Abstracts

On pencils of small genus

Fabrizio Catanese and Roberto Pignatelli

1. The relative canonical algebra

Throughout this abstract X will be a projective surface, $f : X \to B$ a morphism onto a smooth curve of genus b. Without loss of generality, we may assume that f has connected fibres F of genus g.

These maps are studied (see, e.g., [Fuj1], [Fuj2], [Xia]) analyzing their relative canonical algebra.

Definition 1. *Consider the relative dualizing sheaf*

$$\omega_{X|B} := \omega_X(-f^*K_B).$$

Then the relative canonical algebra $\mathcal{R}(f)$ is the commutative graded algebra $\oplus_0^\infty V_n$, where V_n is the vector bundle on B given as the direct image sheaf $f_*(\omega^n_{X|B})$

Definition 2. *The multiplication maps $\mu_{n,m} : V_n \otimes V_m \to V_{n+m}$ yield natural sheaf homomorphisms*

$$S^n(V_1) = S^n(f_*(\omega_{X|B})) \xrightarrow{\sigma_n} V_n = f_*(\omega^n_{X|B}),$$

and we define $\mathcal{T}_n =$ coker σ_n.

Remark 1. *By Noether's theorem on canonical curves, $\mathcal{T}_n$ is a torsion sheaf if the general fibre of f is non-hyperelliptic.*

The previous remark shows that the hyperelliptic and the non hyperelliptic case should be treated separately; assume in fact for the time being that a general fibre is hyperelliptic. Then there is a birational involution σ on X, and σ acts linearly on the space of sections $\mathcal{O}_X(U, \omega^n_{X/B})$, which splits as the direct sum of the $(+1)$-eigenspace and the (-1)-eigenspace. Accordingly, we get direct sums $V_n = V_n^+ \oplus V_n^-$: therefore, in the hyperelliptic case, where obviously $V_1 = V_1^-$, the cokernels $\mathcal{T}_n$ will be bigger than in the non hyperelliptic case.

2. The structure theorems

Let $f : X \to B$ be a genus 2 fibration. The rank 2 vector bundle $V_1 := f_*\omega_{X|B}$ induces a natural factorization of f as $\pi \circ \varphi$, where $\varphi : X \dashrightarrow \mathbb{P}(V_1)$ is a rational map of degree 2, and $\pi : \mathbb{P}(V_1) \to B$ is the natural projection.

The indeterminacy locus of φ is contained in the fibres of f which are not 2-connected, i.e., which split as $\mathcal{E}_1 + \mathcal{E}_2$ with $\mathcal{E}_1\mathcal{E}_2 = 1$. Then $\mathcal{E}_i^2 = -1$, $\mathcal{E}_i$ has arithmetic genus 1 and is called an elliptic cycle. These fibres are recognizable through $\mathcal{T}_2$ as follows.

Lemma 1. *Let $f : X \to B$ be a genus 2 fibration. Then $\mathcal{T}_2$ is the structure sheaf of an effective divisor $\tau \in Div_{\geq 0}(B)$, whose support is given by the points whose corresponding fibres of f are not 2-connected.*

The typical example is given by a fibre consisting of two smooth elliptic curves $\mathcal{E}_1, \mathcal{E}_2$ meeting transversally in a point P'. The blow-up of the point P' maps isomorphically to the fibre F'' of $\mathbb{P}$ over the point $P \in B$, while the elliptic curves $\mathcal{E}_1, \mathcal{E}_2$ are contracted to two distinct points of the fibre F''.

The resolution $\tilde{\varphi}$ of φ is the composition of the contraction of $\mathcal{E}_1, \mathcal{E}_2$ to two simple -2-elliptic singularities, with a finite double cover where the branch curve Δ in $\mathbb{P}$ contains the fibre and has two distinct 4-tuple points on it. More complicated fibres containing elliptic tails can produce different configurations of singularities of the branching divisor of φ: a complete list is the one given by Ogg and by Horikawa in [Ogg],[Hor]. This approach is widely used to construct genus 2 fibrations; the main difficulty is in the construction of Δ, often very singular.

Definition 3. *We denote by $\mathcal{A}$ the graded subalgebra of $\mathcal{R}$ generated by V_1 and V_2; let $\mathcal{A}_n$ be its graded part of degree n, $\mathcal{A}_{even} = \oplus_k \mathcal{A}_{2k}$.*

It is easy to see that the natural map $Sym(V_2) \to \mathcal{A}_{even}$ is surjective with kernel generated by the image of the map $i_2 : \det V_1^2 \hookrightarrow S^2(V_2)$ defined locally by $i_2(x_0 \wedge x_1)^2 = \sigma_2(x_0)^2\sigma_2(x_1)^2 - \sigma_2(x_0x_1)^2$.

Concretely, this gives explicit equations for $\mathbf{Proj}(\mathcal{A})$ as conic subbundle of the $\mathbb{P}^2$-bundle $\mathbb{P}(V_2)$. $\mathbf{Proj}(\mathcal{A})$ and $\mathbb{P}(V_1)$ are clearly birationally equivalent and biregularly equivalent outside the fibers over $\mathrm{supp}(\mathcal{T}_2)$. One can check that the fibres of $\mathrm{supp}(\mathcal{T}_2)$ are in fact the reducible fibres of the conic bundle.

If we consider the natural morphism $\varphi_{\mathcal{A}} : X \to \mathbf{Proj}(\mathcal{A})$ induced by the inclusion $\mathcal{A} \subset \mathcal{R}$ and the natural projection morphism $\pi_{\mathcal{A}} : \mathbf{Proj}(\mathcal{A}) \to B$ we get a new factorization of the fibration ('birational' to the previous one): $f = \pi_{\mathcal{A}} \circ \varphi_{\mathcal{A}}$. The advantage in considering $\varphi_{\mathcal{A}}$ instead of φ is that the branch curve $\Delta_{\mathcal{A}}$ has only simple singularities. In the typical example above described, the elliptic curves $\mathcal{E}_i$ will not be contracted by $\varphi_{\mathcal{A}}$ but they will be double covers of the two lines of the corresponding fibre of the conic bundle.

Lemma 2. *$\mathcal{A}_6$ is the cokernel of the map $\det V_1^2 \otimes V_2 \to S^3(V_2)$ naturally induced by the map i_2 above; note that $\mathcal{A}_6$ depends only on B, V_1 and σ_2. The branch curve $\Delta_{\mathcal{A}}$ is induced by a map $(\det(V_1) \otimes \mathcal{O}_B(\tau))^{\otimes 2} \to \mathcal{A}_6$.*

We can now introduce the building package of a genus 2 fibration:

Definition 4. *Define the* **associated 5-tuple** (B, V_1, τ, ξ, w) *of a genus 2 fibration $f : X \to B$ as follows:*

- *B is the base curve;*
- *$V_1 = f_*(\omega_{X|B})$;*
- *τ is the effective divisor of B with $\mathcal{O}_\tau \cong \mathcal{T}_2$;*
- *$\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ the class induced by σ_2;*
- *$w \in \mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det(V_1) \otimes \mathcal{O}_B(\tau))^{\otimes -2}))$ inducing $\Delta_{\mathcal{A}}$ on* $\mathbf{Proj}(\mathcal{A})$.

Definition 5. *We will say that a a $5-$tuple (B, V_1, τ, ξ, w) is* **admissible** *if*

- *B is a smooth curve;*
- *V_1 is a vector bundle on B of rank 2;*
- *$\tau \in Div^+(B)$;*
- *$\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ yields a vector bundle V_2;*
- *$w \in \mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det(V_1) \otimes \mathcal{O}_B(\tau))^{\otimes -2}))$ inducing $\Delta_{\mathcal{A}}$ on* $\mathbf{Proj}(\mathcal{A})$, *where $\mathcal{A}_6$ is the vector bundle induced by ξ;*

and if moreover they satisfy some open conditions ensuring that the associated double cover has Rational Double Points as singularities.

We do not specify here the open conditions in detail for lack of space. The vector bundle $\mathcal{A}_6$ is 'induced' taking the map σ_2 induced by ξ and defining $\mathcal{A}_6$ as the cokernel of the map in lemma 2.

Theorem 1. *Let f be a relatively minimal genus 2 fibration. Then its associated $5-$tuple is admissible. Viceversa, every admissible $5-$tuple is the associated $5-$tuple of a genus 2 fibration $f : X \to B$, and the surface X has invariants $\chi(\mathcal{O}_X) = \deg(V_1) + (b-1), K^2 = 2\deg V_1 + \deg\tau + 8(b-1)$. Two relatively minimal genus 2 fibration having the same associated $5-$tuple are isomorphic.*

We can prove a very similar statement for a genus 3 fibrations f with non hyperelliptic general fibre, under the assumption that every fibre of f is $2-$connected.

3. Applications

The first application of theorem 1 is a short proof of the following theorem (already proved by Bombieri ([Bom]) using Ogg's list of genus 2 fibres (cf. [Ogg])).

Theorem 2. *Let S be a Godeaux surface, and let $f : S \to \mathbb{P}^1$ be the fibration induced by the bicanonical pencil of S. Then the genus of the fibre can only be 3 or 4.*

We have an interesting application of theorem 1 to minimal surfaces of general type with $p_g = q = 1$. In this case $2 \leq K_S^2 \leq 9$ and the Albanese map is a morphism $f : S \to B$ where B is a smooth elliptic curve.

The case $K_S^2 = 2$ is completely described in [Cat1] where it is proved (among other things) that the moduli space is generically smooth, unirational of dimension 7.

The class of surfaces of general type with $K^2 = 3$, $p_g = q = 1$ is studied in [CC1], [CC2]. In [CC1] it is proved that for this class of surfaces the genus of the Albanese fibre is 2 or 3. The second case is completely classified in [CC2], where it is shown that the corresponding moduli space is generically smooth, unirational of dimension 5.

In [CC1] all surfaces with $p_g = q = 1$, $K^2 = 3$ and genus 2 of the Albanese fibre are described as double covers of $B^{(2)}$. It was conjectured there (see problem 5.5) that this family of surfaces should form an irreducible family of the moduli space. We can disprove this conjecture. More precisely (considering also the family in [CC2])

Theorem 3. *The family, in the moduli space of the minimal surfaces of general type, corresponding to the surfaces S with $p_g(S) = q(S) = 1$, $K_S^2 = 3$ has at least 4 connected components and at most 5 irreducible components, all of dimension 5.*

References

[Bom] Bombieri, E., unpublished manuscript.

[Cat1] Catanese, F., *On a class of surfaces of general type.* Algebraic surfaces, pp. 269–284, Fondazione C.I.M.E., Liguori Editore, Napoli 1981.

[CC1] Catanese, F,, Ciliberto, C., *Surfaces with $p_g = q = 1$.* Problems in the theory of surfaces and their classification (Cortona, 1988), 49–79, Sympos. Math., XXXII, Academic Press, London, 1991.

[CC2] Catanese, F., Ciliberto, C., *Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$.* J. Algebraic Geom. 2 (1993), no. 3, 389–411.

[Fuj1] Fujita, T., *On Kähler fiber spaces over curves.* J. Math. Soc. Japan 30 (1978), no. 4, 779–794.

[Fuj2] Fujita, T., *The sheaf of relative canonical forms of a Kähler fiber space over a curve.* Proc. Japan Acad. Ser. A Math. Sci. 54 (1978), no. 7, 183–184.

[Hor] Horikawa, E., *On algebraic surfaces with pencils of curves of genus* 2. Complex analysis and algebraic geometry, pp. 79–90. Iwanami Shoten, Tokyo, 1977.

[Ogg] Ogg, A. P., *On pencils of curves of genus two.* Topology 5 (1966), 355–362.

[Xia] Xiao, G., *Surfaces fibrées en courbes de genre deux.* Lecture Notes in Mathematics, 1137. Springer-Verlag, Berlin, 1985. x+103 pp. ISBN: 3-540-15662-3