Abstracts

Extrasymmetric matrices and surfaces with $p_g = 4$ and $K^2 = 6$.

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Minimal surfaces with $p_g = 4$ have been studied by several mathematicians since the publication of the famous book of Enriques [Enr]. By the standard inequalities of Noether and Bogomolov-Miyaoka-Yau, for these surfaces it holds $4 \leq K^2 \leq 45$.

The case $K^2 = 4$ is completely described in [Hor2]. All these surfaces are double covers of an irreducible quadric in $\mathbb{P}^3$. Their moduli space is generically smooth, unirational, of dimension 42; its singular locus has codimension 1, and it is exactly the locus corresponding to the double covers of the quadric cone.

In [Hor1] (see also [Rei2], [Gri]) the case $K^2 = 5$ is completely described: the canonical map is either a birational morphism to a quintic in $\mathbb{P}^3$, or a rational map of degree 2 onto an irreducible quadric. Their moduli space has two irreducible unirational components of dimension 40 whose general point corresponds to surfaces with canonical image respectively a quintic or a smooth quadric. The surfaces whose canonical image is a quadric cone form a $39-$dimensional subvariety of this moduli space, the intersection of the two irreducible components.

The case $K^2 = 6$ is the first case not completely solved. In [Hor3] Horikawa listed all possibilities for the canonical map, dividing these surfaces in 11 classes (and therefore their moduli space in 11 strata). He proved that each of these cases occurs, and studying the local deformations of these surfaces (to understand how these strata can 'glue'), Horikawa proved that their moduli space has 4 irreducible components (one of dimension 39, the other three of dimension 38), and at most 3 connected components.

More precisely, Horikawa named the 11 classes as I_a, I_b, II, III_a, III_b, IV_{a_1}, IV_{a_2}, IV_{b_1}, IV_{b_2}, V_1, V_2 (see [Hor3] for precise definitions of each class). According to Horikawa's notation we define

Definition. *Let A and B be two of the above introduced classes. If we write "$A \to B$", it means that there is a flat family with base a small disc $\Delta_\varepsilon \subset \mathbb{C}$ whose central fibre is of type B and whose general fibre is of type A.*

With this notation Horikawa summarized its results in the following picture

$$
\begin{array}{ccccccccc}
 & & & & & & & III_a & \\
 & & & & & & \swarrow & \downarrow & \\
 & & IV_{a_1} & & I_a & V_1 & & III_b & II \\
 & \swarrow & \downarrow & \swarrow & | & \downarrow & \swarrow & & \\
IV_{a_2} & & IV_{b_1} & & | & V_2 & & & \\
\downarrow & & & & \downarrow & & & & \\
IV_{b_2} & & & & I_b & & & &
\end{array}.
$$

He could disprove many other degenerations, but he could neither prove nor disprove the specializations $II \to III_b$, $II \to V$ and $I_a \to V$; we have shown that the degeneration $II \to III_b$ occurs.

Definition. *A minimal surfaces of general type with $p_g = 4$ and $K^2 = 6$ is of type II if the canonical map has degree* 3.

Horikawa proved that in this case the canonical image is a quadric cone.

Surfaces of type III_b are described by Horikawa as follows:

Theorem (5.2 in [Hor3]). *Let S be a surface of type III_b. Then S is birationally equivalent to a double covering of $\mathbb{F}_2$ whose branch locus B consists of the $0-$section Δ_0 and $B_0 \in |7\Delta_0 + 14\Gamma|$ which has a quadruple point at $x \in \Gamma$ and a $2-$fold triple point at $y \in \Gamma$ on a fibre Γ, with x and y being possibly infinitely near.*

The canonical ring of these surfaces is very complicated: it is a quotient of a polynomial ring of big (at least 6, maybe more) codimension. We do not know how to investigate the flat deformations of rings of high codimension. We look then for a 'bigger' and easier ring, a ring containing the canonical ring and of smaller codimension.

By standard computations one can show that the canonical system of S is $|2L| + Z$ where L is the genus 3 pencil pull-back of the ruling of $\mathbb{F}_2$, and Z is a fundamental cycle. Therefore, even if K_S is not $2-$divisible in the Picard group, it can be divided by 2 when considered only as a Weil divisor on the canonical model.

Definition. *Let S be a surface of type III_b, let Z be the fixed part of its canonical system, and let δ be a generator of $H^0(Z)$.*

Let R be the graded ring whose homogeneous components are the spaces $R_d := H^0(dL + \lfloor \frac{d}{2} Z \rfloor)$, $d \in \mathbb{N}$, with product defined on the homogeneous elements as $ab = a \otimes b$ or $a \otimes b \otimes \delta$ according if the product of the degrees of a and b is even or odd.

Note that enlarging the ring 'restricts' the possible deformations. In fact, if the canonical rings induce, given a flat family of surfaces, a flat family of rings, the same does not hold for these 'half-canonical' rings, since the $2-$divisibility of the canonical divisor (as a Weil divisor on the canonical model) is not necessarily preserved by a deformation.

As proved in [MP] (where these surfaces are studied in detail) the canonical system of a surface of type II, can be written again as $2L + Z$ with L genus 3 pencils and Z fundamental cycle. It is then natural to expect, if a family "$II \to III_b$" exists, that this family preserves the genus 3 pencils and the 'half-canonical" rings.

Theorem 1. *$R \cong \mathbb{C}[x_0, x_1, y, z, w, v, u]/I$ with $\deg(x_0, x_1, y, z, w, v, u) = (1, 1, 2, 3, 4, 5, 6)$, where I has codimension 4, generated by 9 equations yoked by 16 syzygies; the 9 generators of I are homogeneous polynomial of respective degrees $(4, 5, 6, 7, 8, 9, 10, 11, 12)$.*

Miles Reid and Duncan Dicks introduced in [Rei1] (see also [Rei2], [Rei3], [BCP]) the 'extrasymmetric format', for some Gorenstein rings of codimension 4 with 9 relations and 16 syzygies.

Roughly speaking, they noticed that the ideal generated by the pfaffians of order 4 of a 6×6 skewsymmetric matrix is, if the matrix has some further symmetry (it is 'extrasymmetric') of codimension 4 with 9 generators and 16 syzygies. This format is flexible, *i.e.* every deformation of the matrix preserving the symmetries induces a flat deformation of the ideal. This property allowed us to prove our main result.

Theorem 2. *Let $(x_0, x_1, y, z, w, v, u)$ variables of degrees $(1, 1, 2, 3, 4, 5, 6)$, Let M be the 6×6 skewsymmetric matrix*

$$M = \begin{pmatrix} 0 & t & z & v & y & x_1 \\ & 0 & w & u & P_3 & y \\ & & 0 & P_9 & u & v \\ & & & 0 & wP_4 & zP_4 \\ & & & & 0 & tP_4 \\ -sym & & & & & 0 \end{pmatrix}.$$

where the P_i's are homogeneous of degree i in the above introduced variables and t is the parameter on a small disc $\Delta_\varepsilon \subset \mathbb{C}$.

For general choice of P_3, P_4 and P_9 the 4×4 pfaffians of M define a variety $X \subset \Delta_\varepsilon \times \mathbb{P}(1,1,2,3,4,5,6)$ whose projection on Δ_ε is flat, with central fibre a surface of type III_b and with general fibre a surface of type II.

Sketch of the proof of Theorem 2. The flatness of the above family (for general entries) follows directly from the flexibility of the format. One can check that for general choice of the polynomials P_i and for t small the above equations define a surface with only rational double points as singularities: the invariants can be easily computed.

Note that the pfaffians Pf_{1235} and Pf_{1236} are of the form $tu - \cdots$ and $tv - \cdots$, and that the pfaffian Pf_{1256} can, for general choice of P_4, be written as $t^2w - \ldots$. Therefore, for $t \neq 0$, we can 'eliminate' the variables u, v, w, and $R \cong \mathbb{C}[x_0, x_1, y, z]/J$ for some ideal J: a straightforward computation shows that J is a principal ideal generated by the equation obtained by Pf_{1234} after 'eliminating' u, v, w using Pf_{1235}, Pf_{1236} and Pf_{1256}.

We get then an hypersurface of degree 9 in $\mathbb{P}(1,1,2,3)$, whose canonical system is induced by $\mathcal{O}(2)$: since for general entries of M the coefficient of the monomial z^3 in its equation does not vanish, we see that the canonical map has degree 3 (and image $\mathbb{P}(1,1,2)$, a quadric cone). This shows that the surface is of type II.

If $t = 0$, the canonical map is given again by the projection on $\mathbb{P}(1,1,2)$, but the surface meets the center of the projection in a point (if $P_4 = w + \cdots$, the point (0,0,0,0,1,0,1)), therefore the projection has only degree 2; one can easily check that the branch locus has the behavior described by Horikawa. □

As a corollary, we can improve Horikawa's bound on the deformation types

Corollary. *The number of deformation types of minimal surfaces of general type with $p_g = 4$ and $K^2 = 6$ is at most 2.*

References

[BCP] Bauer, I.; Catanese, F.; Pignatelli, R.; *Canonical rings of surfaces whose canonical system has base points.* Complex geometry (Göttingen, 2000), 37–72, Springer, Berlin, 2002.

[Enr] Enriques, F., *Le superficie algebriche.* Zanichelli, Bologna, 1949.

[Gri] Griffin, E., II; *Families of quintic surfaces and curves.* Compositio Math. 55 (1985), no. 1, 33–62.

[Hor1] Horikawa, E., *On deformations of quintic surfaces.* Proc. Japan Acad. 49 (1973), 377–379.

[Hor2] Horikawa, E., *Algebraic surfaces of general type with small c_1^2. I.* Ann. of Math. (2) 104 (1976), no. 2, 357–387.

[Hor3] Horikawa, E., *Algebraic surfaces of general type with small c_1^2. III.* Invent. Math. 47 (1978), no. 3, 209–248.

[MP] Lopes, M. Mendes; Pardini, R. *Triple canonical surfaces of minimal degree.* Internat. J. Math. 11 (2000), no. 4, 553–578.

[Rei1] Reid, M.; *Surfaces with $p_g = 3$, $K^2 = 4$ according to E. Horikawa and D. Dicks.* Proc. of Alg. Geometry mini Symposium, Tokyo Univ. Dec. 1989, (1989), 1–22.

[Rei2] Reid, M., *Infinitesimal view of extending a hyperplane section—deformation theory and computer algebra.* Algebraic geometry (L'Aquila, 1988), 214–286, Lecture Notes in Math., 1417, Springer, Berlin, 1990.

[Rei3] Reid, M. *Graded rings and birational geometry.* Proc. of algebraic geometry symposium (Kinosaki, 2000), 1–72.