$, a tangent vector in $T_{(p,t)}(M \times \mathbb{R})$.

We get then a section of the tangent bundle of $M \times \mathbb{R}$ that we denote by $\frac{\partial}{\partial t}$. This is smooth, so $\frac{\partial}{\partial t} \in \mathfrak{X}(M \times \mathbb{R})$, as one easily checks in local coordinates.

Indeed, if (U,φ) is a chart for M giving local coordinates $x_1,\dots,x_n$, $(U \times \mathbb{R}, \varphi \times \mathrm{Id}_{\mathbb{R}})$ is a chart for $M \times \mathbb{R}$, whose corresponding coordinates we denote, by a natural abuse of notation, by $x_1,\dots,x_n,t$. Then the partial derivative with respect to the coordinate t equals the restriction to $U \times \mathbb{R}$ of the just defined vector field $\frac{\partial}{\partial t}$, which is then smooth.

Note $\pi(x_1,\dots,x_n,t) = (x_1,\dots,x_n)$, $s(x_1,\dots,x_n) = (x_1,\dots,x_n,c)$, so $\pi^* dx_i = dx_i$, $s^* dx_i = dx_i$, $s^* dt = ds^* t$ is the differential of the function with constant value c, and so it vanishes: $s^* dt = 0$.

We define K as follows. $\forall k \in \mathbb{R}$ consider the *shift by* k $a_k\colon M \times \mathbb{R} \to M \times \mathbb{R}$ defined by

$$a_k(p,t) = (p, t+k).$$

Then, $\forall q \in \mathbb{N}$, $\forall \omega \in \Omega^q(M \times \mathbb{R})$, $\forall p \in M$, $\forall t \in \mathbb{R}$, $\forall v_1,\dots,v_{q-1} \in T_{(p,t)}(M \times \mathbb{R})$,

$$(K(\omega))_{(p,t)}(v_1,\dots,v_{q-1}) := q \int_c^t (a_{t-u}^* \omega_{p,u}) \left(v_1,\dots,v_{q-1}, \left(\frac{\partial}{\partial t}\right)_{(p,t)} \right) du.$$

The reader can easily check that $K(\omega)$ is a section of the vector bundle $\Lambda^{q-1} T^*(M \times \mathbb{R})$, whose smoothness we check as usual in local coordinates. If $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_q}$ we get $K(\omega) = 0$. If $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge dt$ we get

$$K(\omega) = \left(\int_c^t f(x_1,\dots,x_n,u) du \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}.$$

Smoothness follows since every form in $\Omega^q(M \times \mathbb{R})$ is a sum of forms of the two above considered types.

The formula (7.4) is a local statement, *i.e.* it is enough to prove it in a neighbourhood of every point, so we can check it in local coordinates. Since both sides of (7.4) are linear we only need to check (7.4) for forms of type $f dx_{i_1} \wedge \cdots \wedge dx_{i_q}$ and of type $f dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge dt$.

In the first case, $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_q}$,

$$\begin{aligned}(Id_{\Omega^q(M\times\mathbb{R})} - \pi^* \circ s^*)\omega &= (f - f \circ s \circ \pi) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &= (f(x_1,\ldots,x_n,t) - f(x_1,\ldots,x_n,c)) dx_{i_1} \wedge \cdots \wedge dx_{i_q}\end{aligned}$$

and

$$\begin{aligned}(K\circ d - d\circ K)\omega &= K(d(f dx_{i_1} \wedge \cdots \wedge dx_{i_q})) \\ &= K(df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}) \\ &= K\left(\frac{\partial f}{\partial t} dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}\right) + \sum_i K\left(\frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}\right) \\ &= K\left(\frac{\partial f}{\partial t} dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}\right) \\ &= (-1)^q K(\frac{\partial f}{\partial t} dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt) \\ &= (-1)^q \left(\int_c^t \frac{\partial f}{\partial t}(x_1,\ldots,x_n,u)du\right) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &= (-1)^q (f(x_1,\ldots,x_n,t) - f(x_1,\ldots,x_n,c)) dx_{i_1} \wedge \cdots \wedge dx_{i_q}.\end{aligned}$$

In the second case, $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge dt$, since $s^* dt = 0$ then $s^*\omega = 0$ and

$$(Id_{\Omega^q(M\times\mathbb{R})} - \pi^* \circ s^*)\omega = Id_{\Omega^q(M\times\mathbb{R})}\omega - 0 = \omega.$$

Moreover

$$\begin{aligned}(K\circ d)\omega &= K(d(f dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge dt)) \\ &= K\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge dt\right) \\ &= \sum_{i=1}^n \left(\int_c^t \frac{\partial f}{\partial x_i}\right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}\end{aligned}$$

and

$$\begin{aligned}(d\circ K)\omega &= d(K(f dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge dt)) \\ &= d\left(\left(\int_c^t f\right) dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}\right) \\ &= d\left(\int_c^t f\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \\ &= f dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} + \sum_i \left(\int_c^t \frac{\partial f}{\partial x_i}\right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \\ &= (-1)^{q-1}\omega + \sum_i \left(\int_c^t \frac{\partial f}{\partial x_i}\right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}.\end{aligned}$$ ■

A first consequence is the following

Theorem 7.4.2 — Extended Poincaré Lemma. For every manifold M, the cohomology rings of M and $M \times \mathbb{R}$ are isomorphic. More precisely, the graded algebra homomorphisms

$$\pi^*: H^\bullet(M) \to H^\bullet(M \times \mathbb{R})$$

and

$$s^*: H^\bullet(M \times \mathbb{R}) \to H^\bullet(M)$$

are isomorphisms and $s^* = (\pi^*)^{-1}$.

Proof. Since $\pi \circ s = Id_M$, then $s^* \circ \pi^* = (\pi \circ s)^* = \mathrm{Id}_{H^q(M)}$.

On the other hand, for every closed form $\omega \in \Omega^q(M \times \mathbb{R})$, $(dK - Kd)\omega = dK\omega$ is exact. Then, by Lemma 7.4.1,

$$(\pi^* \circ s^*)[\omega] = [\omega] + (-1)^q[(dK - Kd)\omega] = [\omega],$$

and therefore $\pi^* \circ s^* = \mathrm{Id}_{H^q(M \times \mathbb{R})}$. ■

The classical Poincaré Lemma, claiming that every closed form on $\mathbb{R}^n$ is exact, follows then immediately.

Corollary 7.4.3 — Poincaré Lemma. $\forall q \neq 0$, $h^q(\mathbb{R}^n) = 0$.

Proof. Applying recursively the extended Poincaré lemma

$$h^q(\mathbb{R}^n) = h^q(\mathbb{R}^{n-1}) = \cdots = h^q(\mathbb{R}^0) = 0.$$ ■

A striking application of the extended Poincaré lemma is that the cohomology do not distinguish varieties with the same homotopy type.

To state it properly we need few definitions. First we need a differentiable version of homotopy.

Definition 7.4.4 Let M, N be manifolds, and let $F, G\colon M \to N$ be smooth maps. We say that F and G are **smoothly homotopic** if there exists a smooth map

$$H\colon M \times \mathbb{R} \to N$$

such that $\forall p \in M$, $H(p,t)$ equals $F(p)$ for $t = 0$ and $G(p)$ for $t = 1$.

H is a **smooth homotopy** among F and G.

The usual definitions of homotopic maps in the topological category (so continous insted of differentiable) uses $M \times [0,1]$ as domain of the homotopy map H. Moving to the category of differentiable manifolds $H \times \mathbb{R}$ it is more convenient to replace $[0,1]$ by $\mathbb{R}$ mainly to allow us to consider manifolds with boundary $\partial M \neq 0$.

Corollary 7.4.5 If $F, G\colon M \to N$ are smoothly homotopic, then the ring homomorphisms $F^*, G^*\colon H^\bullet(N) \to H^\bullet(M)$ are equal: $F^* = G^*$.

Proof. We denote by $s_c\colon M \to M \times \mathbb{R}$ the section $s_c(p) = (p, c)$.

Consider the smooth homotopy $H\colon M \times \mathbb{R} \to M$ among F and G. Then $F = H \circ s_0$, $G = H \circ s_1$. By Theorem 7.4.2 $s_0^* = s_1^*$ (since both are inverse of π^*). Therefore

$$F^* = (H \circ s_0)^* = s_0^* \circ H^* = s_1^* \circ H^* = (H \circ s_1)^* = G^*.$$ ■

We deduce the following definition and corollary

Definition 7.4.6 Two manifolds M, N have the same **homotopy type** if there exist smooth maps $F\colon M \to N$ and $G\colon N \to M$ such that both $F \circ G$ and $G \circ F$ are smoothly homotopic to the identity of the respective manifold.

Corollary 7.4.7 If two manifolds have the same homotopy type then their De Rham cohomology rings are isomorphic as graded rings.

Complement 7.4.1 Prove that the operator K of the Lemma 7.4.1 is well defined, *i.e.* that its definition is independent on the coordinates x_i.

Complement 7.4.2 Prove that the existence of a smooth homotopy defines an equivalence relation on the space of smooth functions from M to N.

Complement 7.4.3 Show that, if there exists a smooth map $H'\colon M \times [0,1] \to N$ such that $\forall p \in M$, $H'(p,0) = F(p)$ and $H'(p,1) = G(p)$, then F and G are smoothly homotopic.

Exercise 7.4.1 Let $\pi\colon E \to B$ be a vector bundle.

Show that the De Rham cohomology ring of E is isomorphic to the De Rham cohomology ring of B.

Compute the De Rham cohomology rings of the interior of the cylinder and of the Moebius band.

Exercise 7.4.2 Compute the De Rham cohomology ring of S^n.

7.5 The Poincaré lemma for the compact support cohomology

The De Rham cohomology do not distinguish among manifolds with the same homotopy type. There is no similar statement for the cohomology with compact support: indeed Exercise 7.1.4 shows that the compact support cohomology ring of $\mathbb{R}$ differs from the one of a point, although they have the same homotopy type.

Still, the argument of the proof of the Poincaré Lemma may be adapted to the compact support cohomology, obtaining a different but still interesting result.

Theorem 7.5.1 For every manifold M, for every $q \in \mathbb{Z}$

$$H^q_c(M \times \mathbb{R}) \cong H^{q-1}_c(M)$$

Proof. Arguing as in the proof of the Theorem 7.4.2, the statement follows from the construction, $\forall q \in \mathbb{Z}$, of two chain maps

$$\begin{aligned} e_*\colon \Omega^\bullet_c(M) &\to \Omega^\bullet_c(M \times \mathbb{R}) && \text{of degree } 1\\ \pi_*\colon \Omega^\bullet_c(M \times \mathbb{R}) &\to \Omega^\bullet_c(M) && \text{of degree } -1 \end{aligned}$$

such that $\pi_* \circ e_* = \mathrm{Id}_{\Omega^\bullet_c(M)}$ and an operator K of degree -1 on $\Omega^q_c(M \times \mathbb{R})$ such that

$$\mathrm{Id}_{\Omega^q_c(M\times\mathbb{R})} - e_* \circ \pi_* = (-1)^q (K \circ d - d \circ K).$$

We start by constructing e_*. We choose a function $e' \in C^\infty_c(\mathbb{R})$ such that $\int_{\mathbb{R}} e'(t)dt = 1$, set $e \in \Omega^1(M\times\mathbb{R})$ be the pull back $(\pi')^*(e'(t)dt)$ via the projection map $\pi' \colon M\times\mathbb{R} \to \mathbb{R}$ and finally define

$$e_*\omega := \pi^*\omega \wedge e$$

where $\pi \colon M\times\mathbb{R} \to M$ is the usual projection map.

The support of $e_*\omega$ is compact, although both the supports of e and $\pi^*\omega$ may be not compact. Indeed, in some sense, $\operatorname{supp} e$ is bounded *vertically* and $\operatorname{supp} \pi^*\omega$ *horizontally*: then $\operatorname{supp} e_*\omega$ is compact.

Notice that e is closed, since $de = d(\pi')^*(e'(t)dt) = (\pi')^*(d(e'(t)dt)) = (\pi')^*0 = 0$. Then e_* is a chain map: $de_*\omega = d(\pi^*\omega\wedge e) = d\pi^*\omega\wedge e \pm \pi^*\omega\wedge de = \pi^*d\omega\wedge e + 0 = e_*d\omega$.

For the sake of simplicity, we give the definition of π_* and K in local coordinates, leaving to the reader to find an intrinsic definition (analogous to the definition of the operator K in the proof of Lemma 7.4) to ensure that the definitions are well posed, *i.e.* independent of the choice of the coordinates.

We fix local coordinates as in the proof of Lemma 7.4: coordinates $x_1,\dots,x_n$ on an open subset $U \subset M$ and corresponding coordinates $(x_1,\dots,x_n,t)$ on $U\times\mathbb{R}$. In particular $\pi(x_1,\dots,x_n,t) = (x_1,\dots,x_n)$. Correspondingly we get forms $dx_i \in \Omega^1(U)$, $dx_i, dt \in \Omega^1(U\times\mathbb{R})$.

Consider a form $\omega \in \Omega^q_c(M\times\mathbb{R})$. If $\omega_{|U\times\mathbb{R}} = f dx_{i_1}\wedge\cdots\wedge dx_{i_q}$ we set $(\pi_*\omega)_{|U} := 0$. If $\omega_{|U\times\mathbb{R}} = f dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt$ we set

$$(\pi_*\omega)_{|U} := \left(\int_{\mathbb{R}} f(x_1,\dots,x_n,t)dt\right) dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}.$$

Since every form in $\Omega^\bullet_c(M\times\mathbb{R})$ is a sum of forms as above, this determines locally the operator $\pi_* \colon \Omega^\bullet_c(M) \to \Omega^\bullet_c(M\times\mathbb{R})$.

We show now that π_* is a chain map. This is also a local property, . *i.e.* it holds if and only if it holds in a neighbourhood of every point, so we can check it in coordinates.

If $\omega_{|U\times\mathbb{R}}$ is of the form $f dx_{i_1}\wedge\cdots\wedge dx_{i_q}$ then $d(\pi_*\omega)_{|U} = d0 = 0$ and

$$\begin{aligned}
(\pi_* d\omega)_{|U} &= \pi_*(df\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_q})\\
&= \pi_*\left(\frac{\partial f}{\partial t}dt\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_q}\right) + \pi_*\left(\sum_i \frac{\partial f}{\partial x_i}dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_q}\right)\\
&= \pi_*\left(\frac{\partial f}{\partial t}dt\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_q}\right)\\
&= (-1)^q\pi_*\left(\frac{\partial f}{\partial t}dx_{i_1}\wedge\cdots\wedge dx_{i_q}\wedge dt\right)\\
&= (-1)^q\left(\int_{\mathbb{R}}\frac{\partial f}{\partial t}dt\right)dx_{i_1}\wedge\cdots\wedge dx_{i_q}\\
&= 0.
\end{aligned}$$

If $\omega_{|U\times\mathbb{R}}$ is of the form $fdx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt$ then

$$
\begin{aligned}
d(\pi_*\omega)_{|U} &= d\pi_*(fdx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt)\\
&= d\left(\left(\int_{\mathbb{R}} fdt\right)dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\right)\\
&= d\left(\int_{\mathbb{R}} fdt\right)\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\\
&= \left(\sum_i \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}} fdt\right)\right)dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\\
&= \sum_i \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_i}dt\right)dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
(\pi_* d\omega)_{|U} &= \pi_*(df\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt)\\
&= \pi_*\left(\sum_i \frac{\partial f}{\partial x_i}dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt\right)+\pi_*\left(\frac{\partial f}{\partial t}dt\wedge dx_{i_1}\wedge\cdots\wedge dt\right)\\
&= \pi_*\left(\sum_i \frac{\partial f}{\partial x_i}dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt\right)\\
&= \left(\int_{\mathbb{R}}\sum_i \frac{\partial f}{\partial x_i}dt\right)dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\\
&= \sum_i \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_i}dt\right)dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}
\end{aligned}
$$

Since locally all forms are sum of forms as above, the statement follows: π_* is a chain map.

We prove now that $\pi_*\circ e_* = \mathrm{Id}_{\Omega_c^\bullet(M)}$. Indeed, since we assumed $\int_{\mathbb{R}} e'(t)dt = 1$ then $\pi_* e_*\omega = \pi_*(\pi^*\omega\wedge e) = \pi_*(e'(t)(\pi^*\omega)\wedge dt) = (\int_{\mathbb{R}} e'(t)dt)\,\omega = \omega$.

We define $K\colon \Omega_c^\bullet(M\times\mathbb{R})\to\Omega_c^\bullet(M\times\mathbb{R})$ in local coordinates, by defining it for forms of type $fdx_{i_1}\wedge\cdots\wedge dx_{i_q}$ and of type $fdx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt$, leaving to the reader the check that the definition is independent from the coordinates by finding an intrinsic definition. We set then

$$K(fdx_{i_1}\wedge\cdots\wedge dx_{i_q}) := 0$$

and

$$
\begin{aligned}
&K(fdx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}\wedge dt) :=\\
&= \left(\left(\int_{-\infty}^t f(x_1,\dots,x_n,u)du\right)-\left(\int_{\mathbb{R}} f(x_1,\dots,x_n,u)du\right)\left(\int_{-\infty}^t e'\right)\right)dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}.
\end{aligned}
$$

We leave to the reader the rather long but straightforward check of the equality

$$\mathrm{Id}_{\Omega_c^q(M\times\mathbb{R})} - e_*\circ\pi_* = (-1)^q(K\circ d - d\circ K). \qquad \blacksquare$$

Corollary 7.5.2 $\forall q \neq n$, $h_c^q(\mathbb{R}^n) = 0$, whereas $h_c^n(\mathbb{R}^n) = 1$.

Proof. Applying recursively the extended Poincaré lemma

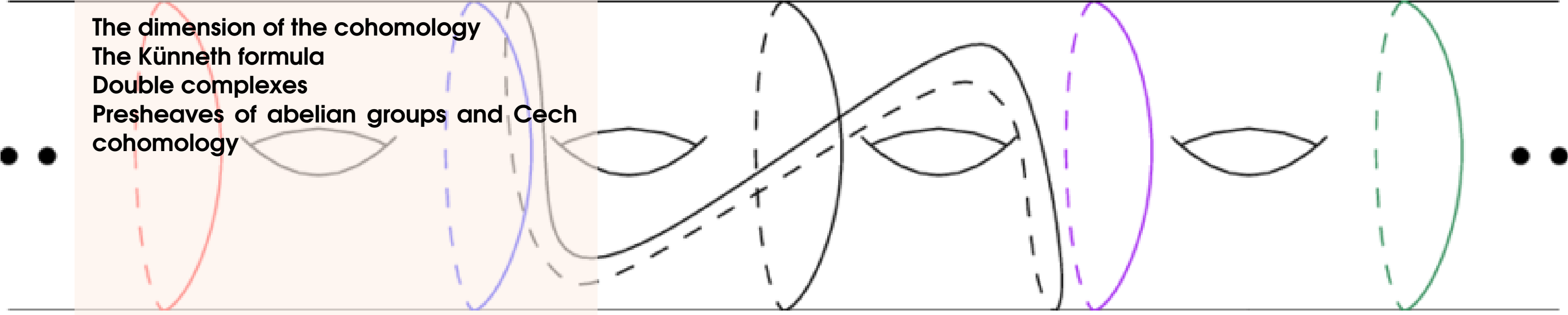
$$h_c^q(\mathbb{R}^n) = h_c^{q-1}(\mathbb{R}^{n-1}) = \cdots = h_c^{q-n}(\mathbb{R}^0) = h^{q-n}(\mathbb{R}^0). \qquad \blacksquare$$

The "shift of exponents" in the statement makes impossible to conjecture generalizations to the compact support cohomology of most of the consequences of the Poincaré lemma for the De Rham cohomology discussed in the previous section.

For example (Exercise 7.4.1), if $\pi\colon E \to B$ is a vector bundle, then the map π^* induces isomorphisms in De Rham cohomology: $\forall q$, $H_{DR}^q(E) \cong H_{DR}^q(B)$. In contrast, even if for the trivial bundle $E = B \times \mathbb{R}^r$ we know by Poincaré Lemma that $H_c^q(B \times \mathbb{R}^r) \cong H_c^{q-r}(B)$, the analogous statement is not true for other vector bundles on B of rank r. A counterexample is provided by the Moebius band, seen as rank 1 vector bundle over S^1: we will see (Theorem 9.3.6) that its second compact support cohomology group has dimension zero, whereas the first cohomology group of S^1 has dimension 1.

A good reason for this *failure* may be that the Moebius band is not orientable as vector bundle over S^1, and therefore there is no way to define an integration along the fibres in this cases. Indeed the Poincaré Lemma for the cohomology with compact support generalizes to orientable vector bundles under some more assumptions on the base, as we will see in the forthcoming Exercise 8.1.2 and later in the crucial (involving a different cohomology theory) Thom isomorphism Theorem 9.5.6.

Exercise 7.5.1 Compute the compact support cohomology of the following manifolds
- $\mathbb{R}^n$
- $\mathbb{R}^n_+$
- the interior of the cylinder
- $S^n \times \mathbb{R}^m$



8. Manifolds of finite type

8.1 The dimension of the cohomology

In all the exercises up to now, all the cohomology groups were finitely dimensional.

Is that true in general? The answer is no: there are manifolds with some cohomology groups infinite dimensional. Anyway these are rare, in some sense, and most of the examples considered in these lectures have all cohomology groups of finite dimension. This property is indeed shared by a large category of manifolds, the manifolds of *finite type*.

Definition 8.1.1 Let M be a manifold of dimension n. An open cover $\mathfrak{U} := \{U_\alpha\}_{\alpha \in I}$ is **good** if

$$\forall k \in \mathbb{N},\ \forall i_1, \ldots, i_k \in I,\ \text{it holds} \bigcap_{j=1}^{k} U_{i_j} \cong \mathbb{R}^n \text{ or } \mathbb{R}^n_+ \text{ or } \emptyset.$$

It is not difficult to construct a good cover in every concrete case (try with your favourite manifold!). Indeed

Theorem 8.1.2 Every manifold has a good cover.

We skip the proof of Theorem 8.1.2, since it needs some Riemannian Geometry.

Definition 8.1.3 A manifold is of **finite type** if it admits a good cover of finite cardinality.

By Theorem 8.1.2 it follows

Corollary 8.1.4 Every compact manifold is of finite type.

Anyway, the category of manifolds of finite type is larger than the category of compact manifolds. For example, all manifolds obtained by removing finitely many points from a compact manifold are of finite type. All the examples of manifolds we have considered up to now are of finite type.

Proposition 8.1.5 All De Rham cohomology groups of a manifold of finite type have finite dimension.

The idea of this proof is very important, since the same inductive procedure will be used in many other proofs in the next sections.

Proof. Let M be a manifold of finite type and let $\mathfrak{U} = \{U_1, \dots, U_k\}$ be a finite good cover of M. We prove the statement by induction on k.

If $k = 1$ then M is isomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$, whose cohomology groups have finite dimension.

Assume then the statement true for all manifolds of finite type admitting a good cover of cardinality strictly smaller than k.

We define $U := U_1 \cup \cdots U_{k-1}$, $V := U_k$. We note that

- $\{U_1, \dots, U_{k-1}\}$ is a good cover of U of cardinality $k-1$;
- $\{U_k\}$ is a good cover of V of cardinality 1;
- $\{U_1 \cap U_k, \dots, U_{k-1} \cap U_k\}$ is a good cover of $U \cap V$ of cardinality $k-1$.

Then the statement holds for U, V and $U \cap V$.

By the Mayer-Vietoris exact sequence

$$H^{q-1}_{DR}(U \cap V) \xrightarrow{d_*} H^q_{DR}(M) \xrightarrow{f_*} H^q_{DR}(U) \oplus H^q_{DR}(V)$$

we deduce[1]

$$\begin{aligned} h^q_{DR}(M) &= \dim \ker f_* + \dim \operatorname{Im} f_* \\ &= \dim \operatorname{Im} d_* + \dim \operatorname{Im} f_* \\ &\leq h^{q-1}_{DR}(U \cap V) + h^q_{DR}(U) + h^q_{DR}(V). \end{aligned}$$

■

Complement 8.1.1 State and prove the analogous of Proposition 8.1.5 for the cohomology with compact support.

Exercise 8.1.1 Construct a connected manifold not of finite type, and prove that it is not of finite type.

Exercise 8.1.2 Let $\pi \colon E \to B$ be a real vector bundle of rank r over a manifold of finite type B given by a smooth cocycle, so that E has a differentiable structure such that π is smooth as in Proposition 3.2.2. Assume moreover that E is orientable as vector bundle. Then

$$\forall q \;\; H^q_c(E) \cong H^{q-r}_c(B).$$

8.2 The Künneth formula

The Künneth formula is a theorem computing the De Rham cohomology ring of a product of manifolds by the De Rham cohomology ring of the factors.

Recall Definition 5.1.3: for every pair of finitely dimensional vector spaces V_1, V_2 their tensor product $V_1 \otimes V_2$ is the space of all bilinear maps $V_1^* \times V_2^* \to \mathbb{K}$.

Recall also that $\forall (v_1, v_2) \in V_1 \times V_2$ we defined the decomposable tensor $v_1 \otimes v_2 \in V_1 \otimes V_2$ as the one such that $\forall (\varphi_1, \varphi_2) \in V_1^* \times V_2^*$,

$$(v_1 \otimes v_2)(\varphi_1, \varphi_2) = \varphi_1(v_1)\varphi_2(v_2).$$

We will need the following two powerful algebraic tools.

[1] Here we use that $\dim V = \dim \ker f + \dim \operatorname{Im} f$ holds for every linear map $f \colon V \to W$, even if the dimension of V is not finite, as then at least one among $\ker f$ and $\operatorname{Im} f$ has infinite dimension too.

Lemma 8.2.1 Let V, A_0, A_1 and A_2 be finitely dimensional vector spaces. Assume be given an exact sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2.$$

Then the induced sequence

$$A_0 \otimes V \xrightarrow{f_0 \otimes \mathrm{Id}_V} A_1 \otimes V \xrightarrow{f_1 \otimes \mathrm{Id}_V} A_2 \otimes V$$

is exact too.

Proof. By Definition 5.1.6 $(f_i \otimes \mathrm{Id}_V)(a \otimes v) = f_i(a) \otimes v$, so the image of $f_i \otimes \mathrm{Id}_V$ is generated by the vectors of the form $a' \otimes v$ for $a' \in \operatorname{Im} f_i$, $v \in V$. It follows that, if $a_1, \ldots, a_m$ and $v_1, \ldots, v_n$ are respective bases of $\operatorname{Im} f_i$ and V, then $\{a_j \otimes v_k\}$ is a basis of $\operatorname{Im}(f_i \otimes \mathrm{Id}_V)$. In particular

$$r(f_i \otimes V) = r(f_i) \cdot (\dim V)$$

It follows that $\operatorname{Im}(f_0 \otimes \mathrm{Id}_V)$ and $\ker(f_1 \otimes \mathrm{Id}_V)$ have the same dimension:

$$\begin{aligned}
&\dim \operatorname{Im}(f_0 \otimes V) = r(f_0 \otimes V) = r(f_0)(\dim V) = (\dim \ker f_1)(\dim V) = \\
&= (\dim A_1 - r(f_1))(\dim V) = (\dim A_1)(\dim V) - r(f_1)(\dim V) = \\
&\qquad\qquad = \dim(A_1 \otimes V) - r(f_1 \otimes \mathrm{Id}_V) = \dim \ker(f_1 \otimes \mathrm{Id}_V)
\end{aligned}$$

It is then enough if we prove the inclusion $\operatorname{Im}(f_0 \otimes \mathrm{Id}_V) \subset \ker(f_1 \otimes \mathrm{Id}_V)$. In other words, we need to prove $(f_1 \otimes \mathrm{Id}_V) \circ (f_0 \otimes \mathrm{Id}_V) = 0$. By Theorem 5.1.5 it is enough if we check that all decomposable elements are in the kernel and in fact

$$(f_1 \otimes \mathrm{Id}_V) \circ (f_0 \otimes \mathrm{Id}_V)(a \otimes v) = (f_1 \otimes \mathrm{Id}_V)\ (f_0(a) \otimes v) = (f_1(f_0(a)) \otimes v) = 0 \otimes v = 0. \blacksquare$$

Lemma 8.2.2 — Five Lemma. Consider a commutative diagram of linear applications

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_A"'] & B \arrow[r] \arrow[d, "f_B"'] & C \arrow[r] \arrow[d, "f_C"'] & D \arrow[r] \arrow[d, "f_D"'] & E \arrow[d, "f_E"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

and assume that both rows are exact sequence and that the "external" vertical maps f_A, f_B, f_D and f_E are isomorphisms. Then also f_C is an isomorphism.

Proof. The proof follows a *diagram chasing* argument like those we have left to the reader in the proof of Theorem 7.2.4. Therefore we leave this proof to the reader as well. ■

R Lemma 8.2.2 holds under the weaker assumption that f_A be just surjective and f_E be just injective, as the reader who writes the proof will easily notice.
The statement of Lemma 8.2.2 is then weaker that its proof. Anyway, this weaker statement is easier to remember and strong enough for all the applications in these notes.

Now we can prove the Künneth formula.

Theorem 8.2.3 — Künneth formula. Let M,N be manifolds of finite type, and assume $\partial M=\emptyset$. Then, $\forall k\in\mathbb{Z}$, there are isomorphisms

$$K_M\colon \bigoplus_{p+q=k} H^p_{DR}(M)\otimes H^q_{DR}(N)\xrightarrow{\cong} H^k_{DR}(M\times N)$$

defined on decomposable tensors as follows: given classes $\omega\in H^p_{DR}(M)$ and $\eta\in H^q_{DR}(N)$,

$$K_M(\omega\otimes\eta)=\pi_1^*\omega\wedge\pi_2^*\eta$$

where the maps π_j are the natural projections $\pi_1\colon M\times N\to M$ and $\pi_2\colon M\times N\to N$.

In particular if $\{\omega_i\}_{i\in I}$ and $\{\eta_j\}_{j\in J}$ are respectively bases of $H^\bullet_{DR}(M)$ and $H^\bullet_{DR}(N)$ then $\{\pi_1^*\omega_i\wedge\pi_2^*\eta_j\}_{(i,j)\in I\times J}$ is a basis of $H^\bullet_{DR}(M\times N)$.

R The assumption $\partial M=\emptyset$ is necessary just to ensure that $M\times N$ has a natural differentiable structure, which we are implicitly using.

Proof. Let $\mathfrak{U}:=\{U_1,\dots,U_h\}$ be a finite good cover of M. We prove the statement by induction on h.

If $h=1$, then $M\cong\mathbb{R}^n$ and

$$\bigoplus_{p+q=k} H^p_{DR}(M)\otimes H^q_{DR}(N)=H^0_{DR}(\mathbb{R}^n)\otimes H^k_{DR}(N)\cong H^k_{DR}(N)$$

where the last isomorphism is given, identifying as usual $H^0_{DR}(\mathbb{R}^n)$ with $\mathbb{R}$ associating the class of a constant function to the corresponding constant, by $\lambda\otimes\omega\mapsto\lambda\omega$. Then the statement for $h=1$ claims that the map $H^k_{DR}(N)\to H^k_{DR}(\mathbb{R}^n\times N)$ mapping η to $\pi_2^*\eta$ is an isomorphism: this is part ot the Extended Poincaré Lemma, Theorem 7.4.2.

Assume now $h>1$. Arguing as in the proof of Proposition 8.1.5 we can find two open subsets $U,V\subset M$ such that $U\cup V=M$ and U, V and $U\cap V$ have finite good covers of cardinality strictly smaller than h. By induction, we may then assume that the statement holds when substituting U, V or $U\cap V$ to M. So by inductive assumption all maps K_U, K_V and $K_{U\cap V}$ are isomorphisms.

Fix two integers p and q and consider the following diagram of linear maps

$$\begin{tikzcd}
(H^{p-1}_{DR}(U)\oplus H^{p-1}_{DR}(V))\otimes H^q_{DR}(N) \arrow[r, "K_U\oplus K_V"] \arrow[d] & H^{p+q-1}_{DR}(U\times N)\oplus H^{p+q-1}_{DR}(V\times N) \arrow[d] \\
H^{p-1}_{DR}(U\cap V)\otimes H^q_{DR}(N) \arrow[r, "K_{U\cap V}"] \arrow[d] & H^{p+q-1}_{DR}((U\cap V)\times N) \arrow[d] \\
H^p_{DR}(M)\otimes H^q_{DR}(N) \arrow[r, "K_M"] \arrow[d] & H^{p+q}_{DR}(M\times N) \arrow[d] \\
(H^p_{DR}(U)\oplus H^p_{DR}(V))\otimes H^q_{DR}(N) \arrow[r, "K_U\oplus K_V"] \arrow[d] & H^{p+q}_{DR}(U\times N)\oplus H^{p+q}_{DR}(V\times N) \arrow[d] \\
H^p_{DR}(U\cap V)\otimes H^q_{DR}(N) \arrow[r, "K_{U\cap V}"] & H^{p+q}_{DR}((U\cap V)\times N)
\end{tikzcd} \tag{8.1}$$

where

- the right column is the cohomology exact sequence induced by the Mayer-Vietoris exact sequence corresponding to the decomposition $M \times N = (U \times N) \cup (V \times N)$;
- the left column is obtained by the cohomology exact sequence induced by the Mayer-Vietoris exact sequence corresponding to the decomposition $M = U \cup V$ by tensoring with $\mathrm{Id}_{H^q(N)}$: it is then exact by Lemma 8.2.1.

We show that the diagram (8.1) commutes. We have to check the commutativity of four squares; the one at the bottom is

$$\begin{tikzcd}
(H^p_{DR}(U) \oplus H^p_{DR}(V)) \otimes H^q_{DR}(N) \arrow[r, "K_U \oplus K_V"] \arrow[d] & H^{p+q}_{DR}(U \times N) \oplus H^{p+q}_{DR}(V \times N) \arrow[d] \\
H^p_{DR}(U \cap V) \otimes H^q_{DR}(N) \arrow[r, "K_{U\cap V}"] & H^{p+q}_{DR}((U \cap V) \times N)
\end{tikzcd}$$

We check it, by taking general elements $(\omega_1, \omega_2) \in H^p(U) \oplus H^p(V)$, $\eta \in H^q(N)$and by computing the two images of $(\omega_1, \omega_2) \otimes \eta$ in $H^{p+q}_{DR}((U \cap V) \times N)$; the one from the "top" way (through $H^{p+q}_{DR}(U \times N) \oplus H^{p+q}_{DR}(V \times N)$) and the one from the "bottom" way (through $H^p_{DR}(U \cap V) \otimes H^q_{DR}(N)$).

Indeed following the top way we obtain

$$(\omega_1, \omega_2) \otimes \eta \mapsto (\pi_1^* \omega_1 \wedge \pi_2^* \eta, \pi_1^* \omega_2 \wedge \pi_2^* \eta) \mapsto (\pi_1^* \omega_2 \wedge \pi_2^* \eta)_{|(U\cap V)\times N} - (\pi_1^* \omega_1 \wedge \pi_2^* \eta)_{|(U\cap V)\times N})$$

and following the bottom way we obtain

$$(\omega_1, \omega_2) \otimes \eta \mapsto ((\omega_2)_{|U\cap V} - (\omega_1)_{|U\cap V}) \otimes \eta \mapsto (\pi_1^* \left((\omega_2)_{|U\cap V} - (\omega_1)_{|U\cap V}\right)) \wedge \pi_2^* \eta$$

that is obviously equal. Then the bottom square commutes, as well as the top square that is identical (substituting p with $p-1$). The proof of the commutativity of the remaining two squares is similar and left to the reader.

Now fix k and consider all diagrams (8.1) for p, q with $p+q=k$; they all have the same right column. *Summing the left columns* we obtain a diagram

$$\begin{tikzcd}
\bigoplus_{p+q=k} (H^{p-1}_{DR}(U) \oplus H^{p-1}_{DR}(V)) \otimes H^q_{DR}(N) \arrow[r, "K_U \oplus K_V"] \arrow[d] & H^{k-1}_{DR}(U \times N) \oplus H^{k-1}_{DR}(V \times N) \arrow[d] \\
\bigoplus_{p+q=k} H^{p-1}_{DR}(U \cap V) \otimes H^q_{DR}(N) \arrow[r, "K_{U\cap V}"] \arrow[d] & H^{k-1}_{DR}((U \cap V) \times N) \arrow[d] \\
\bigoplus_{p+q=k} H^p_{DR}(M) \otimes H^q_{DR}(N) \arrow[r, "K_M"] \arrow[d] & H^k_{DR}(M \times N) \arrow[d] \\
\bigoplus_{p+q=k} (H^p_{DR}(U) \oplus H^p_{DR}(V)) \otimes H^q_{DR}(N) \arrow[r, "K_U \oplus K_V"] \arrow[d] & H^k_{DR}(U \times N) \oplus H^k_{DR}(V \times N) \arrow[d] \\
\bigoplus_{p+q=k} H^p_{DR}(U \cap V) \otimes H^q_{DR}(N) \arrow[r, "K_{U\cap V}"] & H^k_{DR}((U \cap V) \times N)
\end{tikzcd} \tag{8.2}$$

such that

- the columns are exact sequences since the columns of (8.1) are exact;
- the diagram commutes since (8.1) commutes;

- by the inductive hypothesis, the first two horizontal maps and the last two horizontal maps are isomorphisms.

Then the diagram (8.2) commutes as well and the statement follows by the Five Lemma 8.2.2. ■

It is natural to try to generalize the Künneth formula to a formula for computing the cohomology of a general fibre bundle. Indeed, the product of two manifolds is a trivial bundle (in two different ways: $\pi_1 : M \times N \to M$ is a trivial bundle on M with fiber N whereas $\pi_2 : M \times N \to N$ is a trivial bundle on N with fiber M).

We can then say that the Künneth formula produces generators for the cohomology groups of a trivial bundle, from generators of the cohomology groups of its basis and of its fibre. A similar result for every bundle does not hold: for example the Klein bottle in Exercise 9.3.5 is a fibre bundle over S^1 with fibre S^1 whose second cohomology group has dimension 0 (as the reader will show solving Exercise 9.3.5) and not 1 as a Künneth type formula would predict.

Still, if there are cohomology classes in E whose restrictions to every fibre give a basis of the cohomology of the fibre, then one can prove, by the same strategy of the proof of Künneth formula, the following

Theorem 8.2.4 — Leray-Hirsch Theorem. Consider a fibre bundle $\pi \colon E \to B$ with fibre F.

Assume that both F and B are manifolds of finite type and that[a] either $\partial F = \emptyset$ or $\partial B = \emptyset$. Consider E with the natural differentiable structure making π a submersion and the inclusion of each fibre $F \cong F_p \hookrightarrow E$ an embedding.

Assume that there are cohomology classes $e_1, \dots e_r \in H^\bullet_{DR}(E)$ such that $\forall p \in B$, $\{e_{i|F_p}\}$ is a basis of $H^\bullet_{DR}(F_p) \cong H^\bullet_{DR}(F)$.

Then, $\forall k$, $H^k_{DR}(E) \cong H^k_{DR}(B \times F) \cong \bigoplus_{p+q=k} H^p_{DR}(B) \otimes H^q_{DR}(F)$.

More precisely, if $\{\omega_1, \dots \omega_s\}$ is a basis of $H^\bullet_{DR}(B)$, then $\{\pi^* \omega_i \wedge e_j\}$ is a basis of $H^\bullet_{DR}(E)$.

[a] This assumption is necessary to define a differentiable structure on E

Complement 8.2.1 Prove Lemma 8.2.1.

Complement 8.2.2 Prove that there is a canonical isomorphism

$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C).$$

Complement 8.2.3 Complete the proof that the diagram (8.1) commutes.

Complement 8.2.4 State and prove a Künneth formula for the cohomology with compact support.

Exercise 8.2.1 Compute the De Rham cohomology groups of $(S^1)^k$, and compare the result with Pascal's triangle.

Exercise 8.2.2 Prove that $S^{m_1} \times S^{n_1}$ is diffeomorphic to $S^{m_2} \times S^{n_2}$ if and only if $\{m_1, n_1\} = \{m_2, n_2\}$.

Exercise 8.2.3 Let $M_1, \dots, M_k$ be manifolds without boundary of finite type. Prove that

$$\dim H^\bullet(M_1 \times \cdots \times M_k) = \prod_i \dim H^\bullet(M_i).$$

Exercise 8.2.4 Prove that two products of spheres are diffeomorphic if and only if they have the same factors up to the order.

Exercise 8.2.5 Let $M_1, \dots, M_k$ be manifolds without boundary of finite type. Use the Künneth formula to prove that

$$e(M_1 \times \cdots \times M_k) = \prod_i e(M_i).$$

Exercise 8.2.6 Let $\pi\colon E \to B$ be a fibre bundle with fibre F. Assume B compact and F of finite type.

Prove that

$$e(E) = e(B)e(F).$$

8.3 Double complexes

Definition 8.3.1 A double complex is a family of vector spaces $\{K^{p,q}\}_{(p,q)\in\mathbb{N}^2}$ provided, $\forall(p,q)$ of two linear maps

$$d\colon K^{p,q} \to K^{p,q+1}$$

$$\delta\colon K^{p,q} \to K^{p+1,q}$$

such that $d^2 = \delta^2 = 0$ and $d\delta = \delta d$.

Equivalently we can see a double complex as the *bigraded* vector space $K^{\bullet,\bullet} := \bigoplus_{p,q\in\mathbb{N}} K^{p,q}$, where the elements of $K^{p,q}$ are the (bi)homogeneous elements of bidegree (p,q), d is a linear map of bidegree $(0,1)$ and δ is a linear map of bidegree $(1,0)$. Notice that we do not allow negative values for either p or q.

A double complex can be then visualized as a commutative diagram of the form

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
K^{0,4} & \xrightarrow{\delta} & K^{1,4} & \xrightarrow{\delta} & K^{2,4} & \xrightarrow{\delta} & K^{3,4} & \xrightarrow{\delta} & K^{4,4} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
K^{0,3} & \xrightarrow{\delta} & K^{1,3} & \xrightarrow{\delta} & K^{2,3} & \xrightarrow{\delta} & K^{3,3} & \xrightarrow{\delta} & K^{4,3} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
K^{0,2} & \xrightarrow{\delta} & K^{1,2} & \xrightarrow{\delta} & K^{2,2} & \xrightarrow{\delta} & K^{3,2} & \xrightarrow{\delta} & K^{4,2} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
K^{0,1} & \xrightarrow{\delta} & K^{1,1} & \xrightarrow{\delta} & K^{2,1} & \xrightarrow{\delta} & K^{3,1} & \xrightarrow{\delta} & K^{4,1} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
K^{0,0} & \xrightarrow{\delta} & K^{1,0} & \xrightarrow{\delta} & K^{2,0} & \xrightarrow{\delta} & K^{3,0} & \xrightarrow{\delta} & K^{4,0} & \xrightarrow{\delta} & \cdots
\end{array} \tag{8.3}$$

such that all rows and all columns are differential complexes.

We associate to every double complex $(K^{\bullet,\bullet}, d, \delta)$ as above the differential complex $(K^\bullet, D)$ whose graded pieces are the spaces

$$K^n := \bigoplus_{(p,q)|p+q=n} K^{p,q}$$

and whose differential is

$$D := \delta + (-1)^p d, \tag{8.4}$$

in the sense that D is defined as the only linear operator $D : K^\bullet \to K^\bullet$ of degree 1 such that for each $\omega \in K^{p,q} \subset K^{p+q}$, $D\omega = \delta\omega + (-1)^p d\omega \subset K^{p+1,q+1} \oplus K^{p,q+1} \subset K^{p+q+1}$.

By definition $n < 0 \Rightarrow K^n = 0$.

The choice of sign $(-1)^p$ as coefficient of d in the definition of D is necessary to ensure $D \circ D = 0$ as we will see in the proof of the following

Lemma 8.3.2 $(K^\bullet, D)$ is a differential complex.

Proof. The only nontrivial check is $D^2 = 0$. It is enough to prove $DD\omega = 0$ for any $\omega \in K^{p,q}$. Indeed

$$\begin{aligned}
DD\omega &= D\delta\omega + (-1)^p Dd\omega \\
&= (\delta + (-1)^{p+1} d)\delta\omega + (-1)^p(\delta + (-1)^p d)d\omega \\
&= \delta\delta\omega + (-1)^p(-d\delta + \delta d)\omega + dd\omega \\
&= 0 + 0 + 0 = 0.
\end{aligned}$$

■

Consequently we get, for every double complex, a cohomology.

Definition 8.3.3 The **cohomology of a double complex** $(K^{\bullet,\bullet}, d, \delta)$ is the cohomology $H^\bullet_D(K^\bullet)$ of the differential complex $(K^\bullet, D)$.

Notice that $(\ker \delta \cap \ker d) \subset \ker D$.

The converse is not true in general and a general element $\omega \in \ker D$ has $d\omega \neq 0$ and $\delta\omega \neq 0$. However such ω do not belong to any bihomogeneous addendum $K^{p,q}$. Indeed, by definition, for all p and q

$$K^{p,q} \cap \ker D = K^{p,q} \cap \ker \delta \cap \ker d.$$

Since $K^0 = K^{0,0}$ has only one addendum that is not trivial, $K^{0,0}$, then

$$H^0_D(K^\bullet) = K^0 \cap \ker D = K^{0,0} \cap \ker d \cap \ker \delta.$$

Definition 8.3.4 The double complex (8.3) has **exact rows** if its rows are exact, *i.e.*

$$\forall p,q \in \mathbb{N},\ \delta(K^{p,q}) = \ker \delta_{|K^{p+1,q}}.$$

Note that the exactness of the rows of a double complex do not claim anything on the kernels of the maps $\delta_{|K^{0,q}}\colon K^{0,q} \to K^{1,q}$.

We will need the following technical lemma.

Lemma 8.3.5 Consider a double complex with exact rows as in (8.3). Then for every Φ in K^n with the property that $D\Phi$ belongs to $K^{0,n+1}$, there exists a Φ' in $K^{0,n}$ such that $\Phi - \Phi'$ belongs to DK^{n-1}.

Proof. If $\Phi = 0$ the statement is clearly true with $\Phi' = 0$: $\Phi - \Phi' = 0 \in DK^{n-1}$.

We assume then $\Phi \neq 0$.

Set Φ_j for the component of Φ in $K^{j,n-j}$. Since $\Phi \neq 0$ at least one of the Φ_j is different from zero. Set k for the biggest j with $\Phi_j \neq 0$. So $\Phi = \Phi_0 + \ldots + \Phi_k$, $\Phi_k \neq 0$.

We prove the statement by induction on k.

The first case $k = 0$ is trivial since then $K^0 = K^{0,0}$ and therefore we may pick $\Phi' = \Phi$.

Now consider the case $k \geq 1$.

The component of $D\Phi$ in $K^{k+1,n-k}$ equals $\delta\Phi_k$. So, by $D\Phi \in K^{0,n+1}$ follows $\delta\Phi_k = 0$. Then, by the exactness of the rows, $\exists \Psi \in K^{k-1,n-k}$ such that $\delta\Psi = \Phi_k$. Then

$$\Phi - D\Psi = \Phi_0 + \ldots + \Phi_{k-2} + \Phi_{k-1} + \Phi_k - D\Psi = \Phi_0 + \ldots + \Phi_{k-2} + (\Phi_{k-1} \pm d\Psi).$$

Since $(\Phi_{k-1} \pm d\Psi) \in K^{k-1,n-k+1}$ by the inductive hypothesis there exists $\Phi' \in K^{0,n}$ with $(\Phi - D\Psi) - \Phi' \in DK^{n-1}$, and then $\Phi - \Phi' \in DK^{n-1}$. ■

If the double complex (8.3) has exact rows the spaces $A^q := \ker(\delta_{|K^{0,q}})$ naturally build a new column on the left to (8.3). Indeed, if $a \in A^q$, since $\delta da = d\delta a = d0 = 0$ then $dA^q \subset A^{q+1}$. Then $(A^\bullet = \oplus A^q, d)$ is a differential complex.

Set $r\colon A^q \hookrightarrow K^{0,q}$ for the inclusion maps. We have then obtained a bigger commutative diagram

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^4 \arrow[u, "d"] \arrow[r, "r"] & K^{0,4} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,4} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,4} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,4} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,4} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^3 \arrow[u, "d"] \arrow[r, "r"] & K^{0,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,3} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^2 \arrow[u, "d"] \arrow[r, "r"] & K^{0,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,2} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^1 \arrow[u, "d"] \arrow[r, "r"] & K^{0,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^0 \arrow[u, "d"] \arrow[r, "r"] & K^{0,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,0} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
 & 0 \arrow[u] & & & & & &
\end{tikzcd}
\tag{8.5}
$$

This is summarized by the following definition.

Definition 8.3.6 An **augmented double complex with exact rows** is given by
- a double complex $((K^{\bullet,\bullet}, d, \delta)$,
- a differential complex $(A^\bullet, d)$ with $q < 0 \Rightarrow A^q = \{0\}$,
- for all q injective linear maps $r \colon A^q \to K^{0,q}$,

such that the corresponding diagram (8.5) is commutative (*i.e.* $rd = dr$) and has exact rows.

The main result of this section is Proposition 8.3.8 that produces, given an augmented double complex with exact rows, an isomorphism of graded vector spaces among the cohomologies of the differential complexes $(A^\bullet, d)$ and $(K^\bullet, D)$.

Definition 8.3.7 A chain map of degree zero among two differential complexes is a **quasi-isomorphism**, if the induced map among the respective cohomologies is an isomorphism.

Proposition 8.3.8 Consider an augmented double complex with exact rows as in (8.5).
Then the maps $r \colon A^q \to K^{0,q} \subset K^q$ form a quasi-isomorphism

$$r \colon (A^\bullet, d) \to (K^\bullet, D).$$

Proof. For all $q \in \mathbb{N}$, $\forall a \in A^q$, since $\delta \circ r = 0$, by the commutativity of the diagram (8.5)

$$Dra = dra + \delta ra = dra = rda,$$

so $r \colon (A^\bullet, d) \to (K^\bullet, D)$ is a chain map of degree zero.

We have then induced maps in cohomology

$$r_* \colon H^q(A^\bullet) \to H^q_D(K^\bullet).$$

We show first their surjectivity and then their injectivity.

Surjectivity. Consider a class $[\Phi]$ in $H^q_D(K^\bullet)$. Since Φ of it is $D-$closed, we can apply Lemma 8.3.5 and then, replacing Φ with Φ', we can assume Φ in $K^{0,q}$.

Then from $D\Phi = 0$ it follows the vanishing of both $d\Phi \in K^{0,q+1}$ and $\delta\Phi \in K^{1,q}$. The latter vanishing $\delta\Phi = 0$ implies, by the exactness of the rows, that there exists $a \in A^q$ such that $ra = \Phi$. The former vanishing $d\Phi = 0$ and the commutativity of the diagram (8.5) gives then $rda = dra = d\Phi = 0$ that implies, by the injectivity of r, $da = 0$. So there is a class $[a] \in H^q(A^\bullet)$ and $r_*[a] = [ra] = [\Phi]$.

Injectivity. Take $\omega \in A^q$ with $d\omega = 0$ and $[\omega] \in \ker r_*$. Then there exists Φ such that $D\Phi = r\omega \in K^{0,q}$. Then by Lemma 8.3.5 we can assume $\Phi \in K^{0,q-1}$.

Then

$$D\Phi \in K^{0,q} \Leftrightarrow \delta\Phi = 0.$$

Then by the exactness of the rows of the diagram (8.5) there exists $\eta \in A^{q-1}$ such that $r\eta = \Phi$. Finally $r\omega = D\Phi = d\Phi = dr\eta = rd\eta$ that implies, by the injectivity of r, $\omega = d\eta$. So $[\omega] = 0$. ■

An analogous construction may be done by reversing the role of the rows and of the columns of (8.3).

Definition 8.3.9 The double complex (8.3) has **exact columns** if

$$\forall p,q,\ d(K^{p,q}) = \ker d_{|K^{p,q+1}}.$$

In this case we add a row on the bottom of (8.3). We set $B^p := \ker(d\colon K^{p,0} \to K^{p,1})$, $B^\bullet := \oplus B^p$. Since $dB^p \subset B^{p+1}$, $(B^\bullet, \delta)$ is a differential complex.

Definition 8.3.10 An **augmented double complex with exact columns** is given by
- a double complex $((K^{\bullet,\bullet}, d, \delta)$,
- a differential complex $(B^\bullet, \delta)$ wih $p < 0 \Rightarrow B^p = \{0\}$,
- $\forall p$ linear maps $s\colon B^p \to K^{p,0}$,

such that the corresponding diagram is commutative (*i.e.* $s\delta = \delta s$) and has exact columns.

By Proposition 8.3.8, exchanging rows and columns, every augmented double complex with exact columns induces a quasi-isomorphism

$$s\colon (B^\bullet, \delta) \to (K^\bullet, D).$$

If both the rows and the columns of a double complex (8.3) are exact, we may add both the new row and the new column at the same time, getting the following.

Definition 8.3.11 A **doubly augmented double complex with exact rows and columns** is given by
- a double complex $((K^{\bullet,\bullet}, d, \delta)$ with exact rows and columns,
- a differential complex $(A^\bullet, d)$ with $q < 0 \Rightarrow A^q = \{0\}$,
- a differential complex $(B^\bullet, \delta)$ wih $p < 0 \Rightarrow B^p = \{0\}$,
- $\forall q$ linear injective maps $r\colon A^q \hookrightarrow K^{0,q}$,
- $\forall p$ linear injective maps $s\colon B^p \hookrightarrow K^{p,0}$,

such that

$$rd = dr \qquad\qquad s\delta = \delta s$$

We represent it by drawing the commutative diagram

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^3 \arrow[u, "d"] \arrow[r, "r"] & K^{0,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,3} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,3} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^2 \arrow[u, "d"] \arrow[r, "r"] & K^{0,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,2} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,2} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^1 \arrow[u, "d"] \arrow[r, "r"] & K^{0,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,1} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,1} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & A^0 \arrow[u, "d"] \arrow[r, "r"] & K^{0,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{1,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{2,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{3,0} \arrow[u, "d"] \arrow[r, "\delta"] & K^{4,0} \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
 & 0 \arrow[u] \arrow[r] & B^0 \arrow[u, "s"] \arrow[r, "\delta"] & B^1 \arrow[u, "s"] \arrow[r, "\delta"] & B^2 \arrow[u, "s"] \arrow[r, "\delta"] & B^3 \arrow[u, "s"] \arrow[r, "\delta"] & B^4 \arrow[u, "s"] \arrow[r, "\delta"] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{8.6}
$$

Notice that all rows and all columns of (8.6) different by the differential complexes $(A^\bullet, d)$ and $(B^\bullet, \delta)$ are exact sequences.

Applying Proposition 8.3.8 twice, we get that both these differential complexes are quasi-isomorphic to $(K^\bullet, D)$. In particular their cohomologies are both isomorphic, as graded vector spaces, to the same object, the cohomology of $(K^\bullet, D)$.

Theorem 8.3.12 Assume to have a doubly augmented double complex with exact rows and columns as in (8.6). Then

$$H_d^\bullet(A^\bullet) \cong H_\delta^\bullet(B^\bullet)$$

as graded vector spaces.

8.4 Presheaves of abelian groups and Cech cohomology

The sheaf theory is a very powerful tool.

In this section we just sketch the beginning of this theory, by defining the presheaves of abelian groups and their Cech cohomologies.

Presheaves can be done of any algebraic structure (groups, rings, vector spaces, ...) by adapting in the natural way the definition below. Even presheaves of sets exist and are useful.

Anyway, the definition of cohomology does not work in general. The minimal algebraic structure necessary for the definition of cohomology is the structure of abelian group.

We are mainly interested in presheaves of vector spaces. Notice that the vector spaces are abelian groups with a further operation, the multiplication by scalars. So preasheaves of vector spaces are special preasheaves of abelian groups.

Definition 8.4.1 Let X be a topological space.

A **presheaf $\mathscr{F}$ of abelian groups** on X is a *functor* as follows:

- For each open set U of X there corresponds an abelian group $\mathscr{F}(U)$, the **sections** of $\mathscr{F}$

over U;
- For each inclusion of open sets $V \subseteq U$ there corresponds a group homomorphism $res_{V,U} : \mathscr{F}(U) \to \mathscr{F}(V)$.

The homomorphisms $res_{V,U}$ are called restriction morphisms; often $res_{V,U}(s)$ is denoted $s_{|V}$ by analogy with restriction of functions.

The restriction morphisms are required to satisfy two properties:
- For each open set U of X, $res_{U,U}$ is the identity of $\mathscr{F}(U)$.
- If we have three open sets $W \subseteq V \subseteq U$, then $res_{W,V} \circ res_{V,U} = res_{W,U}$.

The latter condition says that it doesn't matter whether we restrict directly to a smaller open subset W or we restrict first to a bigger open subset V, then to W.

Example 8.1 Here are few examples[a] of presheaves of abelian groups on a topological space.
- Consider any abelian group G. The **presheaf of the constant functions with values in *G*** is the presheaf defined by
  for every open subset U, the group G;
  for every pair of open subsets $V \subseteq U$, $res_{V,U} = \mathrm{Id}_G$.
- Consider any abelian group G. The **presheaf *G*** or **presheaf of the locally constant functions with values in *G*** is the presheaf defined by
  for every open subset U, the group[b] of the functions $f : U \to G$ that are locally constant, *i.e.* such that $\forall p \in U$ there exists a neighbourhood V of p in U such that f assumes the same value on all points of V;
  for every pair of open subsets $V \subseteq U$, $res_{V,U}$ is the usual restriction of functions.
- the **presheaf $\mathbf{C^0}$ of the continuos functions** with values in $\mathbb{R}$:
  $C^0(U)$ is the group of the continous functions $f : U \to \mathbb{R}$;
  for every pair of open subsets $V \subseteq U$, $res_{V,U}$ is the usual restriction of functions.

[a] In all cases we set the group of the section over the empty set $\emptyset$ to be the trivial group, the group with one element.

[b] Here the group structure is defined lifting the operations from the codomain G

The presheaf $\mathbb{R}$ of the locally constant functions with real values will be very useful in the following.

If the topologica space is a manifold X, it has some natural presheaves coming from the discussions in these notes.

Example 8.2 In all the following examples of presheaves, to be short, we do not specify the maps $res_{V,U}$: they are the natural restriction maps.

If X is a real manifold, we have
- the **presheaf $\mathbf{C^\infty}$ of the smooth functions**: $\forall U \subset X, C^\infty(U) = \{f : U \to \mathbb{R} | f \text{ is smooth }\}$;
- the **presheaf $\mathbf{\Omega^q}$ of the smooth differential *q*-forms**.

If X is a complex manifold, we have
- the **presheaf $\mathscr{O}$ of the homolomorphic functions**;
- the **presheaf $\mathbf{\Omega^{p,q}}$ of the holomorphic $(\mathbf{p},\mathbf{q})$-forms**;
- the **presheaf $\mathbf{A^{p,q}}$ of the smooth $(\mathbf{p},\mathbf{q})$-forms**.

To every presheaf of abelian groups on X we associate several differential complexes, one for each open covering of X.

For technical reason, we need to fix a total ordering on the open covering.

Definition 8.4.2 Let X be a topological space, and let $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of X, where I is a totally ordered set.

For each p, for each $\alpha_0 \lneqq \alpha_1 \lneqq \ldots \lneqq \alpha_p \in I$ we define

$$U_{\alpha_0\cdots\alpha_p} := U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_p}.$$

Let $\mathscr{F}$ be a presheaf of abelian groups on X.

The **Cech complex** of $\mathscr{F}$ and $\mathscr{U}$ is the differential complex $(C^\bullet(\mathscr{U},\mathscr{F}),\delta)$ where $\forall p \lneqq 0$, $C^p(\mathscr{U},\mathscr{F})$ vanishes whereas $\forall p \geq 0$

$$C^p(\mathscr{U},\mathscr{F}) := \prod_{\alpha_0<\cdots<\alpha_p} \mathscr{F}(U_{\alpha_0\cdots\alpha_p})$$

and the differential δ is defined as follows. $\forall \omega \in C^p(\mathscr{U},\mathscr{F})$, set $\omega_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p}$ for its component in $\mathscr{F}(U_{\alpha_0\cdots\alpha_p})$, $\alpha_0 \lneqq \alpha_1 \lneqq \ldots \lneqq \alpha_p$ Then we define $\delta\omega$ by giving all its components $(\delta\omega)_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}} \in \mathscr{F}(U_{\alpha_0\cdots\alpha_{p+1}})$, $\forall \alpha_0 \lneqq \alpha_1 \lneqq \ldots \lneqq \alpha_{p+1}$.

$$\begin{aligned}(\delta\omega)_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}} &:= (\omega_{\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}})_{|U_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}}} + \\ &- (\omega_{\alpha_0\alpha_2\cdots\alpha_p\alpha_{p+1}})_{|U_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}}} + \ldots + (-1)^{p+1}(\omega_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p})_{|U_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}}}.\end{aligned}$$

Roughly speaking, $(\delta\omega)_{\alpha_0\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}}$, the component of $\delta\omega$ on the intersection of $p+2$ open sets $U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_{p+1}}$, is obtained by taking the restriction of the component of ω on each intersection of $p+1$ of them, and then summing the $p+2$ results with alternating signs.

Note that here we need that our presheaf is a presheaf of groups, as we sum elements in them and take some opposites.

The check that $\delta \circ \delta = 0$ needs the commutativity of the group: it is a straightforward computation that we leave to the reader.

Definition 8.4.3 The **Cech cohomology** of the presheaf of abelian groups $\mathscr{F}$ with respect to the covering $\mathscr{U}$ is the cohomology of the Cech complex $H^\bullet(\mathscr{U},\mathscr{F}) := H^\bullet_\delta(C^\bullet(\mathscr{U},\mathscr{F}))$, a graded vector space whose graded pieces are denoted $H^p(\mathscr{U},\mathscr{F})$.

It is not difficult to prove that the Cech cohomology does not depend, up to isomorphisms, from the choice of the total ordering on I.

Example 8.3 Let $\pi\colon E \to B$ a G-bundle with fibre F, so G is a subgroup of $\mathrm{Aut}(F)$. Assume G abelian.

Choose a trivialization $\left\{\Phi_\alpha\colon E_{|U_\alpha} \to U_\alpha \times F\right\}_{\alpha\in I}$ and fix a total ordedring of I.

The cocycle $\{g_{\alpha\beta}\}$ defines an element of $C^1(\{U_\alpha\},\mathscr{G})$ where G is the presheaf of the constant functions with values in G, which is δ-closed. So it also defines a class in $H^1(\{U_\alpha\},\mathscr{G})$.

Consider a manifold M and an open covering $\mathscr{U} := \{U_\alpha\}_{\alpha\in I}$.

Definition 8.4.4 The **Cech-De Rham complex** is the double complex obtained by taking

- the vector spaces $K^{p,q} := C^p(\mathscr{U},\Omega^q)$;
- as "vertical" differential $d\colon K^{p,q} \to K^{p,q+1}$ the natural map obtained taking the usual differential of forms on each component; so

$$\forall \alpha_0 \lneqq \alpha_1 \lneqq \ldots \lneqq \alpha_p \ (d\omega)_{\alpha_0\cdots\alpha_p} = d(\omega_{\alpha_0\cdots\alpha_p});$$

- as "horizontal" differential $\delta\colon K^{p,q} \to K^{p+1,q}$ the differential of the Cech complex $C^\bullet(\mathscr{U},\Omega^q)$, where Ω^q is the sheaf of the differential $q-$forms.

Here is the Cech-De Rham complex as commutative diagram.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
C^0(\mathscr{U},\Omega^4) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^4) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^4) \arrow[u, "d"] \arrow[r, "\delta"] & C^3(\mathscr{U},\Omega^4) \arrow[u, "d"] \arrow[r] & \cdots \\
C^0(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r, "\delta"] & C^3(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r] & \cdots \\
C^0(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^3(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r] & \cdots \\
C^0(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^3(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r] & \cdots \\
C^0(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^3(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r] & \cdots
\end{tikzcd}
\tag{8.7}
$$

The reader can easily check that Cech-De Rham complex is a double complex fullfilling all requirement of Definition 8.3.1. In particular $d\delta = \delta d$.

In the following Proposition 8.8 we prove that it has exact rows, giving rise to an augmented double complex with, as extra column (the differential complex $(A^\bullet, d)$ of Definition 8.3.6) the De Rham complex $(\Omega^\bullet(M), d)$, and as maps $r\colon \Omega^q(M) \to C^0(\mathscr{U},\Omega^q) = \prod_\alpha \Omega^q(U_\alpha)$ the maps induced by the restrictions, pull-backs for the inclusions $U_\alpha \subset M$.

Proposition 8.4.5 The diagram

$$
\begin{tikzcd}
& \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^3(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^3) \arrow[u, "d"] \arrow[r] & \cdots \\
0 \arrow[r] & \Omega^2(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^2) \arrow[u, "d"] \arrow[r] & \cdots \\
0 \arrow[r] & \Omega^1(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^1) \arrow[u, "d"] \arrow[r] & \cdots \\
0 \arrow[r] & \Omega^0(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\mathscr{U},\Omega^0) \arrow[u, "d"] \arrow[r] & \cdots \\
& 0 \arrow[u] & & & &
\end{tikzcd}
\tag{8.8}
$$

is an augmented double complex with exact rows.

Proof. First note that, if $\mathscr{U}$ is a covering made by exactly two open sets $\{U_0, U_1\}$, the statement follows by Theorem 7.3.2 setting $U := U_0$, $V := U_1$.

The proof follows indeed exactly the same lines of the proof of Theorem 7.3.2.

First of all r is injective. Indeed if a differential form $\omega \in \Omega^q(M)$ has $r\omega = 0$ it means that its restriction to every open subset U_α vanishes, and therefore ω_p vanishes for all $p \in \bigcup U_\alpha$. So $\omega = 0$. This shows the exactness at $\Omega^q(M)$.

$\delta \circ r = 0$. Indeed for every differential form ω, the component $dr\omega$ on $\Omega^\bullet(U_{\alpha\beta})$ is $\left(\omega_{|U_\alpha}\right)_{|U_\beta} - \left(\omega_{|U_\beta}\right)_{|U_\alpha} = \omega_{|U_{\alpha\beta}} - \omega_{|U_{\alpha\beta}} = 0$.

$\ker\delta = \operatorname{Im} r$. Indeed if (ω_α) is in $\ker\delta$, then $\forall(\alpha,\beta)$ $(\omega_\alpha)_{U_\beta} = (\omega_\beta)_{U_\alpha}$. So there exists $\omega \in \Omega^p(M)$ such that $\forall\alpha$, $\forall p \in U_\alpha$, $\omega_p = (\omega_\alpha)_p$: $r\omega = (\omega_\alpha)$. This shows the exactness at $C^0(\mathscr{U},\Omega^q)$.

It remains to prove the exactness at $C^p(\mathscr{U},\Omega^q)$, $p \geq 1$. Since $\delta \circ \delta = 0$, we are left with the proof that for all $\tau \in C^p(\mathscr{U},\Omega^q)$, $p \geq 1$, with $\delta\tau = 0$, there is $\sigma \in C^{p-1}(\mathscr{U},\Omega^q)$ such that $\delta\sigma = \tau$.

This can be proved by constructing explicitely σ by τ using a partition of unity subordinate to $\mathscr{U}$ as in the proof of Theorem 7.3.2. We leave the details to the reader. ■

This in particular implies, by Theorem 8.3.8, that the De Rham cohomology equals the cohomology of the double complex (8.7).

If also the columns of (8.7) were exact, then we would obtain a doubly augmented double complex with exact rows and columns as (8.6) with a new row at the bottom of (8.8); by Theorem 8.3.12 the cohomology of the new row would be isomorphic to the De Rham cohomology of M. Unfortunately, this is not always true.

Our candidate new row is the Cech complex of the presheaf of the locally constant functions. Indeed, since smooth functions are closed if and only if locally constant, the kernel of the map $d\colon C^0(\mathscr{U},\Omega^0) \to C^0(\mathscr{U},\Omega^1)$ equals $C^0(\mathscr{U},\mathbb{R})$.

Adding it we get the following commutative diagram.

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \Omega^3(M) & \xrightarrow{r} & C^0(\mathscr{U},\Omega^3) & \xrightarrow{\delta} & C^1(\mathscr{U},\Omega^3) & \xrightarrow{\delta} & C^2(\mathscr{U},\Omega^3) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \Omega^2(M) & \xrightarrow{r} & C^0(\mathscr{U},\Omega^2) & \xrightarrow{\delta} & C^1(\mathscr{U},\Omega^2) & \xrightarrow{\delta} & C^2(\mathscr{U},\Omega^2) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \Omega^1(M) & \xrightarrow{r} & C^0(\mathscr{U},\Omega^1) & \xrightarrow{\delta} & C^1(\mathscr{U},\Omega^1) & \xrightarrow{\delta} & C^2(\mathscr{U},\Omega^1) & \longrightarrow & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \Omega^0(M) & \xrightarrow{r} & C^0(\mathscr{U},\Omega^0) & \xrightarrow{\delta} & C^1(\mathscr{U},\Omega^0) & \xrightarrow{\delta} & C^2(\mathscr{U},\Omega^0) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow d & & \uparrow d & & \uparrow d & & \\
 & & 0 & \longrightarrow & C^0(\mathscr{U},\mathbb{R}) & \xrightarrow{\delta} & C^1(\mathscr{U},\mathbb{R}) & \xrightarrow{\delta} & C^2(\mathscr{U},\mathbb{R}) & \longrightarrow & \cdots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array} \tag{8.9}$$

If all columns of (8.9) except the first (the De Rham complex) are exact, then (8.9) is a doubly augmented double complex with exact rows and columns.

Those columns are exact if and only if the De Rham cohomology of all the open sets $U_{\alpha_0\cdots\alpha_p}$ is concentrated in degree zero. In other words, if and only if $\forall q > 0$, $\forall p$, $\forall \alpha_0 \lneqq \alpha_1 \lneqq \cdots \lneqq \alpha_p$, $h^q_{DR}(U_{\alpha_0\cdots\alpha_p}) = 0$. This motivates the following definition.

Definition 8.4.6 $\mathscr{U}$ is **acyclic** if $\forall q > 0$, $\forall p$, $\forall \alpha_0 \lneqq \cdots \lneqq \alpha_p$,

$$h^q_{DR}(U_{\alpha_0\cdots\alpha_p}) = 0.$$

Note that all good covers are acyclic, and therefore acyclic covers exist.

$\mathscr{U}$ is acyclic if and only if the columns of (8.9) are exact. Theorem 8.3.12 applies then, if $\mathscr{U}$ is acyclic, to (8.9), showing

Theorem 8.4.7 If $\mathscr{U}$ is an acyclic cover of M, then there is an isomorphism of graded vector spaces

$$H^\bullet_{DR}(M) \cong H^\bullet(\mathscr{U}, \mathbb{R}).$$

Note that the augmenting column (the De Rham complex) does not depend on the cover, and the augmenting row has nothing to do with differential forms, so the cohomology of the double complex (assuming the cover acyclic) does not depend on both things.

This shows that indeed the De Rham cohomology can be computed without using the differential forms (which is a rather surprising conclusion for these notes). If moreover $\mathscr{U}$ is finite (for example a finite good cover of a manifold of finite type), then the Cech complex of the constant presheaf $\mathbb{R}$ relative to $\mathscr{U}$ is finite dimensional and can be indeed explicitely written, giving a concrete method to compute the De Rham cohomology groups of M.

Indeed one can show (and we are not far from that) that the De Rham cohomology ring is a topological invariant.

Complement 8.4.1 Complete the proof of Proposition 8.4.5.

Exercise 8.4.1 Show that $\delta \circ \delta = 0$.

Exercise 8.4.2 Choose two points of $\mathbb{P}^1_{\mathbb{R}}$ and conside the cover $\mathscr{U}$ of $\mathbb{P}^1_{\mathbb{R}}$ formed by the two open subsets complements of those points. Set $G = \{\pm \mathrm{Id}_{\mathbb{R}}\}$ and set $\mathscr{G}$ for the constant presheaf with values in G.

Prove that $H^1(\mathscr{U}, \mathscr{G}) \cong G$.

Write a trivialization of the trivial line bundle and of the tautological bundle with structure group G, and show that their cocycles give different elements (so all elements) of $H^1(\mathscr{U}, \mathscr{G})$.

Exercise 8.4.3 Consider a cover $\mathscr{U}$ of S^1 given by two open subsets that are both the complement of a point. So $\mathscr{U} = \{U_0, U_1\}$, with $U_j = S^1 \setminus p_j$, $p_0 \neq p_1$.

1. Determine if $\mathscr{U}$ is acyclic.
2. Compute the dimensions of all the graded pieces $C^q(\mathscr{U}, \mathbb{R})$ of $C^\bullet(\mathscr{U}, \mathbb{R})$.
3. Write explicitly the differential δ of $C^\bullet(\mathscr{U}, \mathbb{R})$.
4. Compute the cohomology groups of the Cech complex $C^\bullet(\mathscr{U}, \mathbb{R})$ and compare the result with the Hilbert function of the De Rham cohomology of S^1.

Exercise 8.4.4 Construct an acyclic cover $\mathscr{U}$ of S^1 made by three connected open subsets such that each open subset intersects exactly two other open subsets and the intersection of each three of them is empty.

Then

1. Determine if $\mathscr{U}$ is acyclic.
2. Compute the dimensions of all the graded pieces $C^q(\mathscr{U},\mathbb{R})$ of $C^\bullet(\mathscr{U},\mathbb{R})$.
3. Write explicitly the differential δ of $C^\bullet(\mathscr{U},\mathbb{R})$.
4. Compute the cohomology groups of the Cech complex $C^\bullet(\mathscr{U},\mathbb{R})$ and compare the result with the Hilbert function of the De Rham cohomology of S^1.

Exercise 8.4.5 Write $\forall n \geq 4$ an acyclic cover of S^1 made by n connected open subsets, such that each open subset intersects exactly two other open subsets and the intersection of each three of them is empty.

Then

1. compute the dimensions of all the graded pieces $C^q(\mathscr{U},\mathbb{R})$ of $C^\bullet(\mathscr{U},\mathbb{R})$;
2. deduce from it the Euler characteristic of S^1.

Exercise 8.4.6 Consider a cover $\mathscr{U}$ of S^2 given by two open subsets that are both the complement of a point. So $\mathscr{U} = \{U_0, U_1\}$, with $U_j = S^2 \setminus p_j$, $p_0 \neq p_1$.

1. Determine if $\mathscr{U}$ is acyclic.
2. Compute the dimensions of all the graded pieces $C^q(\mathscr{U},\mathbb{R})$ of $C^\bullet(\mathscr{U},\mathbb{R})$.
3. Write explicitly the differential δ of $C^\bullet(\mathscr{U},\mathbb{R})$.
4. Compute the cohomology groups of the Cech complex $C^\bullet(\mathscr{U},\mathbb{R})$ and compare the result with the Hilbert function of the De Rham cohomology of S^2.

Exercise 8.4.7 Consider a homeomorphism of the sphere S^2 onto a tetrahedron, and let $\mathscr{U} = \{U_0, U_1, U_2, U_3\}$ be an acyclic cover such that each U_i is a small neighbourhood of the preimage of a face of the tetrahedron. Here by small neighbourhood we mean the set of points with distance $< \varepsilon$ (in the metric induced by $\mathbb{R}^3$) for some suitably small $\varepsilon > 0$.

1. Compute the dimensions of all the graded pieces $C^q(\mathscr{U},\mathbb{R})$ of $C^\bullet(\mathscr{U},\mathbb{R})$.
2. Deduce from it the Euler characteristic of S^2.
3. Compute the differential of $C^\bullet(\mathscr{U},\mathbb{R})$.
4. Compute the cohomology groups of the Cech complex $C^\bullet(\mathscr{U},\mathbb{R})$.

Exercise 8.4.8 Let S be a compact manifold of dimension 2, and assume that S is homeomorphic to the external surface of a (possibly not convex) polyhedron.

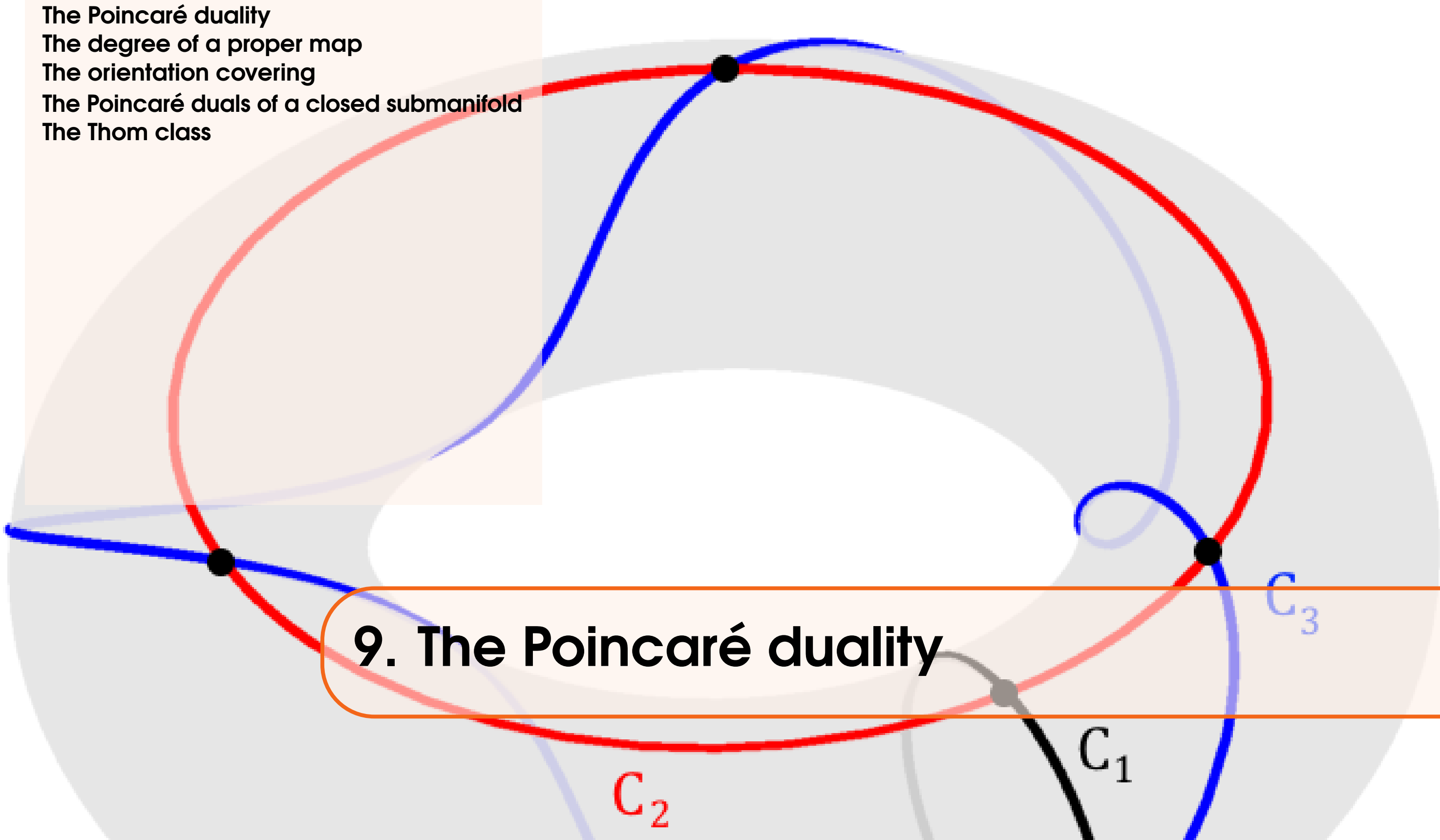
So S is topologically union of f polygons (possibly with different number of sides), which we call *faces*.

We assume that every vertex belongs to exactly three faces, and that the intersection of two polygons is either empty or an edge of both. Let e be the total number of edges and let v be the number of vertices.

Prove that the Euler characteristic $e(S)$ of S equals

$$f - e + v.$$

Exercise 8.4.9 Prove the same statement as in Exercise 8.4.8 without any assumption on the number of faces through any vertex, so allowing four or more faces through the same vertex.



9. The Poincaré duality

9.1 The Poincaré duality

This chapter is devoted to the Poincaré duality and some of its applications.

The Poincaré duality applies only to manifolds M that are orientable and without boundary, since it heavily uses $\int_M$ as element of the vector space dual of $H_c^{\dim M}(M)$.

This includes several interesting cases, for example all manifolds with a complex structure *i.e.* the underlying real manifold of any complex manifold.

Theorem 9.1.1 — Poincaré duality. Let M be an oriented manifold with $\partial M = \emptyset$. Then there exists, $\forall q$, isomorphisms

$$P_M \colon H^q_{DR}(M) \to \left(H_c^{n-q}(M)\right)^*$$

such that for all pairs of closed forms $\omega \in \Omega^q(M)$ and $\eta \in \Omega_c^{n-q}(M)$

$$P_M([\omega])([\eta]) = \int_M \omega \wedge \eta.$$

Proof. We first show that P_M is well defined. In other words, we show that $\int_M \omega \wedge \eta$ depends only on the classes $[\omega] \in H^q(M)$ and $[\eta] \in H_c^{n-q}(M)$.

Indeed, chosen forms $\overline{\omega} \in \Omega^{q-1}(M)$, $\overline{\eta} \in \Omega_c^{n-q-1}(M)$, by Stokes' Theorem 6.2.9, since by assumption $\partial M = \emptyset$ and $d\omega = d\eta = 0$

$$\begin{aligned}
\int_M (\omega + d\overline{\omega}) \wedge (\eta + d\overline{\eta}) &= \int_M \omega \wedge \eta + \int_M d\overline{\omega} \wedge \eta + \int_M (\omega + d\overline{\omega}) \wedge d\overline{\eta} \\
&= \int_M \omega \wedge \eta + \int_M d(\overline{\omega} \wedge \eta) \pm \int_M d((\omega + d\overline{\omega}) \wedge \overline{\eta}) \\
&= \int_M \omega \wedge \eta + \int_{\partial M} (\overline{\omega} \wedge \eta) \pm \int_{\partial M} ((\omega + d\overline{\omega}) \wedge \overline{\eta}) \\
&= \int_M \omega \wedge \eta.
\end{aligned}$$

Therefore the map P_M is well defined.

We are left with the proof that all maps P_M, obviously linear, are isomorphisms.

By sake of simplicity[1], we prove it only for manifolds of finite type, so we take a finite good cover $\mathfrak{U} = \{U_1, \dots, U_k\}$ and argue by induction on k.

If $k = 1$ then M is diffeomorphic to $\mathbb{R}^n$, whose De Rham and compact support cohomology groups we know by the Poincaré lemmas: Corollaries 7.4.3 and 7.5.2.

Then for all $q \neq 0$ both the domain and the codomain of $P_{\mathbb{R}^n}$ have dimension zero, and therefore $P_{\mathbb{R}^n}$ is the unique map amon them, an isomorphism.

In the case $q = 0$, $P_{\mathbb{R}^n}$ is a map among two spaces of dimension 1, and therefore either it is the zero map or it is an isomorphism. If it were the zero map, then $P_{\mathbb{R}^n}([1]) = 0 \in H^n_c(\mathbb{R}^n)^*$, and so $\forall \eta \in \Omega^n_c(\mathbb{R}^n) \int_{\mathbb{R}^n} \eta = 0$, which is obviously[2] false.

Therefore $P_{\mathbb{R}^n}$ is an isomorphism for all q, and the starting step of the induction is proved.

By induction, arguing as in the proof of Proposition 8.1.5 we find two open subsets U and V of M such that $M = U \cup V$ and the statement holds for U, V and $U \cap V$; all maps P_U, P_V and $P_{U\cap V}$ are isomorphisms.

We consider the diagrams

$$\begin{tikzcd}
H^{q-1}(U) \oplus H^{q-1}(V) \arrow[r, "\pm(P_U \oplus P_V)"] \arrow[d] & H^{n-q+1}_c(U)^* \oplus H^{n-q+1}_c(V)^* \arrow[d] \\
H^{q-1}(U \cap V) \arrow[r, "\pm P_{U\cap V}"] \arrow[d] & H^{n-q+1}_c(U \cap V)^* \arrow[d] \\
H^q(M) \arrow[r, "P_M"] \arrow[d] & H^{n-q}_c(M)^* \arrow[d] \\
H^q(U) \oplus H^q(V) \arrow[r, "\pm(P_U \oplus P_V)"] \arrow[d] & H^{n-q}_c(U)^* \oplus H^{n-q}_c(V)^* \arrow[d] \\
H^q(U \cap V) \arrow[r, "\pm P_{U\cap V}"] & H^{n-q}_c(U \cap V)^*
\end{tikzcd} \tag{9.1}$$

where

- the left column is the long cohomology exact sequence associated to the Mayer-Vietoris exact sequence corresponding to the decomposition of M as union of U and V;
- the right column is the dual of the analogous exact sequence for the cohomology with compact support.

Note that we are indeed considering 16 different diagrams, depending on the choice of 4 signs.

Since the dual of an exact sequence is exact (Exercise 7.2.2), both columns of the diagram (9.1) are exact sequences. By the inductive assumption all maps $\pm(P_U \oplus P_V)$ and $\pm P_{U\cap V}$ are isomorphisms. Therefore, if there is a choice of the signs $\pm$ such that the diagram (9.1) commutes, the Five Lemma 8.2.2 implies that P_M is an isomorphism, concluding our proof.

We complete then the proof by proving that there is a choice of the signs $\pm$ in the diagram (9.1) making it commutative.

[1] The Poincaré duality holds indeed also for manifolds not of finite type. Its proof in the general case follows the same idea, and uses transfinite induction.

[2] Take $\eta = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ for any $f \in C^\infty_c(\mathbb{R}^n)$, $f \geq 0$, $f \neq 0$.

We have to check the commutativity of four squares; the one at the bottom is

$$\begin{array}{ccc}
H^q(U)\oplus H^q(V) & \xrightarrow{\pm(P_U\oplus P_V)} & H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^* \\
\downarrow & & \downarrow \\
H^q(U\cap V) & \xrightarrow{\pm P_{U\cap V}} & H_c^{n-q}(U\cap V)^*
\end{array}$$

We check it by computing the two images of a general element $([\omega_1],[\omega_2])\in H^q(U)\oplus H^q(V)$ in $H_c^{n-q}(U\cap V)^*$: the one from the "top" way (through $H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^*$) and the one from the "bottom" way (through $H^q(U\cap V)$).

Top:

$$\begin{aligned}
([\omega_1],[\omega_2]) &\mapsto \pm\left(([\eta_1],[\eta_2])\mapsto \int_U \omega_1\wedge\eta_1+\int_V \omega_2\wedge\eta_2\right)\\
&\mapsto \pm\left([\eta]\mapsto \int_{U\cap V}(\omega_2-\omega_1)\wedge\eta\right)
\end{aligned}$$

Bottom:

$$\begin{aligned}
([\omega_1],[\omega_2]) &\mapsto ([(\omega_2)_{|U\cap V}-(\omega_1)_{|U\cap V}])\\
&\mapsto \pm\left([\eta]\mapsto \int_{U\cap V}(\omega_2-\omega_1)\wedge\eta\right)
\end{aligned}$$

Then the bottom square of diagram (9.1) commutes for a suitable choice of the signs. The same proof shows that the same holds also for the top square.

We study the commutativity of the square

$$\begin{array}{ccc}
H^q(M) & \xrightarrow{P_M} & H_c^{n-q}(M)^* \\
\downarrow & & \downarrow \\
H^q(U)\oplus H^q(V) & \xrightarrow{\pm(P_U\oplus P_V)} & H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^*
\end{array}$$

in a similar way.

Top:

$$\begin{aligned}
[\omega] &\mapsto \left([\eta]\mapsto \int_M \omega\wedge\eta\right)\\
&\mapsto \left(([\eta_1],[\eta_2])\mapsto \int_M \omega\wedge(j_M^U\eta_1+j_M^V\eta_2)\right)
\end{aligned}$$

Bottom:

$$\begin{aligned}
[\omega] &\mapsto ([\omega_{|U}],[\omega_{|V}])\\
&\mapsto \pm\left(([\eta_1],[\eta_2])\mapsto \int_U \omega\wedge\eta_1+\int_V \omega\wedge\eta_2\right)
\end{aligned}$$

Since clearly $\int_M \omega \wedge (j_M^U \eta_1 + j_M^V \eta_2) = \int_U \omega \wedge \eta_1 + \int_V \omega \wedge \eta_2$ also this square commutes for a suitable choice of the signs.

The last square is

$$\begin{tikzcd} H^{q-1}(U \cap V) \arrow[r, "\pm P_{U\cap V}"] \arrow[d] & H_c^{n-q+1}(U \cap V)^* \arrow[d] \\ H^q(M) \arrow[r, "P_M"] & H_c^{n-q}(M)^* \end{tikzcd}$$

Here we need the coboundary maps of the long cohomology exact sequence induced by the short exact sequences of Mayer-Vietoris, and then the function f_V in the proof of Theorem 7.3.2.

Bottom:

$$\begin{aligned} ([\omega]) &\mapsto [j_M^{U\cap V} d(-f_V \omega)] = [j_M^{U\cap V}(-df_V \wedge \omega)] \\ &\mapsto \pm \left([\eta] \mapsto -\int_{U\cap V} df_V \wedge \omega \wedge \eta\right) \end{aligned}$$

Top:

$$\begin{aligned} [\omega] &\mapsto \pm \left([\eta] \mapsto \int_{U\cap V} \omega \wedge \eta\right) \\ &\mapsto \pm \left([\eta] \mapsto \int_{U\cap V} \omega \wedge d(f_V \eta)\right) \end{aligned}$$

and the commutativity up to a sign follows because η is closed, which implies $d(f_V \eta) = df_V \wedge \eta$. ■

The first simple consequence of the Poincaré duality is the following

Corollary 9.1.2 Let M be a connected orientable manifold without boundary of dimension n.

Then the map $\int_M : H_c^n(M) \to \mathbb{R}$ is an isomorphism.

In particular $h_c^n(M) = 1$. Moreover a form ω is in the image $d\Omega_c^{n-1}$ if and only if $\int_M \omega = 0$.

Proof. Since we assumed M connected, by definition $H^0(M) \cong \mathbb{R}$ is the space of the constant functions. Theorem 9.1.1 implies then that $H_c^n(M)^*$, and therefore also $H_c^n(M)$, has dimension 1.

More precisely, the isomorphism in Theorem 9.1.1 for $q = 0$ is the map

$$H^0(M) \ni c \mapsto \left([\eta] \mapsto c \int_M \eta\right) \in H_c^n(M)^*.$$

This gives for each c a different (as P_M is an isomorphism) linear map among two vector spaces of dimension 1, $H_c^n(M)$ and $\mathbb{R}$. Since for $c = 0$ it is the zero map, for $c = 1$ it is a different map, and a linear map among vector spaces of dimension 1 is either zero or an isomorphism. ■

Exercise 9.1.1 Compute the De Rham cohomology groups of a torus with g holes (the Riemann surface of genus g).

Exercise 9.1.2 Compute the De Rham cohomology groups of a torus with g holes minus n points.

Exercise 9.1.3 Compute the De Rham cohomology groups of a torus with g holes minus n small open[a] discs pairwise disjoint.

[a]Be careful, this manifold has a boundary!

Exercise 9.1.4 Prove that every good cover of an orientable manifold of dimension n has cardinality at least $n+1$.

9.2 The degree of a proper map

Corollary 9.1.2 gives an interesting interpretation of the number $h_c^{\dim M}(M)$ for a manifold without boundary: it is the number of the connected components of M that are orientable.

The second part of the statement, claiming that two forms represent the same class if and only if they have the same integral on each orientable component, leads naturally to the definition of the degree of a proper map.

Let $F\colon M \to N$ be a smooth proper map among two connected oriented (possibly not compact) manifolds without boundary of the same dimension. Then the pull-back induces a map $F^*\colon H_c^n(N) \to H_c^n(M)$. Corollary 9.1.2 shows that both spaces have dimension 1, and more precisely the linear maps $\int_M\colon H_c^n(M) \to \mathbb{R}$ and $\int_N\colon H_c^n(N) \to \mathbb{R}$ are isomorphisms. Composing F^* with these isomorphism, we get a linear map $\mathbb{R} \to \mathbb{R}$ which is then the multiplication by a constant, the degree of F.

In other words:

Definition 9.2.1 Let M, N be oriented manifolds without boundary of the same dimension n, N connected, and let $F\colon M \to N$ be a smooth proper map.

Choose $\omega \in \Omega_c^n(N)$ with $\int_N \omega = 1$. We define the **degree** of F as

$$\deg F := \int_M F^*\omega.$$

Note that we have required only the connectedness of N. This is necessary in the definition to ensure that the definition of the degree of F does not depend on the choice of ω: if N is connected the forms $\omega \in \Omega_c^n(N)$ with $\int_N \omega = 1$ form a cohomology class; then their pull-backs belong to the same cohomology class and therefore they all have the same integral.

In contrast the connectedness of M is not necessary.

We deduce immediately by the definition

$$\forall \omega \in \Omega_c^n(N) \qquad \int_M F^*\omega = (\deg F)\int_N \omega$$

R The degree is a **multiplicative** function. Indeed, if F, G are proper maps such that the codomain of G equals the domain of G, then $F \circ G$ exists and directly by the definition we deduce

$$\deg(F \circ G) = (\deg F)(\deg G)$$

R If we change the orientation of M, the degree of F change sign.
If we change the orientation of N, the degree of F change sign.
If we change the orientation of both N and M the degree does not change.
In particular, if we consider a proper map from an orientable manifold to itself, its degree does not depend on the choice[3] of its orientation.

R If F is a diffeomorphism that preserves the orientation, then $\deg F = 1$.
If F is a diffeomorphism that reverses the orientation, then $\deg F = -1$.

Proposition 9.2.2 Let M,N be compact oriented manifolds without boundary of the same dimension. If $F,G\colon M \to N$ are smoothly homotopic maps, then $\deg F = \deg G$.

Proof. This follows easily by Corollary 7.4.5. ■

It follows the following Theorem, known as "Hairy Ball Theorem", since it determines, "roughly speaking", if one can comb a hairy ball such that each hair lies flat.

Theorem 9.2.3 — The Hairy Ball Theorem. A sphere S^n admits a smooth vector field without any zero if and only if its dimension n is odd.

Proof. The readed should have already combed flat all odd-dimensional spheres solving Exercise 3.3.4. We only need then to show that if a sphere can be combed flat, its dimension is odd.

Consider a sphere $S^n := \{x = (x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} | \sum x_i^2 = 1\}$ and a vector field v on it. We can write it as

$$v(x) = \sum v_i(x)\frac{\partial}{\partial x_i}$$

where "v is orthogonal to x": $\sum x_i v_i(x) = 0$ for all x.

If v never vanish (combing the sphere), we can divide it by $\sqrt{\sum v_i^2}$. So we can assume without loss of generalities that $\sum v_i^2 = 1$.

Set $\mathfrak{v} := (v_1,\dots,v_{n+1})$ for the point in $\mathbb{R}^{n+1}$ whose coordinates correspond to the components of v. Note that $\mathfrak{v} \in S^n$.

Consider the following induced function on $S^n \times \mathbb{R}$

$$H(x,t) = \cos(\pi t)x + \sin(\pi t)\mathfrak{v}.$$

Note that $H(x,0) = x$, $H(x,1) = -x$. Since x and $\mathfrak{v}$ are orthogonal and of norm 1, $H(x,t)$ belongs to S^n for all x and t, so H can be seen as a smooth homotopy

$$H\colon S^n \times \mathbb{R} \to S^n.$$

It follows that the antipodal map $A\colon S^n \to S^n$ is smoothly homotopic to the identity.

Then by Proposition 9.2.2 $\deg A = 1$, so A preserves the orientation. It follows n odd by the forthcoming Lemma 9.2.4. ■

[3] if we agree to use the same orientation as orientation of the domain and of the codomain

Lemma 9.2.4 Let $A\colon S^n \to S^n$ be the antipodal map $A(p) = -p$. Then
if n is odd then A is a diffeomorphism that preserves the orientation;
if n is even then A is a diffeomorphism that reverses the orientation.

Proof. Since A is a diffeomorphism and S^n is connected, then A either preserves or reverses the orientation.

Notice that A is the restriction to the boundary of the diffeomorphism $B\colon B^{n+1} \to B^{n+1}$ analogously defined by $B(p) = -p$. By definition of orientation induced on the boundary A preserves the orientation if and only if B does. It is enough then if we prove the statement for B instead of A.

B has a fixed point, the origin O. Since B^{n+1} is connected, B preserves the orientation if and only if $dB_O \in \mathrm{Aut}(T_O B^{n+1})$ preserves the orientation as well, *i.e.* if and only if its Jacobi matrix has positive determinant.

The statement follows then by $dB_O = -\mathrm{Id}_{T_O\mathbb{R}^{n+1}} \Rightarrow \deg B = (-1)^{n+1}$. ■

In the next proposition we show that the degree has a geometrical interpretation that makes it usually easy to compute; roughly speaking, it counts (in some sense) the cardinality of the general fibre.

Recall that by Sard's Lemma a smooth map among manifolds of the same dimension has always at least a regular value.

Proposition 9.2.5 Let $F: M \to N$ be a smooth proper map among oriented manifolds without boundary of the same dimension, and let $q \in N$ be a regular value of F. Then

$$\deg F = \sum_{p \in F^{-1}(q)} \varepsilon(p)$$

where $\varepsilon(p) = 1$ if F preserves the orientation in a neighbourhood of p, $\varepsilon(p) = -1$ if F reverses the orientation in a neighbourhood of p. In particular $\deg F \in \mathbb{Z}$.

Proof. Since q is regular, then $\forall p \in F^{-1}(q)$, dF_p is invertible and then $\varepsilon(p)$ is well defined. It follows that $F^{-1}(q)$ is discrete: by the properness of F, $F^{-1}(q)$ is also compact, and therefore finite. Then $\sum_{p\in F^{-1}(q)} \varepsilon(p)$ is a finite sum of 1's and -1's, an integer.

We write $F^{-1}(q) = \{p_1, \dots, p_k\}$. By the local diffeomorphism theorem, there are open neighborhoods U_i of p_i such that $F_{|U_i}$ is a diffeomorphism onto a neighborhood of q. By restricting the U_i we may assume that they are pairwise disjoint and that $\forall i\, F(U_i) = V$ for a fixed open neighborhood of q. We may also assume that V is contained in a chart (V', ψ) inducing local coordinates $x_1, \dots, x_n$.

Finally, we may also assume, up to shrinking V, that $F^{-1}(V) = \bigcup U_i$. Indeed, if this were false, there would be a sequence $\{z_i\}$ in $M \setminus \bigcup U_i$ such that $\{F(z_i)\}$ converges to q. Then, since $\{f(z_i)\} \cup \{q\}$ is compact, its preimage is a compact containing the sequence $\{z_i\}$. Therefore, up to passing to a subsequence, $\{z_i\}$ converges to some $z \in M$ and by continuity of F, $F(z) = q$, so z is one of the p_i. In particular $\{z_i\}$ intersects U_i, a contradiction.

We choose a form ω with $\int_N \omega = 1$ and $\operatorname{supp}\omega \subset V$. This can be done for example by picking any nonnegative function $0 \neq f \in C^\infty_c(N)$ with $\operatorname{supp} f \subset V$. Then $\int_N f dx_1 \wedge \cdots \wedge x_n \neq 0$ and we can define the form

$$\omega := j_N^V \left(\frac{f dx_1 \wedge \cdots \wedge x_n}{\int_N f dx_1 \wedge \cdots \wedge x_n} \right).$$

Then $\int_N \omega = \int_V \omega = 1$. Note that, since $\operatorname{supp}\omega \subset V$, then $\operatorname{supp} F^*\omega \subset \bigcup_i U_i$.

By Definition 9.2.1 and Complement 6.2.1 of Chapter 2

$$\deg F = \int_M F^*\omega = \sum_1^k \int_{U_i} F^*\omega = \sum_1^k \varepsilon(p_i) \int_V \omega = \sum_1^k \varepsilon(p_i). \qquad \blacksquare$$

There are several easy consequences of this results. First of all, recalling that a point of the codomain of a map that is not in the image is always a regular value, we see[4] that every map that is not surjective has degree zero. Conversely

Corollary 9.2.6 Let $F: M \to N$ be a smooth proper map among oriented manifolds without boundary of the same dimension. If $\deg F \neq 0$ then F is surjective.

If F is an holomorphic proper map among complex manifolds of the same dimension, then we can consider F as a smooth map among real oriented manifolds without boundary. In this case

Corollary 9.2.7 Let $F: M \to N$ be an holomorphic proper map among complex manifolds of the same dimension, and let $q \in N$ be a regular value of F. Then

$$\deg F = \#F^{-1}(q) \geq 0.$$

Proof. By Theorem 6.1.7 and its proof in this case $\varepsilon(p)$ equals always 1, and then the statement follows immediately from Proposition 9.2.5. ■

In particular the cardinality of the fibre of a regular value of a holomorphic proper map does not depend on the choice of the regular value. This is not true in general for a smooth map among real oriented manifolds where the same argument just proves that the parity of the cardinality of the fibre is constant. In fact, the map $f: \mathbb{R} \to \mathbb{R}$ given by $f(t) = t^2$ is not surjective, so it has degree 0, although the preimage of each positive regular value has cardinality 2.

Corollary 9.2.8 Let $F: M \to N$ be a smooth proper map among oriented manifolds without boundary of the same dimension.

Let $q \in N$ be a regular value of F. Then $\#F^{-1}(q) - |\deg F| \in 2\mathbb{N}$.

For example, the map $F: \mathbb{R} \to \mathbb{R}$ given by $F(x) = x^3 - x$ has two critical values $\pm\frac{\sqrt{3}}{3}$, dividing $\mathrm{Reg}(F)$ in three connected components.

A straightforward explicit computation shows that the preimage of a regular value q has cardinality 1 if q belongs to one of the two unbounded components, so if $|q| > \frac{\sqrt{3}}{3}$, and then that $\deg F = 1$. Still if $|q| < \frac{\sqrt{3}}{3}$, $\#F^{-1}(q) = 3$.

Exercise 9.2.1 Show that a real projective space can be combed flat if and only if its dimension is odd.

Exercise 9.2.2 Let $F: M \to N$ be a smooth map among compact oriented manifolds without boundary of the same dimension.

Show that if $\deg F \neq 0$ then the map $F^*: H^\bullet(N) \to H^\bullet(M)$ is injective.

[4] At a first glance, the proof of Proposition 9.2.5 requires that $F^{-1}(q)$ is not empty, so $k \neq 0$, but this is not true. For $k = 0$ the requirement $F_{|U_i}$ to be a diffeomorphism is an empty condition, so one may first take any open subset V of N containing q, for example $V = N$. Then the following argument, showing that, up to shrinking V, $F^{-1}(V) = \bigcup U_i$, shows in fact that one can choose V so that $F^{-1}(V)$ is empty. This implies $F^*\omega = 0$ when $\mathrm{supp}\,\omega \subset V$, so $\deg F = 0$.

Exercise 9.2.3 Show that if $F\colon M \to N$ is a holomorphic map among compact complex manifolds of dimension 1 and the genus of M is strictly smaller than the genus of N, then F is not surjective.

Note: with some standard complex analysis one can conclude that F is constant.

Exercise 9.2.4 Let P be a real polynomial of degree d, and consider it as smooth function $P\colon \mathbb{R} \to \mathbb{R}$.

1) Prove that P is proper if and only if $d > 0$.
2) Prove that if d is even, then the degree of P as smooth proper map is 0.
3) Prove that if d is odd, then the degree of P as smooth proper map is either 1 or -1.

Exercise 9.2.5 Let P be a complex polynomial of degree d, and consider it as holomorphic function $P\colon \mathbb{C} \to \mathbb{C}$.

1) Prove that P is proper if and only if $d > 0$.
2) Use Proposition 9.2.5 to prove that $\deg F \neq 0$.
3) Deduce that P is surjective (this is the fundamental theorem of algebra).
4) Prove that the degree of P as smooth proper map equals its degree as a polynomial, d.

9.3 The orientation covering

The Poincaré duality holds only for orientable manifolds, as its proof shows: we need to be able to integrate. One could think that this is only a technical problem: maybe there is a different duality which works more generally?

In this section we will see that the answer to this question is negative. Actually, this negative answer is very useful, since it produces a cohomological criterion for orientability.

We first need a couple of Lemmas.

Lemma 9.3.1 Let $A\colon M \to M$ be a smooth involution[a] without fixed points and consider the quotient $N := M/\sim$ by the equivalence relation generated by $p \sim A(p)$.

Then there is a unique differentiable structure on N such that the projection map $\pi\colon M \to N$ is a local diffeomorphism. Moreover

$$\pi^*\Omega^\bullet(N) = \{\omega \in \Omega^\bullet(M) | A^*\omega = \omega\} \tag{9.2}$$

[a] This means $A \circ A = \mathrm{Id}_M$.

The formula (9.2) says that the $A-$invariant forms on M are exactly the forms coming from N.

Proof. We first notice that, since $A^{-1} = A$, A is a diffeomorphism.

Since M is Hausdorff, for all p in M there exists two disjoint open subsets U_1 and U_2 containing respectively p and $A(p)$. Then $U := U_1 \cap A(U_2)$ is an open subset of M containing p such that $U \cap A(U) = \emptyset$. Shrinking U if necessary, we find a chart (U,φ) in p for M with $U \cap A(U) = \emptyset$.

We define the differentiable structure on N as follows: for each q in N choose p in M such that $\pi(p) = q$ and take a chart (U,φ) in p as above. Then π maps injectively U onto $V := \pi(U)$. Then we say that $(V, \varphi \circ \pi^{-1})$ is a chart for N. Varying q in M we obtain an atlas for N such that π is a local diffeomorphism. The uniqueness of the structure is obvious.

We still have to prove (9.2). One inclusion is easy: since $\pi \circ A = \pi$ then for every $\omega \in \Omega^\bullet(N)$, $A^*\pi^*\omega = (\pi \circ A)^*\omega = \pi^*\omega$, and then $\pi^*\omega$ is A-invariant.

For the other inclusion, take $\omega \in \Omega^\bullet(M)$ such that $A^*\omega = \omega$. For all point q in N take a chart (V,ψ) as above, so that $\pi^{-1}(V)$ is the disjoint union of an open subset U with $A(U)$. Since π defines a diffeomorphism among U and V we can pull-back $\omega_{|U}$ to a form on V using π^{-1}.

Notice that, since $A^*\omega = \omega$, the induced form on V does not change if we replace U with $A(U)$.

Now consider two open subsets V_1 and V_2 of N as above. The above construction gives two forms $\eta_j \in \Omega^\bullet(V_j)$ by setting $\eta_j := (\pi^{-1}_{|U_j})^*\omega_{|U_j}$. The reader can easily show that η_1 and η_2 coincide on the common domain $V_1 \cap V_2$. Therefore all the forms obtained in this way are restriction of the same global form $\eta \in \Omega^\bullet(N)$. The equality $\pi^*\eta = \omega$ holds since it holds on each open subset U as above by definition of η. ■

An interesting example of the above situation is given by the antipodal map of a sphere, in which case the quotient is a projective space.

We can now prove that the Poincaré duality fails for the real projective plane $\mathbb{P}^2_{\mathbb{R}}$.

Proposition 9.3.2 If n is even then $h^n_{DR}(\mathbb{P}^n_{\mathbb{R}}) = h^n_c(\mathbb{P}^n_{\mathbb{R}}) = 0$

Proof. Consider the natural projection map $\pi\colon S^n \to \mathbb{P}^n_{\mathbb{R}}$.

Pick any $n-$form $\omega \in \Omega^n(\mathbb{P}^n_{\mathbb{R}})$. Consider $\bar\omega := \pi^*\omega \in \Omega^n(S^n)$. By Lemma 9.3.1 $\bar\omega$ is *A-invariant*: $A^*\bar\omega = \bar\omega$.

Since A reverses the orientation (Lemma 9.2.4) then

$$\int_{S^n} \bar\omega = -\int_{S^n} A^*\bar\omega = -\int_{S^n} \bar\omega \Rightarrow \int_{S^n} \bar\omega = 0.$$

Then, by Corollary 9.1.2, $\bar\omega$ is exact. Pick then $\bar\eta'$ such that $d\bar\eta' = \bar\omega$, and define $\bar\eta := \frac{\bar\eta' + A^*\bar\eta'}{2}$ *averaging $\bar\eta'$ with respect to A.*

Notice that $A^*\bar\eta = \bar\eta$ so by Lemma 9.3.1 $\bar\eta = \pi^*\eta$ for some $\eta \in \Omega^\bullet(\mathbb{P}^n_{\mathbb{R}})$.

Since π is a local diffeomorphism by

$$d\bar\eta = d\left(\frac{\bar\eta' + A^*\bar\eta'}{2}\right) = \frac{d\bar\eta' + A^* d\bar\eta'}{2} = \frac{\bar\omega + A^*\bar\omega}{2} = \frac{\bar\omega + \bar\omega}{2} = \bar\omega$$

it follows $d\eta = \omega$ completing the proof. ■

Proposition 9.3.2 implies, since by Corollary 9.1.2 for every connected orientable manifold M of dimension n, $h^n_c(M) = 1$.

Corollary 9.3.3 Every real projective space of even dimension is not orientable.

The proof of Proposition 9.3.2 relies on the use of the Poincaré duality on an orientable manifold, S^n, strictly related (through the maps A and π) with the variety under investigation, $\mathbb{P}^n_{\mathbb{R}}$.

The argument fails for odd dimensional real projective spaces since the antipodal map of the corresponding sphere preserves the orientation. In fact odd dimensional real projective spaces are orentable, see Exercise 9.3.3.

We give now an analogous construction for every manifold, the orientation covering.

Definition 9.3.4 Let M be a connected manifold.

The **orientation covering** of M is defined, set-theoretically, as

$$\tilde M := \{(p,o) | p \in M,\ o \text{ is an orientation}^a \text{ of } T_pM\}.$$

Denote by $\pi\colon \tilde M \to M$ the projection $\pi(p,o) = p$. $\tilde M$ has a natural structure of (possibly disconnected) differentiable manifold making π a local diffeomorphism, as follows.

Let $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ be the maximal atlas of M. Every chart $(U_\alpha,\varphi_\alpha)$ gives local coordinates $x_1,\dots,x_n$; $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ determining, for each $p\in U_\alpha$ an orientation of $T_p(M)$. This gives a subset, say V_α, of $\tilde{M}$, such that π maps V_α bijectively onto U_α.

We give to $\tilde{M}$ the topology generated by the V_α and the differentiable structure obtained by restricting the "atlas" $\{(V_\alpha,\varphi_\alpha\circ\pi)\}_{\alpha\in I}$.

[a]We are using here Definition 6.1.1: an orientation of a vector space V is an equivalence class of bases of V.

R Notice that by definition the transition functions $(\varphi_\alpha\circ\pi)\circ(\varphi_\beta\circ\pi)^{-1}$ preserve the orientation, so $\tilde{M}$ is an oriented manifold!

R Since for every $p\in M$ T_pM admits exactly two orientations, we have a natural map

$$A\colon \tilde{M}\to\tilde{M}$$

defined by $A(p,o):=(p,\bar{o})$ where $\bar{o}$ is the opposite orientation $\bar{o}\neq o$.
A is a diffeomorphism that reverses the orientation with $A\circ A=\mathrm{Id}_{\tilde{M}}$, $\pi\circ A=\pi$.

R By the definition of the differentiable structure on $\tilde{M}$, π is smooth, and for all $(p,o)\in\tilde{M}$, $d\pi_{(p,o)}$ is an isomorphism.
Therefore π is a local diffeomorphism and then, by Theorem 2.4.4, an open map. Moreover π is obviously a proper map. Since M and $\tilde{M}$ are both locally compact Hausdorff spaces, then π is a closed map too.

Lemma 9.3.5 Let M be a connected manifold.
If M is not orientable then $\tilde{M}$ is connected.
If M is orientable then $\tilde{M}$ has two connected components, say $\tilde{M}_1$ and $\tilde{M}_2$, and $\forall i$, $\pi_{|\tilde{M}_i}\colon\tilde{M}_i\to M$ is a diffeomorphism. More precisely, if we fix an orientation on M, one of the diffeomorphisms $\pi_{|\tilde{M}_i}$ preserves the orientation whereas the other one reverses the orientation.

Proof. If M is orientable fix an orientation of M and denote as usual by $\bar{M}$ the same manifold with the opposite orientation. We have a natural map $F\colon M\coprod\bar{M}\to\tilde{M}$ mapping each point p in the disjoint union $M\coprod\bar{M}$ to the pair (p,o) where o is the orientation of p as point of M or $\bar{M}$. It is easy to show that F is a preserving orientation diffeomorphism. Then $\tilde{M}$ has two connected components, $F(M)$ and $F(\bar{M})$ and the rest of the statement for the orientable case follows easily.

We conclude the proof by showing that, if $\tilde{M}$ is disconnected, then M is orientable.

So assume $\tilde{M}$ disconnected. Then there exists an open and closed proper nonempty subset $\tilde{M}_1$ of $\tilde{M}$.

Since π is at the same time an open map and a closed map, $\pi(\tilde{M}_1)$ is an open and closed nonempty subset of M, so $\pi(\tilde{M}_1)=M$.

We recall that for every point $p\in M$ there are exactly two possible distinct orientations, so $\pi^{-1}(p)=\{(p,o),(p,\bar{o})\}$. It follows that if (p,o) does not belong to $\tilde{M}_1$ then $(p,\bar{o})$ will belong to it, since otherwise p would not belong to $\pi(\tilde{M}_1)$ contradicting $\pi(\tilde{M}_1)=M$. In other words $\tilde{M}=\tilde{M}_1\cup A(\tilde{M}_1)$: for any point q in the complementary subset $\tilde{M}\setminus\tilde{M}_1$, $A(q)\in\tilde{M}_1$.

Now we show that $\tilde{M}_1\cap A(\tilde{M}_1)$ is the empty set. Since $\tilde{M}_1$ is a proper open and closed subset and A is a diffeomorphism, $A(\tilde{M}_1)$ is a proper and closed subset too, and then $\tilde{M}_1\cap A(\tilde{M}_1)$ is a proper and closed subset of $\tilde{M}$ as well. If it were not empty, then by the argument above $\pi\left(\tilde{M}_1 \cap A(\tilde{M}_1)\right) = M$. However, since $\tilde{M}_1 \cap A(\tilde{M}_1)$ is $A-$invariant, that would imply $\tilde{M}_1 \cap A(\tilde{M}_1) = \tilde{M}$, a contradiction.

Finally consider the restriction of the map π to the open subset $\tilde{M}_1$. We have shown that it is a surjective local diffeomorphism from $\tilde{M}_1 \to M$. The fact that $\tilde{M}_1 \cap A(\tilde{M}_1)$ is the empty set implies that it is injective too. So M is diffeomorphic to an open subset of the oriented manifold $\tilde{M}$ and then it is orientable too. ■

It follows the following criterion for orientability.

Theorem 9.3.6 Let M be a connected manifold without boundary of dimension n. Then

$$\begin{cases} h^n_c(M) = 1 & \text{if M is orientable} \\ h^n_c(M) = 0 & \text{if M is not orientable} \end{cases}$$

Proof. When M is orientable, this is just Corollary 9.1.2.

Assume now M not orientable, and consider the reversing orientation diffeomorphism $A\colon \tilde{M} \to \tilde{M}$ defined already as $A(p,o) = (p,\bar{o})$. The proof now follows now exactly the strategy of the proof of Proposition 9.3.2 just by substituting S^n with the orientation covering $\tilde{M}$ of M. ■

Complement 9.3.1 Prove that the orientation covering of $\mathbb{P}^2_{\mathbb{R}}$ is diffeomorphic to S^2.

Exercise 9.3.1 Let $M_1,\dots,M_k$ be compact manifolds without boundary. Prove that $M_1 \times \cdots \times M_k$ is orientable if and only if all M_i are orientable.

Exercise 9.3.2 Compute the De Rham cohomology ring of the real projective plane.

Exercise 9.3.3 Show that all real projective spaces of odd dimension are orientable.

Hint: The Mayer-Vietoris exact sequence may be useful

Exercise 9.3.4 Compute the De Rham cohomology rings of all real projective spaces $\mathbb{P}^n_{\mathbb{R}} = S^n/x \sim -x$.

Exercise 9.3.5 Compute the De Rham cohomology ring of the Klein bottle $\mathbb{R}^2/\sim$ where the equivalence relation is given by $(x_0,y_0) \sim (x_1,y_1) \Leftrightarrow (x_0 - x_1, y_0 - (-1)^{x_0 - x_1} y_1) \in \mathbb{Z}^2$.

Exercise 9.3.6 Let $\pi\colon E \to B$ be a fibre bundle with fibre $\mathbb{P}^r_{\mathbb{R}}$ on a manifold B of finite type. Prove that if r is even then the map $\pi^*\colon H^\bullet_{DR}(B) \to H^\bullet_{DR}(E)$ is a ring isomorphism.

9.4 The Poincaré duals of a closed submanifold

Let M be an oriented manifold without boundary of dimension n and let S be a closed oriented submanifold without boundary of dimension k.

In other words S is an oriented manifold without boundary and the inclusion

$$i\colon S \hookrightarrow M$$

is an embedding whose image $i(S)$ is closed.

Then, $\forall \omega \in \Omega_c^k(M)$, the support of $\omega_{|S} := i^*\omega$ is compact. Integrating along S we get a linear application

$$[\omega] \mapsto \int_S \omega := \int_S \omega_{|S}$$

in $\Omega_c^k(M)^*$ that vanishes, by Stokes' Theorem 6.2.9, on $d\Omega_c^{k-1}(M)$.

So $\int_S$ defines an element of $H_c^k(M)^*$. Since by the Poincaré duality 9.1.1 $H_c^k(M)^* \cong H_{DR}^{n-k}(M)$ this associates to S a De Rham cohomology class on M.

Definition 9.4.1 The **Poincaré dual** or the **closed Poincaré dual** of S in M is the unique cohomology class $\eta_S' \in H_{DR}^{n-k}(M)$ representing $\int_S$.

In other words $\eta_S' \in H_{DR}^{n-k}(M)$ is the unique De Rham cohomology class such that for every compact support cohomology class $\omega \in H_c^k(M)$

$$\int_S \omega = \int_M \omega \wedge \eta_S'.$$

R Note we are considering the wedge product $\omega \wedge \eta_S'$ of two cohomology classes belonging to different cohomology theories, this at a first glance does not make much sense!

The meaning is the following.

We consider a representative of ω in $\Omega_c^k(M)$ and a representative of η_S' in $\Omega^{n-k}(M)$, their wedge product is a closed $n-$form whose support is compact, so we can integrate it. A standard argument shows that the integral does not depend on the choice of the two representatives.

The closed Poincaré dual behaves well under diffeomorphisms, as follows.

Proposition 9.4.2 Let M, S be oriented manifolds without boundary of respective dimension n, k, and let $i\colon S \hookrightarrow M$ be an embedding with closed image, and set η_S' for the closed Poincaré dual of S in M. Let $F\colon M \to M$ be a diffeomorphism.

If F preserves the orientation, then $\eta_S' = F^*\eta_{F(S)}'$, else $\eta_S' = -F^*\eta_{F(S)}'$.

Proof. By the characterizing property of the Poincaré dual, for all cohomology classes $\omega \in H_c^k(M)$,

$$\int_M \omega \wedge \eta_{F(S)}' = \int_{F(S)} \omega = \int_S (F \circ i)^*\omega = \int_S i^*F^*\omega = \int_M F^*\omega \wedge \eta_S'.$$

If F preserves the orientation the left hand term equals $\int_M F^*(\omega \wedge \eta_{F(S)}') = \int_M F^*\omega \wedge F^*\eta_{F(S)}'$. Since this holds for all ω, then $\eta_S' = F^*\eta_{F(S)}'$.

Similarly we obtain the equality $\eta_S' = -F^*\eta_{F(S)}'$ when F reverses the orientation. ■

Corollary 9.4.3 Let M be an oriented manifold without boundary, and let $F\colon M \to M$ be an orientation preserving diffeomorphism which is smoothly homotopic to the identity.

Let S be a closed oriented submanifold without boundary. Then S and $F(S)$ have the same closed Poincaré dual in M.

Proof. By assumption F preserves the orientation, so Proposition 9.4.2 gives $\eta_S' = F^*\eta_{F(S)}' \in H_{DR}^{n-k}(M)$. On the other hand, by Corollary 7.4.5, $F^*\colon H_{DR}^{n-k}(M) \to H_{DR}^{n-k}(M)$ is the identity map. So $\eta_{F(S)}' = \eta_S'$. ■

If we further assume S compact, we similarly also associate to S a De Rham cohomology class. We need however to assume that all De Rham cohomology group are finitely dimensional (as for manifolds of finite type) so that dualizing the Poincaré duality 9.1.1 we obtain the following

Theorem 9.4.4 Let M be an oriented manifold with $\partial M = \emptyset$ and assume moreover that $H^\bullet_{DR}(M)$ is finitely dimensional. Then there are isomorphisms

$$P'_M \colon H^q_c(M) \to \left(H^{n-q}_{DR}(M)\right)^*$$

defined by

$$\forall [\eta] \in H^q_c(M) \quad \forall [\omega] \in H^{n-q}_{DR}(M) \quad P'_M([\eta])([\omega]) = \int_M \omega \wedge \eta.$$

Proof. The defition of P'_M may be written simply as

$$P'_M([\eta])([\omega]) = P_M([\omega])([\eta]).$$

and therefore, since P_M is well defined, P'_M is well defined too.

Moreover

$$\begin{aligned}
[\eta] \in \ker P'_M &\Leftrightarrow \forall [\omega] \in H^{n-q}_{DR}(M) \quad \int_M \omega \wedge \eta = 0 \\
&\Leftrightarrow \forall [\omega] \in H^{n-q}_{DR}(M) \quad [\eta] \in \ker P_M([\omega]) \\
&\Leftrightarrow \forall \varphi \in (H^q_c(M))^* \quad [\eta] \in \ker \varphi \\
&\Leftrightarrow [\eta] = 0
\end{aligned}$$

So P'_M is injective. Since its domain and its codomain are by assumption finitely dimensional, and of the same dimension by Poincaré duality, P'_M is an isomorphism. ■

So, if all the De Rham cohomology groups of M are finitely dimensional, then we can exchange the role of the De Rham cohomology and of the compact support cohomology in the discussion above.

Definition 9.4.5 Let M be an oriented manifold of dimension n without boundary whose De Rham cohomology is finitely dimensional. Let S be a compact oriented manifold of dimension k without boundary embedded in M.

By Poincaré duality, there is a unique cohomology class $\eta \in H^{n-k}_c(M)$ representing $\int_S$; in other words there is a unique $\eta_S \in H^{n-k}_c(M)$ such that $\forall \omega \in H^k_{DR}(M)$

$$\int_S \omega = \int_M \omega \wedge \eta_S.$$

We will say that η_S is the **compact Poincaré dual** of S in M.

Of course, if M is compact, then closed and compact Poincaré duals coincide.

The proof of Proposition 9.4.2 gives in the case of compact Poincaré duals the following analogous statement.

Proposition 9.4.6 Let M be an oriented manifold without boundary of dimension n such that $H^\bullet_{DR}(M)$ is finitely dimensional, and let $F \colon M \to M$ be a diffeomorphism.

Let S be an oriented compact submanifold of dimension k. Set η_S for the compact Poincaré dual of S in M.

If F preserves the orientation, then the compact Poincaré dual of $F^{-1}(S)$ is $F^*\eta_S$.
If F reverses the orientation, then the compact Poincaré dual of $F^{-1}(S)$ is $-F^*\eta_S$.

We cannot prove an analogous of Corollary 9.4.3 for compact Poincaré duals, since its proof uses Corollary 7.4.5, which does not generalize to the compact support cohomology.

The compact Poincaré dual has the very useful property that we can shrink the support of it in arbitrarily small neighbourhoods of S in M.

Theorem 9.4.7 — Localization principle. Let M be an oriented manifold without boundary whose De Rham cohomology is finitely dimensional.

Let S be a compact oriented submanifold without boundary of M.

Let $W \subset M$ be an open subset containing S such that $H^\bullet_{DR}(W)$ is finitely dimensional.

Then there is a representative $\eta \in \Omega^\bullet_c(M)$ of the compact Poincaré dual η_S of S in M such that $\operatorname{supp}\eta \subset W$.

R The finite dimension of $H^\bullet_{DR}(W)$ is automatic for the tubular neighbourhoods in Theorem 3.3.5, since in that case W is diffeomorphic to a vector bundle over S and then its De Rham cohomology is isomorphic to the De Rham cohomology of S and then finitely dimensional by the compactness of S.

Proof. Consider S as compact submanifold of the manifold W. Then S has a compact Poincaré dual in $H^\bullet_c(W)$. Choose a representative $\tilde{\eta} \in \Omega^\bullet_c(W)$ of it.

Since $\tilde{\eta}$ has compact support, we can extend $\tilde{\eta}$ to a smooth form $\eta \in \Omega^\bullet_c(M)$ vanishing on $M \setminus W$.

Notice that $\forall \omega \in \Omega^k(M)$, $\int_S \omega = \int_W \omega \wedge \tilde{\eta} = \int_M \omega \wedge \eta$. Then η is a representative of the compact Poincaré dual η_S of S in M. Since its support is contained in W the proof is complete. ■

Exercise 9.4.1 For each of the following oriented manifolds without boundary find closed embedded submanifolds whose closed Poincaré duals form a basis of their De Rham cohomology.
- $\mathbb{R}^n$;
- S^n;
- the torus $S^n \times S^m$;
- $\mathbb{R}^n \setminus \{0\}$.

Exercise 9.4.2 For each of the oriented manifolds without boundary of the previous exercise find compact embedded submanifolds whose compact Poincaré duals form a basis of their compact support cohomology.

Exercise 9.4.3 Show that if S is the boundary of a closed orientable manifold T embedded in M, then its closed Poincaré dual is 0.

9.5 The Thom class

In this section we show how to concretely construct a representative of the Poincaré dual of a closed oriented manifold S without boundary embedded in an oriented manifold without boundary M.

By the tubular neighbourhood Theorem 3.3.5 there is a neighbourhood W of S that is diffeomorphic to the normal bundle $\mathscr{N}_{S|M}$.

If S is compact, by the localization principle Theorem 9.4.7 we can find a representative of the compact Poincaré dual of S in M with support contained in W.

In this section we describe this representative rather explicitly as form on the manifold $\mathscr{N}_{S|M}$. As we will see in Proposition 9.5.10 we will be able to do it even for the closed Poincaré dual, without any compactness assumption on S.

We need to consider a new cohomology theory, coming from a differential complex contained in the De Rham complex $\Omega^\bullet(\mathscr{N}_{S|M})$ that contains $\Omega_c^\bullet(\mathscr{N}_{S|M})$. This is a cohomology theory defined on every vector bundle. Recall that, from Proposition 3.2.2 on, we are implicitly assuming that all vector bundles have the natural differentiable structure considered there.

Definition 9.5.1 Let $\pi\colon E \to B$ be a real vector bundle of rank r over a manifold B.

A form $\omega \in \Omega^\bullet(E)$ has **compact support in the vertical direction** if $\forall K \subset B$, K compact, $\pi^{-1}(K) \cap \operatorname{supp} \omega$ is compact.

The subspace of $\Omega^\bullet(E)$ of the forms with compact support in the vertical direction is denoted by $\Omega_{cv}^\bullet(E)$. It is invariant by the standard differential d of the De Rham complex, whose restriction makes then $\Omega_{cv}^\bullet(E)$ a differential complex with graded pieces $\Omega_{cv}^q(E) := \Omega_{cv}^\bullet(E) \cap \Omega^q(E)$.

We denote by $H_{cv}^\bullet(E)$ its cohomology and by $H_{cv}^q(E)$ the graded piece of degree q of $H_{cv}^\bullet(E)$.

The forms with compact support in the vertical direction are a natural place for generalizing to vector bundles the integration along the fibres $\pi_*\colon \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^\bullet(M)$ considered in the proof of the Poincaré Lemma for the cohomology with compact support Theorem 7.5.1. Indeed if $\omega \in \Omega_{cv}^r(E)$, $\forall p \in B$, $\operatorname{supp} \omega_{|E_p}$ is compact. Since we want to integrate on each E_p, we need to consider oriented vector bundles.

For the sake of simplicity, as in the proof of Theorem 7.5.1, we give the definition of π_* in local coordinates, leaving to the reader to find an intrinsic definition to ensure that the definitions are well posed, *i.e.* independent from the choice of the coordinates.

Definition 9.5.2 Let $\pi\colon E \to B$ be an oriented vector bundle.

Choose a trivialization $\{\phi_\alpha\}$ associated to a cover $\{U_\alpha\}$ of B made of charts $(U_\alpha, \varphi_\alpha)$. For each chart $(U_\alpha, \varphi_\alpha)$ let $x_1, \dots, x_n$ be the induced coordinates on U_α.

Consider the chart of E induced by Φ_α and φ_α

$$E_{|U_\alpha} \xrightarrow{\Phi_\alpha} U_\alpha \times \mathbb{R}^r \xrightarrow{\varphi_\alpha \times \mathrm{Id}_{\mathbb{R}^r}} D_\alpha \times \mathbb{R}^r$$

inducing coordinates $x_1, \dots, x_n, t_1, \dots, t_r$ on $E_{|U_\alpha}$ such that $\pi^* x_i = x_i$.

For every form ω of type $f(x_i, t_j) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_{j_1} \wedge \ldots \wedge dt_{j_s}$, $s \neq r$, we define $\pi_* \omega = 0$.

For every form ω of type $f(x_i, t_j) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \ldots \wedge dt_r$ we define

$$\pi_* \omega = \left(\int_{\mathbb{R}^r} f(x_i, t_j) dt_1 \cdots dt_r \right) dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

Since each form in $\Omega_{cv}^\bullet(E)$ is a sum of forms as above, this defines a map of graded vector spaces

$$\pi_*\colon \Omega_{cv}^\bullet(E) \to \Omega^\bullet(B)$$

of degree $-r$ called **integration along the fibres**.

Please notice that if the bundle is the trivial bundle of rank 1 then the map π_* coincides exactly with the map considered in the proof of the Theorem 7.5.1.

We will later need the following result

Lemma 9.5.3 — Projection formula. Let $\pi\colon E \to$ be an oriented real vector bundle over a manifold B.

Then $\forall \omega \in \Omega^q(B)$, $\forall \tau \in \Omega^{q'}_{cv}(E)$

$$\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \pi_*\tau.$$

Moreover, if B is oriented of dimension n, $\operatorname{supp}\omega$ is compact and E has rank $q+q'-n$, then

$$\int_E \pi^*\omega \wedge \tau = \int_B \omega \wedge \pi_*\tau$$

Proof. The first statement is local on B. Since every bundle is locally trivial, it is enough if we prove it for the trivial bundle $U \times \mathbb{R}^r \to U$ where U is a chart with coordinates $x_1, \dots, x_n$. Then, setting $t_1 \dots, t_r$ for the vertical coordinates, by linearity it is enough if we prove our statement for every form τ of type $f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_a} \wedge dt_{j_1} \wedge \dots \wedge dt_{j_s}$, $s \neq r$ or of type $f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_a} \wedge dt_1 \wedge \dots \wedge dt_r$.

This is a straightforward computation: more precisely in the first case both $\pi_*(\pi^*\omega \wedge \tau)$ and $\omega \wedge \pi_*\tau$ vanish, whereas in the second case both are equal to

$$\left(\int f dt_1 \cdots dt_r\right) \omega \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_a}.$$

The second statement, comparing the value of two integrals, is global. We consider a trivialization $\{\phi_\alpha\}$ related to a cover $\{U_\alpha\}$ made by charts, and a partition of unity ρ_i subordinate to it. Setting $\omega_i := \rho_i\omega$ then $\omega = \sum_i \omega_i$ and

$$\int_E \pi^*\omega \wedge \tau = \sum_i \int_{E_{|U_{\alpha(i)}}} \pi^*\omega_i \wedge \tau, \qquad \int_B \omega \wedge \pi_*\tau = \sum_i \int_{U_{\alpha(i)}} \omega_i \wedge \pi_*\tau.$$

Therefore it suffices to prove $\int_{E_{|U_{\alpha(i)}}} \pi^*\omega_i \wedge \tau = \int_{U_{\alpha(i)}} \omega_i \wedge \pi_*\tau$. In other words, we can assume that the bundle is trivial and the base is a chart.

We can then conclude by two explicit computations as in the previous case, checking the equality for every form τ of type $f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_a} \wedge dt_{j_1} \wedge \dots \wedge dt_{j_s}$, $s \neq r$ or of type $f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_a} \wedge dt_1 \wedge \dots \wedge dt_r$.

Indeed in the first case both integrals vanish since both integrands vanish, whereas in the second case both integrals equal

$$\int_{U_\alpha} \left(\int_{\mathbb{R}^r} f(x_i,t_j) dt_1 \cdots dt_r\right) \omega \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_a}. \qquad \blacksquare$$

We show that π_* is a chain map.

Proposition 9.5.4 Let $\pi\colon E \to B$ be an oriented real vector bundle. Then $d\pi_* = \pi_* d$.

Proof. It is enough to prove $d\pi_*\omega = \pi_* d\omega$ for every form ω of type $f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_{j_1} \wedge \dots \wedge dt_{j_s}$, $s \neq r$ or of type $f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \dots \wedge dt_r$.

If $\omega = f(x_i,t_j)dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_{j_1} \wedge \dots \wedge dt_{j_s}$, $s \neq r$ then $d\pi_*\omega = d0 = 0$.

If $s \neq r-1$ then $d\omega$ is a sum of forms of the same type, and then $\pi_* d\omega = 0$. If $s = r-1$ then $\omega = f(x_i,t_j) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \ldots \wedge dt_{k-1} \wedge dt_{k+1} \wedge \cdots dt_r$ and therefore

$$\begin{aligned}
\pi_* d\omega &= \pi_* d\left(f(x_i,t_j) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \ldots \wedge dt_{k-1} \wedge dt_{k+1} \wedge \cdots dt_r\right) \\
&= \pi_* \left(\frac{\partial f(x_i,t_j)}{\partial t_k} dt_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \ldots \wedge dt_{k-1} \wedge dt_{k+1} \wedge \cdots dt_r\right) \\
&= \pm \pi_* \left(\frac{\partial f(x_i,t_j)}{\partial t_k} dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \ldots \wedge dt_r\right) \\
&= \pm \left(\int_{\mathbb{R}^r} \frac{\partial f(x_i,t_j)}{\partial t_k} dt_1 \cdots dt_r\right) dx_{i_1} \wedge \cdots \wedge dx_{i_q}
\end{aligned}$$

vanishes because, since ω has compact support in the vertical direction, $\int_{\mathbb{R}} \frac{\partial f}{dt_k} dt_k = 0$.

If $\omega = f(x_i,t_j) dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dt_1 \wedge \ldots \wedge dt_r$ then a straightforward computation shows that both $d\pi_*\omega$ and $\pi_* d\omega$ equal

$$\sum_k \left(\int_{\mathbb{R}^r} \frac{\partial f(x_i,t_j)}{\partial x_k} dt_1 \cdots dt_r\right) dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}. \qquad \blacksquare$$

It follows

Corollary 9.5.5 The integration along the fibres defines a morphisms of graded vector spaces

$$\pi_* \colon H^{q+r}_{cv}(E) \to H^q_{DR}(B)$$

of degree $-r$ such that, for every closed form $\eta \in \Omega^\bullet_{cv}(E)$, $\pi_*[\eta] = [\pi_*\eta]$.

A key very important result is the following generalization of the Poincaré lemma for forms with compact support, whose proof we skip.

Theorem 9.5.6 — Thom isomorphism. If E is an oriented vector bundle on a manifold B of finite type, then the integration along the fibres $\pi_* \colon H^{q+r}_{cv}(E) \to H^q_{DR}(B)$ is an isomorphism.

The Poincaré Lemma for the cohomology with compact support Theorem 7.5.1 corresponds to the special case when B is compact (so $\Omega^q_{cv}(E) = \Omega^q_c(E)$ and $H^{q+r}_{cv}(E) = H^{q+r}_c(E)$) and E is trivial.

Theorem 9.5.6 allows us to give the following definition.

Definition 9.5.7 The **Thom class** of an oriented vector bundle $\pi \colon E \to B$ is

$$\Phi(E) := \pi_*^{-1}(1) \in H^r_{cv}(E).$$

The following proposition shows how to recognize a representative of the Thom class.

Proposition 9.5.8 Let $\pi \colon E \to B$ be an oriented vector bundle of rank r and let $\Phi \in \Omega^r_{cv}(E)$ be a closed form. Then the map $f \colon B \to \mathbb{R}$ defined by $f(p) = \int_{E_p} \Phi$ is locally constant.

Moreover the following are equivalent
- the cohomology class of Φ is the Thom class of E;
- $\forall p \in B$, $\int_{E_p} \Phi = 1$;
- $\exists p \in B$ such that $\int_{E_p} \Phi = 1$.

Proof. By definition of integration along the fibres $\pi_*\Phi = f$. Therefore, by Proposition 9.5.4 $df = 0$, and therefore f is locally constant. Then Φ is the Thom class if and only if $f \equiv 1$. $\blacksquare$

We consider now a closed oriented submanifold without boundary S of dimension k of an oriented manifold without boundary M of dimension n.

By the tubular neighbourhood Theorem 3.3.5 we can see every form ω on $\mathcal{N}_{S|M}$ as a form on a tubular neighbourhood W of S in M. If ω has compact support on the vertical direction, then it vanishes near the boundary of its closure $\overline{W}$, so we can extend it to a form in $\Omega^\bullet(M)$ that vanishes on $M\setminus W$. This defines a chain map from $\Omega^\bullet_{cv}(\mathcal{N}_{S|M})$ to $\Omega^\bullet(M)$ inducing a graded ring homomorphism

$$i'_*\colon H^\bullet_{cv}(\mathcal{N}_{S|M}) \to H^\bullet_{DR}(M). \tag{9.3}$$

If S is compact, then $\operatorname{supp}\omega$, that equals the support of its extension in $\Omega^\bullet(M)$, is compact, and then we get also a graded ring homomorphism

$$i_*\colon H^\bullet_{cv}(\mathcal{N}_{S|M}) \to H^\bullet_c(M). \tag{9.4}$$

The normal bundle of S in M is an orientable vector bundle (see Exercise 6.1.14). We need to fix one orientation on it.

Definition 9.5.9 Let S be an oriented submanifold without boundary of the oriented manifold without boundary M, $\dim S \neq \dim M$.

Choose an oriented basis $v_1,\dots,v_k$ of T_pS and complete it to an oriented basis $v_1,\dots,v_n$ of T_pM.

We define the **induced orientation** as vector bundle of $\mathcal{N}_{S|M}$ as the one corresponding to the basis of $(\mathcal{N}_{S|M})_p$ given by the classes of $v_{k+1},\dots,v_n$.

Notice that the definition is well posed since it does not depend on the chosen bases, but only on the orientations they induce.

Consider the Thom class $\Phi := \Phi(\mathcal{N}_{S|M}) \in H^{n-k}_{cv}(\mathcal{N}_{S|M})$ of the normal bundle $\mathcal{N}_{S|M}$ oriented as in Definition 9.5.9.

Proposition 9.5.10 Let S be a closed oriented submanifold without boundary of dimension k of an oriented manifold without boundary M of dimension n.

Let $\Phi \in H^{n-k}_{cv}(\mathcal{N}_{S|M})$ be the Thom class of $\mathcal{N}_{S|M}$ oriented as in Definition 9.5.9.

Then $i'_*\Phi$ is the closed Poincaré dual η'_S of S in M, where i'_* is the map in (9.3).

If S is compact and M is of finite type then $i_*\Phi$ is the compact Poincaré dual η_S of S in M, where i_* is the map in (9.4).

Proof. By the tubular neighbourhood Theorem 3.3.5 there is an open neighbourhood W of S in M such that $W \cong \mathcal{N}_{S|M}$, and the inclusion of S in M is the composition of the zero section $s_0\colon S \to \mathcal{N}_{S|M}$ with the inclusion $W \subset M$.

In the following we identify W with $\mathcal{N}_{S|M}$, so getting maps $s_0\colon S \to W$ (the inclusion) and $\pi\colon W \to S$. Since $s_0\circ\pi$ is smoothly homotopic to the identity, by Corollary 7.4.5 $(s_0\circ\pi)^* = \mathrm{Id}_{H^\bullet_{DR}(W)}$.

Then, for every closed form $\omega \in \Omega^k(M)$, $[\omega_{|W}] = \pi^* s_0^*[\omega_{|W}] \in H^k_{DR}(W)$. In other words $\exists \eta \in \Omega^\bullet(W)$ such that $\omega_{|W} = \pi^* i^* \omega_{|W} + d\eta = \pi^*\omega_{|S} + d\eta$.

If moreover $\operatorname{supp}\omega$ is compact, by the Projection Formula 9.5.3 and Stokes' Theorem 6.2.9, choosing a representative $\Psi \in \Omega^\bullet(M)$ of $i_*\Phi$

$$\int_M \omega\wedge\Psi = \int_W \omega\wedge\Psi = \int_W (\pi^*\omega_{|S} + d\eta)\wedge\Psi = \int_W \pi^*\omega_{|S}\wedge\Psi + \int_W d\eta\wedge\Psi =$$
$$= \int_W \pi^*\omega_{|S}\wedge\Psi + \int_W d(\eta\wedge\Psi) = \int_S \omega\wedge\pi_*\Psi + 0 = \int_S \omega$$

and therefore $[\Psi] = i'_*\Phi$ is the closed Poincaré dual of S in M.

If S is compact, then $H^\bullet_{cv}(W) = H^\bullet_c(W)$ and therefore we can choose Ψ with compact support. Then the same chain of equalities holds for every $\omega \in \Omega^{\dim S}(M)$ showing that the class of Ψ in $H^\bullet_c(M)$ is the compact Poincaré dual of S in M. ■

An useful property of the Thom class is its good behavior with respect to the direct sum of line bundles.

Proposition 9.5.11 Let E, F be two oriented vector bundles over the same base B and consider the vector bundle $E \oplus F$ with the induced orientation given in Definition 6.1.18.

Consider the natural projections $\pi_E \colon E \oplus F \to E$, $\pi_F \colon E \oplus F \to F$. Let $\Phi_E \in \Omega^r_{cv}(E)$, $\Phi_F \in \Omega^{r'}_{cv}(F)$ be representatives of the respective Thom classes of E and F. Then

$$\pi_E^*\Phi_E \wedge \pi_F^*\Phi_F$$

is a representative of the Thom class of $E \oplus F$.

Proof. First of all we notice that $\pi_E^*\Phi_E \wedge \pi_F^*\Phi_F$ has compact support in the vertical direction (even if neither $\pi_E^*\Phi_E$ nor $\pi_F^*\Phi_F$ have compact support in the vertical direction) since its support is contained in the fibre product of the supports of $\pi_E^*\Phi_E$ and $\pi_F^*\Phi_F$.

Then $\pi_E^*\Phi_E \wedge \pi_F^*\Phi_F$ is closed. Indeed

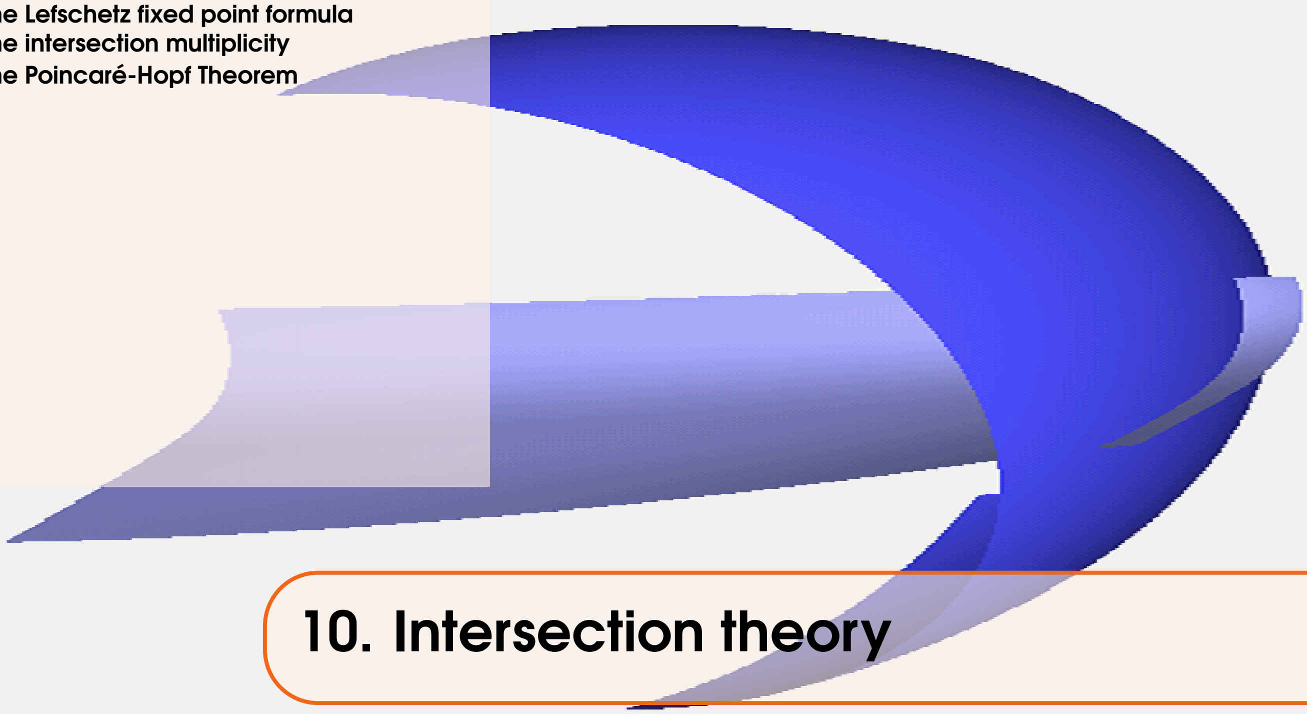
$$\begin{aligned} d(\pi_E^*\Phi_E \wedge \pi_F^*\Phi_F) = d\pi_E^*\Phi_E \wedge \pi_F^*\Phi_F \pm \pi_E^*\Phi_E \wedge d\pi_F^*\Phi_F = \\ = \pi_E^* d\Phi_E \wedge \pi_F^*\Phi_F \pm \pi_E^*\Phi_E \wedge \pi_F^* d\Phi_F = 0 \pm 0 = 0. \end{aligned}$$

Finally

$$\int_{(E\oplus F)_p} \pi_E^*\Phi_E \wedge \pi_F^*\Phi_F = \int_{E_p \oplus F_p} \pi_E^*\Phi_E \wedge \pi_F^*\Phi_F = \left(\int_{E_p} \Phi_E\right)\left(\int_{F_p} \Phi_F\right) = 1 \cdot 1 = 1$$

and the statement follows by Proposition 9.5.8. ■

Complement 9.5.1 Show that the integration along the fibres is well defined. More precisely, show that the given definition of π_* does not depend on the choice of the local coordinates $x_1, \dots, x_n$.



10. Intersection theory

10.1 Transversal intersections

Let R, S be two submanifolds embedded in a manifold M. By sake of simplicity we ask that all three manifolds R, S and M are without boundary.

Definition 10.1.1 Let $p \in R \cap S$.
We say that R and S are **transversal at *p*** if $T_pR + T_pS = T_pM$.
We say that R and S are **transversal** if they are trasversal at every point $p \in R \cap S$.

R By Definition 10.1.1, if R and S are transversal at a point $p \in R \cap S$ then $\dim R + \dim S \geq \dim M$. In particular, if $\dim R + \dim S \lneqq \dim M$, R and S are transversal if and only if $R \cap S = \emptyset$.

Notice that when $\dim R + \dim S = \dim M$ the trasversality at a point p gives $T_pM = T_pR \oplus T_pS$.

We will use the following Lemma, generalization of Proposition 2.4.6, without proving it.

Lemma 10.1.2 — Transversality Lemma. Let R and S be two embedded submanifolds without boundary of a manifold M without boundary.

Assume that R and S are transversal at p. Set $r = \dim R$, $s = \dim S$, $n = \dim M$.

Then there is a chart of M in p giving local coordinates $x_1, \dots, x_n$ such that locally

$$R = \{x_{r+1} = \ldots = x_n = 0\} \text{ and } S = \{x_1 = \ldots = x_{n-s} = 0\}.$$

From the Transversality Lemma 10.1.2 easily follows

Theorem 10.1.3 — Transversality theorem. Let R and S be two embedded transversal submanifolds without boundary of a manifold M without boundary.

Then $N := R \cap S$ has a structure of manifold embedded in R, in S and in M so that $\forall p \in N$, $T_pN = T_pR \cap T_pS$ as vector subspaces of T_pM. In particular

$$\dim(R \cap S) = \dim R + \dim S - \dim M.$$

Moreover, if R, S and M are orientable, then N is orientable too.

As in other similar situations, since every connected component of $R \cap S$ has two possible orientations, it is convenient to choose once and for all an orientation on a transversal intersection $R \cap S$ (so on every component N) *induced* by the orientations of R, S and M.

Let us start with the case $\dim R + \dim S = \dim M$. Then by Theorem 10.1.3 $R \cap S$ is a discrete set, and the orientation of a component (=point) p of $R \cap S$ is the choice of a sign.

Definition 10.1.4 Assume R and S are oriented transversal submanifolds without boundary of the oriented manifold M without boundary such that $\dim R + \dim S = \dim M$.

For every $p \in R \cap S$ we define the **induced orientation in p** as follows: we pick an oriented basis $v_1, \dots, v_r$ of T_pR and an oriented basis $v'_1, \dots, v'_s$ of T_pS and

- if $v_1, \dots, v_r, v'_1, \dots, v'_s$ is an oriented basis of T_xM we choose the sign $+$;
- else we choose the sign $-$.

In local coordinates, by Lemma 10.1.2 (up to exchange the sign of few coordinate functions x_i), we can assume that p is the origin of our system of coordinates, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$ is an oriented basis of T_pR and $\frac{\partial}{\partial x_{r+1}}, \dots, \frac{\partial}{\partial x_n}$ is an oriented basis of T_pS.

Then we orient p with $+$ if $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is an oriented basis of T_pM, with $-$ else.

In the general case

Definition 10.1.5 Assume R and S are oriented transversal submanifolds without boundary of the oriented manifold without boundary M, and assume moreover $\dim R + \dim S \gneqq \dim M$. The **induced orientation on $R \cap S$** is the one such that, $\forall p \in R \cap S$

- if $v_1, \dots, v_a$ is an oriented basis of $T_p(R \cap S)$
- once completed it to an oriented basis $v_1, \dots, v_a, v_{a+1}, \dots, v_r$ of T_pR
- and to an oriented basis $v_1, \dots, v_a, v'_{a+1}, \dots, v'_s$ of T_pS
- then $v_1, \dots, v_r, v'_{a+1}, \dots, v'_s$ is an oriented basis of T_pM,

This gives an orientation of $T_p(R \cap S)$ which only depends on the orientations of T_pR, T_pS and T_pM, and then is globally "coherent", producing an orientation on $R \cap S$.

By Lemma 10.1.2 we can assume that

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is an oriented basis of T_pM;

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$ is an oriented basis of T_pR;

$\frac{\partial}{\partial x_{n-s+1}}, \dots, \frac{\partial}{\partial x_n}$ is an oriented basis of T_pS;

Then $\frac{\partial}{\partial x_{n-s+1}}, \dots, \frac{\partial}{\partial x_r}$ is an oriented basis of $T_p(R \cap S)$.

Indeed following Definition 10.1.5 we complete if first to the oriented basis

$$\frac{\partial}{\partial x_{n-s+1}}, \dots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-s-1}}, (-1)^{ra} \frac{\partial}{\partial x_{n-s}}$$

of T_pR and then to the oriented basis

$$\frac{\partial}{\partial x_{n-s+1}}, \dots, \frac{\partial}{\partial x_n}$$

of T_pS. We conclude observing that the resulting basis of T_pM,

$$\frac{\partial}{\partial x_{n-s+1}}, \dots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-s-1}}, (-1)^{ra} \frac{\partial}{\partial x_{n-s}}, \frac{\partial}{\partial x_{r+1}}, \dots, \frac{\partial}{\partial x_n},$$

is in the same orientation class of $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$.

R Note that $R \cap S$ equals $S \cap R$ as submanifold but possibly not as oriented submanifold. More precisely the orientation is the same if and only if $(\dim M - \dim R)(\dim M - \dim S)$ is even.

In particular if M is a compact complex manifold and R, S are complex manifolds holomorphically embedded in M transversally, then $R \cap S = S \cap R$ as real oriented manifolds. Indeed in this case one can use a complex version of the Trasversality Lemma 10.1.2 to show that $R \cap S$ has a structure of complex manifold holomorphically embedded in R, S and M, and the orientation we have obtained is exactly the one induced by this complex structure.

We can now prove

Lemma 10.1.6 Let R, S be transversal oriented submanifolds without boundary of the manifold without boundary M, and consider $R \cap S$ with the induced orientation.

There is an isomorphism of vector bundles

$$\mathscr{N}_{R\cap S|M} \cong (\mathscr{N}_{R|M})_{|R\cap S} \oplus (\mathscr{N}_{S|M})_{|R\cap S} \tag{10.1}$$

Moreover, orienting the normal bundles $\mathscr{N}_{R\cap S|M}$, $\mathscr{N}_{R|M}$ $\mathscr{N}_{S|M}$ as in Definition 9.5.9. consider the natural induced orientations on $(\mathscr{N}_{R|M})_{|R\cap S}$, $(\mathscr{N}_{S|M})_{|R\cap S}$ by restriction and then on their direct sum $(\mathscr{N}_{R|M})_{|R\cap S} \oplus (\mathscr{N}_{S|M})_{|R\cap S}$ following Definition 6.1.18.

Then the isomorphism (10.1) preserves the orientation on each fibre.

Proof. For every point $p \in R \cap S$ let us consider oriented basis of $T_p(R \cap S)$, T_pR, T_pS and T_pM as in Definition 10.1.5 whose notation we borrow here.

Then by Definition 9.5.9, the classes of $v'_{a+1}, \dots, v'_s$ form an oriented basis of $(\mathscr{N}_{R|M})_p$. Similarly an oriented basis of $\mathscr{N}_{R\cap S|M}$ is given by the classes of $v_{a+1}, \dots, v_r, v'_{a+1}, \dots, v'_s$.

An oriented basis of $(\mathscr{N}_{S|M})_p$ is given on each point by the classes of $v_{a+1}, \dots, v_{r-1}$, $(-1)^{(r-a)(s-a)} v_r$ (can you see why $(-1)^{(r-a)(s-a)}$?), and the statement follows since the basis we gave for $\mathscr{N}_{R\cap S|M}$ is orientedly equivalent to $v'_{a+1}, \dots, v'_s, v_{a+1}, \dots, v_{r-1}, (-1)^{(r-a)(s-a)} v_r$. ■

We are now able to give an idea of the proof of the main result of this section, namely that *the wedge product is the Poincaré dual of the transversal intersection*, that is almost a complete proof. The only missing point, as you will read, is an argument at the beginning of the proof that comes from the proof of the Tubular Neighbourhood Theorem 3.3.5, a Theorem that we did not prove.

Theorem 10.1.7 Let M be an oriented manifold of finite type without boundary.

Let R, S be compact oriented manifolds without boundary transversally embedded in M.

Set η_R for the compact Poincaré dual of R in M, η_S for the compact Poincaré dual of S in M and $\eta_{R\cap S}$ for the compact Poincaré dual of $R \cap S$ (with the induced orientation) in M. Then

$$\eta_{R\cap S} = \eta_R \wedge \eta_S.$$

Proof. (*Sketch*) By the proof of Theorem 3.3.5 one can choose a tubular neighbourhood W_R of R in M such that the isomorphism among W_R and $\mathscr{N}_{R|M}$ maps $W_R \cap S$ onto $(\mathscr{N}_{R|M})_{|R\cap S}$. We choose an analogous tubular neighbourhood $W_S \cong \mathscr{N}_{S|M}$ of S in M.

More precisely near any point $p \in R \cap S$ there are local coordinates $x_1, \dots, x_n$ such that R and S are locally given as in Lemma 10.1.2, coordinates chosen related with the orientations of R, S and M as in the local description we gave of Definitions 10.1.4 and 10.1.5.

Moreover we can choose those coordinates and the tubular neighbourhoods so that

$$W_R = \{x_i^2 \lneqq 1 | \ \forall i \geq r+1\}, \qquad W_S = \{x_i^2 \lneqq 1 | \ \forall i \leq n-s+1\},$$

and the bundle maps $\pi_R\colon W_R \to R$, $\pi_S\colon W_S \to S$ are the projections

$$\pi_R(x_1,\dots,x_n) = (x_1,\dots,x_r,0,\dots,0), \qquad \pi_S(x_1,\dots,x_n) = (0,\dots,0,x_{n-s+1},\dots,x_n).$$

We set $W := W_R \cap W_S$. W is a vector bundle over $R\cap S$, with bundle map given locally by the projection on the *central coordinates* $x_{n-s+1},\dots x_r$, isomorphic as vector bundle to $(\mathcal{N}_{R|M})_{|R\cap S} \oplus (\mathcal{N}_{S|M})_{|R\cap S}$. Therefore, by Lemma 10.1.6, W is a tubular neighbourhood of $R\cap S$ in M.

The projections on the former addendum $W \to (\mathcal{N}_{R|M})_{|R\cap S}$ is the restriction of π_S to W. The projections on the latter addendum $W \to (\mathcal{N}_{S|M})_{|R\cap S}$ is the restriction of π_R to W. In particular the former addendum coincides, as subset of W, with $R\cap W_S$, and the latter addendum with $S\cap W_R$.

Let us now choose a representative of the Thom class of $\mathcal{N}_{R|M}$. By the characterizing property of the Thom class, we can pick any form Φ_R of degree $n-r$ such that the integral along the fibres $\int \Phi_R dx_{r+1}\cdots dx_n$, a function in the r variables $x_1,\dots,x_r$, is the constant function 1. If Φ_R has this property then $\pi_S^*(\Phi_R)_{|S\cap W}$ has the same property too. Then one can choose Φ_R such that $(\Phi_R)_{|W} = \pi_S^*(\Phi_R)_{|S\cap W}$.

In other words we choose Φ_R so that in the chosen local coordinates

$$\Phi_R = f_R(x_{n-s+1},\dots,x_n)dx_{n-s+1}\wedge\cdots\wedge dx_n$$

do not depend on the first $n-s$ variables; then $\int f_R(x_{n-s+1},\dots,x_n)dx_{n-s+1}\cdots dx_n = 1$.

Similarly, we can choose a representative Φ_S of the Thom class of $\mathcal{N}_{S|M}$ such that $(\Phi_S)_{|W} = \pi_R^*(\Phi_S)_{|R\cap W}$, so locally not depending on the last $n-r$ variables:

$$\Phi_S = f_S(x_1,\dots,x_r)dx_1\wedge\cdots\wedge dx_r$$

with $\int f_S(x_1,\dots,x_r)dx_1\cdots dx_r = 1$.

Then, by Proposition 9.5.11, a representative for the Thom class of $\mathcal{N}_{R\cap S|M}$ is

$$\Phi := \pi_S^*(\Phi_R)_{|S\cap W}\wedge \pi_R^*(\Phi_S)_{|R\cap W} = \Phi_R\wedge\Phi_S$$

The thesis follows now from Proposition 9.5.10. ■

As first application, we compute the cohomology ring of all complex projective spaces.

Proposition 10.1.8 The graded ring $H^\bullet(\mathbb{P}^n_{\mathbb{C}})$ is isomorphic to the polynomial ring $\mathbb{R}[t]/(t^{n+1})$ with the grading induced by setting $\deg t = 2$.

Proof. Let $(x_0:\cdots:x_n)$ be homogeneous coordinates on $\mathbb{P}^n_{\mathbb{C}}$ and consider the hyperplanes $H_i := \{x_i = 0\}$. They are holomorphically embedded submanifolds of $\mathbb{P}^n_{\mathbb{C}}$ biholomorphic to $\mathbb{P}^{n-1}_{\mathbb{C}}$, so their Poincaré duals η_{H_i} belong to $H^2_{DR}(\mathbb{P}^n_{\mathbb{C}})$.

First of all we show that all η_{H_i} are equal. Let F be the biholomorphism

$$F(x_0:x_1:x_2\cdots:x_n) = (x_0+x_1:x_1:x_2\cdots:x_n).$$

F is homotopically equivalent to the identity, a homotopy being given[1] by

$$H((x_0:x_1:x_2\cdots:x_n),t) = (x_0+tx_1:x_1:x_2\cdots:x_n).$$

By Corollary 9.4.3 η_{H_0} equals the Poincaré dual of $F(H_0) = H_{01} = \{x_0 = x_1\}$. Replacing F with the biholomorphism $(x_0:x_1:x_2\cdots:x_n)\mapsto(x_0:x_0+x_1:x_2\cdots:x_n)$, the same argument shows that η_{H_1} equals the Poincaré dual of H_{01}. Then $\eta_{H_0} = \eta_{H_1}$.

[1] H is well defined since, among other things, for all $t\in[0,1]$, $x_0+tx_1 = x_1 = x_2 = \cdots = x_n = 0$ implies $x_0 = x_1 = x_2 = \cdots = x_n = 0$.

Iterating the same argument for all pairs of variables we get[2] $\forall i, j\ \eta_{H_i} = \eta_{H_j}$.

Since H_0 and H_1 are transversal, by Theorem 10.1.7 $\eta_{H_0} \wedge \eta_{H_0} = \eta_{H_0} \wedge \eta_{H_1} = \eta_{H_0 \cap H_1}$. Since H_2 is transversal to $H_0 \cap H_1$ then $\eta_{H_0}^{\wedge 3} = \eta_{H_0 \cap H_1} \wedge \eta_{H_2} = \eta_{H_0 \cap H_1 \cap H_2}$.

Iterating this argument we obtain $\eta_{H_0}^{\wedge n} = \eta_{H_0 \cap \cdots \cap H_{n-1}} = \eta_p$ where p is the point of homogeneous coordinates $(0 : \cdots : 0 : 1)$. So $\int_{\mathbb{P}^n_{\mathbb{C}}} \eta_{H_0}^{\wedge n} = \pm 1$, sign depending[3] on the orientation induced on the point p.

Since $\eta_{H_0}^{\wedge n+1} = 0$ there is a graded homomorphism from $\mathbb{R}[t]/(t^{n+1})$ (with $\deg t = 2$) to $H^\bullet(\mathbb{P}^n_{\mathbb{C}})$ mapping t to η_{H_0}. Since $\int_{\mathbb{P}^n_{\mathbb{C}}} \eta_{H_0}^{\wedge n} \neq 0$ then $\eta_{H_0}^{\wedge n} \neq 0$ and then $\eta_{H_0}^{\wedge k} \neq 0$ for all $1 \leq k \leq n$. This implies that our graded homomorphism is injective.

We conclude the proof showing that the two rings have the same Hilbert function, which means that their graded pieces corresponding to the same degrees have the same dimension.

We have then to prove

$$\begin{cases} h^q(\mathbb{P}^n_{\mathbb{C}}) = 1 & \text{if } q \text{ is even and } 0 \leq q \leq 2n \\ h^q(\mathbb{P}^n_{\mathbb{C}}) = 0 & \text{else} \end{cases}. \tag{10.2}$$

We prove it by induction on n. For $n = 0$, $\mathbb{P}^n_{\mathbb{C}}$ is a point and the statement is trivial.

Then assume the statement true for all complex projective spaces of smaller dimension.

Consider the open subset $U_0 = \{x_0 \neq 0\}$ complement of the hyperplane H_0. Note that $U_0 \cong \mathbb{C}^n$, via the biholomorphism

$$(x_0 : \cdots : x_n) \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right).$$

Consider the point $p' \in U_0$ of homogeneous coordinates $(1 : 0 : \cdots : 0)$ and set $V_0 = \mathbb{P}^n_{\mathbb{C}} \setminus p'$. Then V_0 is homotopically equivalent to the hyperplane H_0, with homotopy equivalence given by[4] the map $(x_0 : x_1 : x_2 \cdots : x_n) \mapsto (0 : x_1 : x_2 \cdots : x_n)$.

Then $\mathbb{P}^n_{\mathbb{C}}$ is the union $U_0 \cup V_0$, with $U_0 \cong \mathbb{C}^n \sim p'$, $V_0 \sim \mathbb{P}^{n-1}_{\mathbb{C}}$, the intersection $U_0 \cong V_0$ as subset of $U \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$ is the complement of a point and therefore homotopically equivalent to a sphere S^{2n-1}.

Then the cohomology exact sequence induced by the Mayer-Vietoris exact sequence for the decomposition $\mathbb{P}^n_{\mathbb{C}} = U_0 \cup V_0$ has the form

$$\begin{array}{ccccccc}
 & & & \cdots & \to & H^{q-1}_{DR}(S^{2n-1}) & \to \\
\to & H^q_{DR}(\mathbb{P}^n_{\mathbb{C}}) & \to & H^q_{DR}(p') \oplus H^q_{DR}(\mathbb{P}^{n-1}_{\mathbb{C}}) & \to & H^q_{DR}(S^{2n-1}) & \to \\
\to & H^{q+1}_{DR}(\mathbb{P}^n_{\mathbb{C}}) & \to & \cdots & & &
\end{array}$$

and reader can easily complete the proof by induction on n. ■

Exercise 10.1.1 Show that $\mathbb{P}^3_{\mathbb{C}}$ is not diffeomorphic as real manifold to $S^2 \times S^4$.

[2] Actually the reader can now easily prove that every hyperplane has the same Poincaré dual.

[3] As in the Remark after Definition 10.1.5, since all H_i are complex manifold holomorphically embedded, one can show that the orientation of p is $+$ and therefore $\int_{\mathbb{P}^n_{\mathbb{C}}} \eta_{H_0}^{\wedge n} = 1$. This is not necessary for this proof, so we do not run this computation here.

[4] Note that the map cannot be extended continously to p'. This map is a retraction, the homotopy with the identity of V_0 being $(x_0 : x_1 : x_2 \cdots : x_n) \mapsto (tx_0 : x_1 : x_2 \cdots : x_n)$.

Exercise 10.1.2 Show that if $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 1$, is diffeomorphic as a real manifold to a product of k spheres S^{m_i} (possibly of different dimensions), then $n = k = 1$.

10.2 The Lefschetz fixed point formula

Let M be a compact oriented manifold without boundary.

Consider the manifold $M \times M$ with the induced orientation and the diagonal

$$\Delta = \{(p,p) | p \in M\}.$$

The projections $\pi_1, \pi_2 \colon M \times M \to M$ on the two factors restrict to Δ to the same map, a diffeomorphism onto M. We consider Δ with the orientation induced by it, and compute its Poincaré dual. Note that since $M \times M$ is compact, closed and compact Poincaré dual coincide.

Lemma 10.2.1 Let M be a compact oriented manifold without boundary, consider $M \times M$ with the induced orientation and the diagonal $\Delta = \{(p,p)|p \in M\}$ embedded in $M \times M$ oriented so that both projections $\pi_1, \pi_2 \colon M \times M \to M$ preserve the orientation.

Fix a basis $\{\omega_i\}$ of $H^\bullet_{DR}(M)$ of homogeneous elements, and set $q_i = \deg \omega_i$. Consider its dual (respect[a] to Poincaré duality) basis $\{\tau_i\}$ of $H^\bullet_{DR}(M)$.

Then the Poincaré dual of Δ in $M \times M$ is

$$\eta_\Delta = \sum (-1)^{q_i} \pi_1^* \omega_i \wedge \pi_2^* \tau_i.$$

[a] In other words, $\deg \tau_i = n - q_i$, $\int_M \omega_i \wedge \tau_i = 1$, and, if $q_i = q_j$, $i \neq j \Rightarrow \int_M \omega_i \wedge \tau_j = 0$. Note that if $q_i \neq q_j$, $\deg(\omega_i \wedge \tau_j) \neq n$, so we can't integrate $\omega_i \wedge \tau_j$ on M.

Proof. By Künneth formula 8.2.3, a basis of $H^{\dim M}_{DR}(M \times M)$ is $\{\pi_1^* \omega_i \wedge \pi_2^* \tau_j | q_i = q_j\}$. So there are constants c_{ij} such that

$$\eta_\Delta = \sum_{(i,j) \text{ such that } q_i = q_j} c_{ij} \pi_1^* \omega_i \wedge \pi_2^* \tau_j.$$

Choose now k, l with $q_k = q_l$. Then, since $(\pi_1)_{|\Delta} = (\pi_2)_{|\Delta}$

$$\int_\Delta \pi_1^* \tau_k \wedge \pi_2^* \omega_l = \int_\Delta \pi_1^* \tau_k \wedge \pi_1^* \omega_l = \int_\Delta \pi_1^* (\tau_k \wedge \omega_l) = \int_M \tau_k \wedge \omega_l = (-1)^{q_k(n-q_k)} \delta_{kl},$$

so, using Complement 6.2.2,

$$\begin{aligned}
\delta_{kl} &= (-1)^{q_k(n-q_k)} \int_\Delta \pi_1^* \tau_k \wedge \pi_2^* \omega_l \\
&= (-1)^{q_k(n-q_k)} \int_{M \times M} \pi_1^* \tau_k \wedge \pi_2^* \omega_l \wedge \eta_\Delta \\
&= (-1)^{q_k(n-q_k)} \sum_{(i,j) \text{ such that } q_i = q_j} c_{ij} \int_{M \times M} \pi_1^* \tau_k \wedge \pi_2^* \omega_l \wedge \pi_1^* \omega_i \wedge \pi_2^* \tau_j \\
&= (-1)^{q_k(n-q_k)} \sum_{(i,j) \text{ such that } q_i = q_j} (-1)^{nq_i} c_{ij} \int_{M \times M} \pi_1^* \omega_i \wedge \pi_1^* \tau_k \wedge \pi_2^* \omega_l \wedge \pi_2^* \tau_j \\
&= (-1)^{q_k(n-q_k)} \sum_{(i,j) \text{ such that } q_i = q_j} (-1)^{nq_i} c_{ij} \int_{M \times M} \pi_1^* (\omega_i \wedge \tau_k) \wedge \pi_2^* (\omega_l \wedge \tau_j)
\end{aligned}$$

If $q_i = q_j$ is different from $q_k = q_l$ then either $\omega_i \wedge \tau_k$ or $\omega_l \wedge \tau_j$ has degree strictly bigger than the dimension of M and therefore equals zero; in particular in that case $\pi_1^*(\omega_i \wedge \tau_k) \wedge \pi_2^*(\omega_l \wedge \tau_j) = 0$. So

$$\begin{aligned}
\delta_{kl} &= (-1)^{q_k(n-q_k)} \sum_{(i,j) \text{ such that } q_i=q_j=q_k} (-1)^{nq_i} c_{ij} \left(\int_M \omega_i \wedge \tau_k\right)\left(\int_M \omega_l \wedge \tau_j\right) \\
&= (-1)^{q_k(n-q_k)} \sum_{(i,j) \text{ such that } q_i=q_j=q_k} (-1)^{nq_i} c_{ij} \delta_{ik} \delta_{jl} \\
&= (-1)^{q_k} c_{kl}
\end{aligned}$$

and then $\eta_\Delta = \sum_{(i,j) \text{ such that } q_i=q_j} c_{ij} \pi_1^* \omega_i \wedge \pi_2^* \tau_j = \sum_i (-1)^{q_i} \pi_1^* \omega_i \wedge \pi_2^* \tau_i$. ∎

Let $F\colon M \to M$ be a smooth map and consider its graph

$$\Gamma_F := \{(p, F(p))\} \subset M \times M,$$

oriented so that the diffeomorphism $(\pi_1)_{|\Gamma_F}\colon \Gamma_F \to M$ preserves the orientation.

We have then two oriented submanifolds Δ and $\Gamma := \Gamma_F$ of $M \times M$ and corresponding Poincaré duals η_Δ and $\eta_\Gamma = \eta_{\Gamma_F}$.

Definition 10.2.2 Let M be a compact[a] manifold , and let $F\colon M \to M$ be a smooth map; consider its pull-back maps

$$H^q_{DR}(F) := F^*\colon H^q_{DR}(M) \to H^q_{DR}(M)$$

The **Lefschetz number** of F is defined by

$$L(F) := \sum (-1)^q \operatorname{trace}(H^q_{DR}(F)).$$

[a] In fact the definition extends obviously to manifolds of finite type and more generally to manifolds with finitely dimensional De Rham cohomology

R The last summand in the definition of $L(F)$ is, up to a sign, the degree of F!
In fact, consider a connected oriented manifold without boundary M of dimension n and a proper smooth map $F\colon M \to M$. Then $h^n_c(M) = 1$ and by Definition 9.2.1, $\forall \omega \in H^n_c(M)$,

$$F^*\omega = (\deg F)\omega$$

so $H^n_c(F)$ is the multiplication by $\deg F$. In particular $\det H^n_c(F) = \operatorname{trace} H^n_c(F) = \deg F$.
If M is a compact oriented manifold without boundary of dimension n, for every smooth map $F\colon M \to M$,

$$\det(H^n_{DR}(F)) = \operatorname{trace}(H^n_{DR}(F)) = \deg F$$

R The first summand in the definition of $L(F)$ equals 1. It does not depend on F!
Indeed, $H^0(M)$ is the space of the constant functions, on which F^* (that acts on functions by $F^*f = f \circ F$) acts trivially!

Proposition 10.2.3 Let M be a compact oriented manifold without boundary and let $F\colon M \to$

M be a smooth map. Then

$$\int_\Delta \eta_{\Gamma_F} = L(F).$$

Proof. By the definition 9.4.1 of Poincaré dual, using the explicit formula for η_Δ in Lemma 10.2.1

$$\begin{aligned}
\int_\Delta \eta_{\Gamma_F} &= \int_{M\times M} \eta_{\Gamma_F} \wedge \eta_\Delta \\
&= (-1)^n \int_{M\times M} \eta_\Delta \wedge \eta_{\Gamma_F} \\
&= (-1)^n \int_{\Gamma_F} \sum (-1)^{q_i} \pi_1^* \omega_i \wedge \pi_2^* \tau_i \\
&= \sum (-1)^{n+q_i} \int_{\Gamma_F} \pi_1^* \omega_i \wedge \pi_2^* \tau_i \\
&= \sum (-1)^{n+q_i} \int_M \left((\pi_1)^{-1}_{|\Gamma_F} \right)^* \pi_1^* \omega_i \wedge \pi_2^* \tau_i \\
&= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge \left(\pi_2 \circ (\pi_1)^{-1}_{|\Gamma_F} \right)^* \tau_i \\
&= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F^* \tau_i \\
&= \sum_q (-1)^q \sum_{i|q_i=n-q} \int_M \omega_i \wedge F^* \tau_i.
\end{aligned}$$

In the last equality we have grouped the terms by $q := n - q_i$.

By definition of the basis $\{\tau_i\}$, for all cohomology class η of degree $q = n - q_i$, $\int_M \omega_i \wedge \eta$ equals the coefficient of the term τ_i in the expression of η in the basis $\{\tau_j\}$.

Applying it to $\eta = F^* \tau_i$

$$\sum_{i|q_i=n-q} \int_M \omega_i \wedge F^* \tau_i = \operatorname{trace}(H^q(F)).$$ ■

It follows that the Lefschetz number of a function is related to its fixed points: indeed, if it does not vanish, there is at least a fixed point somewhere!

Definition 10.2.4 — Fixed Locus. Let $F: M \to M$ be a function of a set on itself. Then the fixed locus of F is

$$\operatorname{Fix}(F) = \{p \in M | F(p) = p\}.$$

Corollary 10.2.5 — Weak version of Lefschetz Fixed-point Formula. Let M be a compact oriented manifold without boundary and let $F: M \to M$ be a smooth map. Assume that $L(F) \neq 0$.

Then $\operatorname{Fix}(F) \neq \emptyset$.

Proof. We argue by contradiction. If F has no fixed points, then $\Gamma_F \cap \Delta = \emptyset$, so $(M \times M) \setminus \Delta$ is an open subset of $M \times M$ containing Γ_F. By the localization principle[5] 9.4.7 we can assume that $\operatorname{supp} \eta_{\Gamma_F} \subset (M \times M) \setminus \Delta$, so $L(F) = \int_\Delta \eta_\Gamma = 0$. ■

[5] It is not difficult to show that $(M \times M) \setminus \Delta$ has finitely dimensional cohomology by using the cohomology exact sequence induced by the Mayer-Vietoris exact sequence obtained by writing M as union of it and a tubular neighbourhood of Δ.

We can say more: the Lefschetz number 'counts', in some sense, the fixed points of F.

We consider smooth maps $F\colon M \to M$ whose graph Γ_F is transversal to the diagonal Δ. Notice that if p is a point of M such that $F(p) = p$, then dF_p is an operator on T_pM and therefore we can consider its determinant, trace, characteristic polynomial, spectrum, eigenvalues and eigenvectors.

Definition 10.2.6 Let $F\colon M \to M$ be a smooth map, $p \in \mathrm{Fix}(F)$.

We say that p is a **non-degenerate fixed point** of F if 1 is not in the spectrum of dF_p.

If p is a non-degenerate fixed point then $\mathrm{Id}_{T_pM} - dF_p$ is invertible and therefore we can define

$$\sigma_p := \operatorname{sign}\det(\mathrm{Id}_{T_pM} - dF_p) \in \{\pm 1\}.$$

The main motivation for Definition 10.2.6 comes from the following Lemma.

Lemma 10.2.7 Let M be a manifold without boundary, and let $\Delta \subset M \times M$ be the diagonal. Let $\Gamma_F \subset M \times M$ be the graph of a smooth function $F\colon M \to M$.

Then Γ_F and Δ are transversal if and only if all $p \in \mathrm{Fix}(F)$ are non-degenerate.

If moreover M is oriented, consider Γ_F and Δ with the induced orientation, so that the diffeomorphisms $\pi_1|_\Delta$ and $\pi_1|_{\Gamma_F}$ preserve the orientation, and $M \times M$ with the natural product orientation.

Then if Γ_F and Δ are transversal, $\Gamma_F \cap \Delta$ is a discrete set and the induced orientation on each point $(p,p) \in \Gamma_F \cap \Delta$ equals σ_p.

Proof. The points of $\Gamma_F \cap \Delta$ are the points (p,p) for $p \in \mathrm{Fix}(F)$.

Choose a chart of $M \times M$ near (p,p) given by a chart in p of M (for the given orientation) taken twice. This gives local coordinates $u_1, \ldots, u_n$ on M and $x_1, \ldots, x_n, y_1, \ldots, y_n$ on $M \times M$ such the $\Delta = \{x_j = y_j\}$.

In these coordinates

$$T_{(p,p)}\Delta = \bigcap_{j=1}^{n} \ker\left((dx_j)_{(p,p)} - (dy_j)_{(p,p)}\right)$$

has a basis of the form

$$\left(\frac{\partial}{\partial x_1}\right)_{(p,p)} + \left(\frac{\partial}{\partial y_1}\right)_{(p,p)}, \ldots, \left(\frac{\partial}{\partial x_n}\right)_{(p,p)} + \left(\frac{\partial}{\partial y_n}\right)_{(p,p)}. \tag{10.3}$$

Similarly, since $\Gamma_F = \{F_j(x_1, \ldots, x_n) = y_j\}$

$$T_{(p,p)}\Gamma_F = \bigcap_{j=1}^{n} \ker\left((dF_j)_{(p,p)} - (dy_j)_{(p,p)}\right)$$

has a basis of the form

$$\left(\frac{\partial}{\partial x_1}\right)_{(p,p)} + \sum_i \frac{\partial F_i}{\partial u_1}(p)\left(\frac{\partial}{\partial y_i}\right)_{(p,p)}, \ldots, \left(\frac{\partial}{\partial x_n}\right)_{(p,p)} + \sum_i \frac{\partial F_i}{\partial u_n}(p)\left(\frac{\partial}{\partial y_i}\right)_{(p,p)}. \tag{10.4}$$

Then Γ_F and Δ are transversal at (p,p) if and only the $2n$ vectors in (10.3) and (10.4) are linearly independent in T_pM, *i.e* if the block matrix

$$A = \begin{pmatrix} I_n & I_n \\ J(F)_p & I_n \end{pmatrix}$$

is invertible, where I_n is the identity matrix of order n, and $J(F)_p$ is the Jacobi matrix of F in p with respect to the coordinates $u_1,\dots,u_n$.

By a standard Gauss elimination $\det A$ equals the determinant of the matrix

$$\begin{pmatrix} I_n & I_n \\ J(F)_p - I_n & 0 \end{pmatrix}.$$

So Γ_F and Δ are transversal at (p,p) if and only if $J(F)_p - I_n$ is invertible. In other words if and only if 1 is not in the spectrum of dF_p.

Moreover, since the basis

$$\left(\frac{\partial}{\partial x_1}\right)_{(p,p)},\dots,\left(\frac{\partial}{\partial x_n}\right)_{(p,p)},\left(\frac{\partial}{\partial y_1}\right)_{(p,p)},\dots,\left(\frac{\partial}{\partial y_n}\right)_{(p,p)},$$

is compatible with the chosen orientation of $M \times M$ and the bases (10.3) and (10.4) are compatible with the chosen orientations of Δ and Γ then if Γ_F and Δ are transversal at (p,p), its induced orientation equals, by Definition 10.1.4, the sign of $\det A$, *i.e.* of

$$\begin{aligned}\det\begin{pmatrix} I_n & I_n \\ J(F)_p - I_n & 0 \end{pmatrix} &= (-1)^n \det\begin{pmatrix} I_n & I_n \\ 0 & J(F)_p - I_n \end{pmatrix} \\ &= (-1)^n \det(J(F)_p - I_n) \\ &= \det(I_n - J(F)_p).\end{aligned}$$ ■

Let us now further assume the compactness of M, in order to apply Proposition 10.2.3.

Theorem 10.2.8 — Lefschetz fixed point formula for nondegenerate fixed points. Let M be a compact oriented manifold without boundary, and let $F\colon M \to M$ be a smooth map with only non-degenerate fixed points. Then

$$L(F) = \sum_p \sigma_p.$$

Proof. By Lemma 10.2.7, Γ_F and Δ are transversal and each point (p,p) in $\Gamma_F \cap \Delta$ has induced orientation, as connected component of $\Gamma_F \cap \Delta$, equal to σ_p.

Then by Proposition 10.2.3

$$L(F) = \int_\Delta \eta_{\Gamma_F} = \int_{M\times M} \eta_{\Gamma_F} \wedge \eta_\Delta = \int_{M\times M} \eta_{\Gamma_F \cap \Delta} = \int_{\Gamma_F\cap\Delta} 1 = \sum_p \sigma_p.$$ ■

If M is a complex manifold and F is holomorphic, $\det(\mathrm{Id}_{T_pM} - dF_p)$ can't be negative by the argument of the proof of Theorem 6.1.7. Therefore in this special case, always under the assumption that 1 be not in the spectrum of dF_p, the number of fixed points of F equals exactly $L(F)$.

Exercise 10.2.1 Use Corollary 10.2.5 to write a simple proof that the antipodal map $A\colon S^n \to S^n$ preserves the orientation if and only if n is odd.

Exercise 10.2.2 Let M be a connected manifold, $F\colon M \to M$ any smooth function.

Show that $H^0(F) = \mathrm{Id}_{H^0_{DR}(M)}$.

Exercise 10.2.3 Show that every holomorphic map from $\mathbb{P}^n_{\mathbb{C}}$ to itself has a fixed point.

Exercise 10.2.4 Show that, if $F: M \to M$ is smoothly homotopic to Id_M, then $L(F) = e(M)$.

Exercise 10.2.5 Let F be a biholomorphism of $\mathbb{P}^1_{\mathbb{C}}$ with only non-degenerate fixed points.
Show that F has exactly 2 fixed points.
Construct an example of a biholomorphism of $\mathbb{P}^1_{\mathbb{C}}$ with exactly two fixed points.
Construct an example of a biholomorphism of $\mathbb{P}^1_{\mathbb{C}}$ with exactly one fixed point.

Exercise 10.2.6 Let F be a biholomorphism of $\mathbb{P}^n_{\mathbb{C}}$ with only non-degenerate fixed points.
Show that F has $n+1$ fixed points. Construct an example of a biholomorphism of $\mathbb{P}^n_{\mathbb{C}}$ with exactly $n+1$ fixed points.

10.3 The intersection multiplicity

Let us now consider two compact oriented manifold without boundary R and S embedded in an oriented manifold without boundary M of finite type, such that $\dim R + \dim S = \dim M$, without any transversality assumption.

Then we may not consider $\eta_R \wedge \eta_S \in \Omega_c^{\dim M}(M)$ as the Poincaré dual of a submanifold of M, but we can still integrate it on M.

Definition 10.3.1 If R and S are compact oriented manifolds without boundary embedded in an oriented manifold without boundary M of finite type of dimension $\dim R + \dim S$ we define the **intersection number of R and S** to be

$$R \cdot S = \int_M \eta_R \wedge \eta_S.$$

Note that $R \cdot S = S \cdot R$ unless both R and S have odd dimension, in which case $R \cdot S = -S \cdot R$. If M is a complex manifold, and R and S are complex manifolds holomorphically embedded in M, then $R \cdot S = S \cdot R$.

If R and S are transversal, then by Theorem 10.1.7, $R \cdot S = \sum_{p \in R \cap S} \varepsilon_p$ where ε_p equals 1 or -1 according to the orientation of p. In particular $R \cdot S \in \mathbb{Z}$. This is however still true even in weaker hypotheses.

Definition 10.3.2 Let R and S be compact oriented manifolds without boundary embedded in an oriented manifold without boundary M of finite type of dimension $\dim R + \dim S$.
Assume that $p \in R \cap S$ is an isolated intersection point, so open in the topology of $R \cap S$.
Then we define the **intersection multiplicity** of R and S at p as

$$\mathrm{mult}_p(R,S) = \int_W \eta_R \wedge \eta_S.$$

where W is a connected component of the intersection of a tubular neighbourhood of R and a tubular neighbourhood of S chosen small enough so that $W \cap R \cap S = \{p\}$.

If R and S are transversal at p, then by Theorem 10.1.7, $\mathrm{mult}_p(R,S)$ equals 1 or -1 according to the orientation of p.

In general $\mathrm{mult}_p(R,S) \in \mathbb{Z}$. Indeed choose a tubular neighbourhood W_R of R small enough such that, if S_p is the connected component of $W_R \cap S$ containing p, then $R \cap S_p = \{p\}$.

Let $\pi: W_R \to R$ be the bundle map given by the identification of W_R with $\mathscr{N}_{R|M}$.

Let $U \subset R$ be an open subset diffeomorphic to a disc centered in p, small enough so that $\left(\mathcal{N}_{R|M}\right)_{|U}$ is a trivial bundle. Let $W_U := \pi^{-1}(U)$. Then $W_U \cong U \times \mathbb{R}^{\dim S}$. Here we choose a diffeomorphism compatible with the orientation of the bundle $\mathcal{N}_{R|M}$.

Then we have a *second projection* $\bar{\pi}\colon W_U \to \mathbb{R}^{\dim S}$.

Since S is compact $\bar{\pi}_{|S\cap W_U} : S \cap W_U \to U$ is proper, so its degree (Definition 9.2.1) is well defined and

Proposition 10.3.3 $\operatorname{mult}_p(R,S) = \deg \bar{\pi}_{|S\cap W_U}$.

Proof. By the localization principle we can choose representatives

η_R of the compact Poincaré dual of R in M and

η_S of the compact Poincaré dual of S in M

with support shrinked in suitably small tubular neighbourhoods of R and S respectively.

Arguing as in the proof of Theorem 10.1.7, we can assume that there esists $\eta \in \Omega^\bullet_c(\mathbb{R}^{\dim S})$

such that $(\eta_R)_{|S\cap W_U} = \bar{\pi}^*_{|S\cap W_U}(\eta)$ and $\int_{\mathbb{R}^{\dim S}} \eta = 1$.

Then

$$\begin{aligned}
\operatorname{mult}_p(R,S) &= \int_W \eta_R \wedge \eta_S \\
&= \int_{W_U} \eta_R \wedge \eta_S \\
&= \int_{S\cap W_U} \eta_R \\
&= \int_{S\cap W_U} \bar{\pi}^*_{|S\cap W_U}(\eta_R) \\
&= \left(\deg \bar{\pi}_{|S\cap W_U}\right) \int_{\mathbb{R}^{\dim S}} \eta \\
&= \deg \bar{\pi}_{|S\cap W_U}.
\end{aligned}$$

■

Example 10.1 Assume that M, R and S are complex manifolds of complex dimensions $\dim_{\mathbb{C}} R = \dim_{\mathbb{C}} S = 1$. If locally $R = \{x = 0\}$ and $S = \{x = y^k\}$, then the intersection point p has coordinates $(0,0)$ and $\bar{\pi}(x,y) = y$.

It follows $\operatorname{mult}_p(R,S) = \deg \bar{\pi}_{|S\cap W_U} = k$.

This produces a bunch of straightforward consequences.

Corollary 10.3.4 Let R and S be compact oriented manifolds without boundary embedded in an oriented manifold without boundary M of finite type of dimension $\dim R + \dim S$.

Assume that $R \cap S$ is finite. Then $R \cdot S \in \mathbb{Z}$.

If R and S are transversal, then the cardinality of $R \cap S$ is at least $|R \cdot S|$, and their difference is even.

Corollary 10.3.5 Let R and S be compact oriented complex manifolds holomorphically embedded in a complex manifold without M of finite type of dimension $\dim R + \dim S$. Assume that $p \in R \cap S$ is an isolated intersection point.

Then $\operatorname{mult}_p(R,S) \in \mathbb{N}$.

Corollary 10.3.6 Let R and S be compact oriented complex manifolds holomorphically embedded in a complex manifold M of finite type of dimension $\dim R + \dim S$.

Assume that $R \cap S$ is finite. Then $R \cdot S \in \mathbb{N}$.
The cardinality of $R \cap S$ is at most $R \cdot S$.
If R and S are transversal, then $R \cdot S$ equals the cardinality of $R \cap S$.

We can now state a Lefschetz fixed point formula in weaker assumptions. First we need

Definition 10.3.7 — Multiplicity of an isolated fixed point. Let M be a compact oriented manifold without boundary, let $F: M \to M$ be a smooth map and let p be an isolated fixed point of F (in other words $\{p\}$ is a connected component of $\mathrm{Fix}(F)$).

Then define the **multiplicity** of p as fixed point

$$\sigma_p := \mathrm{mult}_{(p,p)}(\Gamma_F, \Delta).$$

If 1 is not an eigenvalue of dF_p then Definition 10.3.7 reduces to Definition 10.2.6.

If M is a complex manifold and p is an isolated fixed point of a holomorphic function $F: M \to M$ then $\sigma_p \geq 1$. Rewriting the proof of Theorem 10.2.8 in the weaker assumption that $\mathrm{Fix}(F)$ be discrete using Definition 10.3.7 we obtain the following two results.

Theorem 10.3.8 — Real Lefschetz fixed point formula for isolated fixed points. Let M be a compact oriented manifold without boundary, and let $F: M \to M$ be a smooth map such that $\mathrm{Fix}(F)$ is a discrete set.

Then the Lefschetz number of F is an integer.

More precisely $L(F)$ equals the sum on the fixed points of F of their multiplicity as fixed points of F:

$$L(F) = \sum_p \sigma_p.$$

Theorem 10.3.9 — Complex Lefschetz fixed point formula for isolated fixed points. Let M be a complex manifold and let $F: M \to M$ be a holomorphic map such that, $\mathrm{Fix}(F)$ is a discrete set.

Then the Lefschetz number of F is a natural number and more precisely it equals the sum on the fixed points of F of their multiplicity as fixed points of F:

$$L(F) = \sum_{p \in \mathrm{Fix}(F)} \sigma_p.$$

A further application is a proof of the following well known result.

Theorem 10.3.10 — Bézout theorem on the plane. Let $R, S \subset \mathbb{P}^2_{\mathbb{C}}$ be holomorphically embedded compact submanifold set-theoretically defined as zero locus of a homogeneous polynomial (in the homogeneous variables z_0, z_1, z_2) of respective degrees d_R and d_S.

Then $R \cdot S = d_R d_S$.

In particular,

- if $R \cap S$ is finite then its cardinality is at most $d_R d_S$.
- If R and S are transversal then $R \cap S$ is a set of $d_R d_S$ points.

Proof. By the proof of Proposition 10.1.8 all hyperplanes H, defined by the vanishing of a homogeneous polynomial of degree 1, have the same Poincaré dual, η_{H_0}, who generates the whole cohomology ring.

Then there are constants $a_R, a_S \in \mathbb{R}$ such that the Poincaré dual of R, η_R, equals $a_R \eta_H$ and he Poincaré dual of S, η_S, equals $a_S\eta_H$.

Since $H\cdot H=1$, $a_R=a_RH\cdot H=a_r\int\eta_H\wedge\eta_H=\int\eta_R\wedge\eta_H=R\cdot H$. Chosing H general, H and R are transversal and eliminating one variable using the equationg of H one sees that $H\cap R$ is defined by the vanishing of a homogeneous polynomial of degree a_R without multiple roots. So $a_R=d_R$. Similarly $a_S=d_S$.

Finally

$$R\cdot S=\int\eta_R\wedge\eta_S=\int d_R\eta_H\wedge d_S\eta_H=d_Rd_S\int\eta_H^{\wedge 2}=d_Rd_S. \qquad \blacksquare$$

Exercise 10.3.1 Let $\Delta\subset M\times M$ be the diagonal. Show that the tangent bundle of Δ is isomorphic to the normal bundle of Δ in $M\times M$.

Exercise 10.3.2 Let M be a Riemann surface of genus g, *i.e.* a torus with g holes, and let Δ be the diagonal in $M\times M$ with the natural orientation. Show

$$\Delta\cdot\Delta=2-2g.$$

Exercise 10.3.3 Constructs , fo all $n\geq 1$, a biholomorphic map $\varphi\colon\mathbb{P}^n_{\mathbb{C}}\to\mathbb{P}^n_{\mathbb{C}}$ with exactly one fixed point. Compute the multiplicity of it.

Exercise 10.3.4 Let M be a compact complex manifold of dimension 1.

Assume that there exists a complex biholomorphism $\varphi\colon M\to M$ smoothly homotopic to the identity, $\varphi\neq\mathrm{Id}_M$.

Show that then the genus of M is 0 or 1.

Exercise 10.3.5 — The first Hirzebruch surface $\mathbb{F}_1$. Consider $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}}$ with coordinates $((t_0:t_1),(x_0:x_1:x_2))$, $\mathbb{F}_1\subset\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}}$ defined as $\{t_0x_1=t_1x_0\}$.

1) Show that $\mathbb{F}_1$ is a complex manifold of complex dimension 2 embedded in $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}}$;
2) Show that the formula $f(t_0:t_1),(x_0:x_1:x_2))=((t_0:t_1),(t_0\bar{x}_2:t_1\bar{x}_2:t_0\bar{x}_0+t_1\bar{x}_1))$ defines a reversing orientation diffeomorphism.

Consider $E\subset\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}}$ defined as $\{x_0=x_1=0\}$. Then

3) Show that E is a complex manifold of complex dimension 1 embedded in $\mathbb{F}_1$.
4) Show[a] that the self intersection of E as submanifold of $\mathbb{F}_1$ is

$$E\cdot E=-1.$$

[a]Hint: use 2)

Exercise 10.3.6 Let $M=\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$. Consider homogeneous coordinates $(x_0:x_1)$ on the first factor, $(y_0:y_1)$ on the second factor.

Let $F\in\mathbb{C}[x_0,x_1,y_0,y_1]$. We will say that F is *bihomogeneous* of bidegree (d,e) if F is homogeneous of degree d as polynomial in the variables (x_0,x_1) (with coefficients in $\mathbb{C}[y_0,y_1]$) as well as homogeneous of degree e as polynomial in the variables (y_0,y_1) (with coefficients in $\mathbb{C}[x_0,x_1]$).

Show that the zero locus $\{F=0\}$ is well defined if and only if F is bihomogeneous.

Let $R,S\subset M$ be holomorphically embedded compact submanifold set-theoretically defined as zero locus of a bihomogeneous polynomial of respective bidegrees (d_R, e_R) and (d_S, e_S).

Show that $R \cdot S = d_R e_S + e_R d_S$.

Exercise 10.3.7 — Bézout Theorem in higher dimension. Let $X_1, \dots, X_n \subset \mathbb{P}^n_{\mathbb{C}}$ be embedded submanifold, and assume that $\forall i$, X_i is exactly the zero locus of a homogeneous polynomial (in the homogeneous variables $z_0, \dots, z_n$) of degree d_i.

Assume that $X_1 \cap \cdots \cap X_n$ is finite. Show that its cardinality is at most $d_1 \cdots d_n$.

Assume that each X_i is transversal to the intersection $X_1 \cap \ldots \cap X_{i-1}$. Show that then $X_1 \cap \cdots \cap X_n$ is a set of $d_1 \cdots d_n$ points.

10.4 The Poincaré-Hopf Theorem

Definition 10.4.1 Let $\pi\colon E \to M$ be an oriented vector bundle and consider its Thom class $\Phi(E) \in H^r_{cv}(E)$.

Let $s_0\colon M \to E$ be the zero section. Then the pull back s_0^* of forms define a degree zero chain map $s_0^*\colon \Omega^\bullet_{cv}(E) \to \Omega^\bullet(M)$, and therefore a degree zero graded ring homomorphism $s_0^*\colon H^\bullet_{cv}(E) \to H^\bullet_{DR}(M)$.

The **Euler class** $e(E)$ of E is the cohomology class $s_0^*\Phi \in H^r_{DR}(M)$.

The Euler class is connected to the Euler number as follows.

Theorem 10.4.2 Let M be a compact oriented manifold without boundary. Then

$$e(M) = \int_M e(TM).$$

Proof. Let $\Delta \cong M$ be the diagonal of $M \times M$.

As the reader can easily prove (Exercise 10.3.1) the tangent bundle $T\Delta$ is isomorphic as oriented vector bundle to the normal bundle $\mathscr{N}_{\Delta|M\times M}$.

Every bundle is a tubular neighbourhhod of the image of its zero section, and therefore we can write $e(\mathscr{N}_{\Delta|M\times M}) = \Phi(\mathscr{N}_{\Delta|M\times M})_{|\Delta}$.

By Proposition 9.5.10 any representative of the Thom class $\Phi(\mathscr{N}_{\Delta|M\times M})$ is also a representative of the Poincaré dual η_Δ of Δ in $M \times M$.

Summing up, by the expression of η_Δ in Lemma 10.2.1

$$\begin{aligned}\int_M e(TM) &= \int_\Delta e(T\Delta) = \int_\Delta e(\mathscr{N}_{\Delta|M\times M}) = \int_\Delta \Phi(\mathscr{N}_{\Delta|M\times M}) = \\ &= \int_\Delta \eta_\Delta = \sum_i (-1)^{\deg\omega_i} \int_\Delta \pi_1^*\omega_i \wedge \pi_2^*\tau_i = \sum_i (-1)^{\deg\omega_i} \int_M \omega_i \wedge \tau_i = \sum_i (-1)^{\deg\omega_i}\end{aligned}$$

and the result follows since the number of ω_i in each $H^q_{DR}(M)$ equals its dimension. ■

As the Lefschetz number is an obstruction to the existence of smooth maps $F\colon M \to M$ without fixed points, the Euler number is an obstruction to the existence of vector fields without zeroes on M. This allows us to prove the following generalization of the Hairy Ball Theorem 9.2.3.

Theorem 10.4.3 — Weak version of Hopf's Theorem. Let M be a compact orientable manifold without boundary. Assume that M can be combed flat, *i.e.*, it admits a smooth vector field without zeroes. Then $e(M) = 0$.

Proof. The vector field is a smooth section of the tangent bundle, so it is an embedding $v\colon M \to TM$, and the condition about the zeroes ensures $v(M) \cap s_0(M) = 0$, where as usual s_0 denotes the zero section. We write η_v for the compact Poincaré dual of $v(M)$, and η_0 for the compact Poincaré dual of $s_0(M)$: η_v and η_0 are both elements in $H_c^{\dim M}(TM)$ which equals, M being compact, $H_{cv}^{\dim M}(TM)$.

Considering the embedding s_0 of M in TM, a tubular neighbourhood of $s_0(M)$ in TM is TM itself and Proposition 9.5.10 implies

$$\eta_0 = \Phi(N_{s_0(M)|TM}) = \Phi(TM) \in H^n_{cv}(TM) = H^n_c(TM).$$

Let now $\Phi \in \Omega^n_c(TM)$ be a representative of $\Phi(TM)$. Since both $s_0 \circ \pi$ and $v \circ \pi$ are smoothly homotopic to the identity, $s_0^* = v^*\colon H^q_{DR}(TM) \to H^q_{DR}(M)$. Since the integral of a closed form only depends on its De Rham cohomology class, it follows $\int_M v^*\Phi = \int_M s_0^*\Phi$.

Then, by the localization principle 9.4.7,

$$0 = \int_{v(M)} \eta_0 = \int_{v(M)} \Phi = \int_M v^*\Phi = \int_M s_0^*\Phi = \int_M e(TM) = e(M).$$

■

As in the case of the Lefschetz fixed point formula, the statement can be refined by considering the case when $s_0(M)$ and $v(M)$ intersect trasversally: in that case we say that the vector field has only *nondegenerate zeroes*.

Under such assumption the same argument shows that $e(M)$ equals a sum on the zeroes of v of 1s and -1s, where the sign is the orientation of the point as transversal intersection of $s_0(M)$ and $v(M)$.

More generally, we can count the number of zeroes of a vector field with only isolated zeroes.

Definition 10.4.4 — Index of a vector field at a zero. Let $v \in \mathfrak{X}(M)$ be a vector field and $p \in M$ be an isolated zero of v; in other words we are assuming that $\{p\}$ is a connected component of its zero locus.

Then we define **the index** of the vector field v at p as the intersection multiplicity at the zero of T_pM, say 0_p, of s_0 and v:

$$i(v)_p := \mathrm{mult}_{0_p}(s_0(M), v(M)).$$

The index $i(v)_p$ in an integer, and equals -1 or 1 if $s_0(M)$ and $v(M)$ are transversal at 0_p.

Following the same ideas one proves the following

Theorem 10.4.5 — Hopf's Theorem. Let M be a compact orientable manifold without boundary, and let $v \in \mathfrak{X}(M)$. Then

$$s_0(M) \cdot v(M) = e(M).$$

In particular, if v has only isolated zeroes. Then

$$e(M) = \sum_{p|v(p)=0} i(v)_p$$

So every vector field on a sphere of even dimension with isolated zeroes has exactly two zeroes *if counted with multiplicity* (here multiplicity=index). Indeed in the picture in the front page of these notes you see represented a vector field on S^2 with just one zero, and you may deduce by the picture that the index at that point is two.

As usual, as for most results in this notes, we deduce also a complex version of the statement, by recalling that the holomorphic tangent bundle is isomorphic, as a real vector bundle, to the real tangent bundle (Proposition 3.4.3). Then for a holomoprhic vector field v the index $i(v)_p$ is positive and so

Corollary 10.4.6 A holomorphic vector field on a compact complex manifold of dimension 1 and genus g has $2-2g$ zeroes, counted with multiplicity.

In particular,

- holomorphic vector fields on $\mathbb{P}^1$ have exactly two zeroes (counted with multiplicity) or vanish identically;
- holomorphic vector fields on a complex torus either vanish identically or have no zeroes (*combing them also from a complex point of view, the homolorphic tangent bundle of a complex torus is trivial*);
- every holomorphic vector field on a curve of genus $g \geq 2$ is identically zero.

Proof. If a holomorphic vector field has a zero that is not isolated then it must be identically zero by standard complex analysis. Else we apply Theorem 10.4.5. ■

Exercise 10.4.1 Prove that every compact orientable manifold without boundary of odd dimension has Euler number zero.

Find a compact manifold with Euler number zero and one with Euler number different from zero for every possible even dimension.

Exercise 10.4.2 Let $M_1, \ldots, M_k$ be real manifolds diffeomorphic to complex projective spaces (possibly of different dimensions). Show that $M_1 \times \cdots \times M_k$ cannot be combed flat.

Exercise 10.4.3 Show that a product of spheres can be combed flat if and only if one of the factors has odd dimension.

Bibliography

[1] Bott, Raoul; Tu, Loring W. Differential forms in algebraic topology. Grad. Texts in Math., 82 Springer-Verlag, New York-Berlin, 1982. xiv+331 pp.

[2] Narasimhan, R. Analysis on real and complex manifolds. Reprint of the 1973 edition North-Holland Math. Library, 35 North-Holland Publishing Co., Amsterdam, 1985. xiv+246 pp

[3] Warner, Frank W.(1-PA) Foundations of differentiable manifolds and Lie groups. Corrected reprint of the 1971 edition Grad. Texts in Math., 94 Springer-Verlag, New York-Berlin, 1983. ix+272 pp.