UNPROJECTION AND DEFORMATIONS OF TERTIARY BURNIAT SURFACES

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ABSTRACT. We construct a 4-dimensional family of canonical models of surfaces of general type with $p_g = 0$ and $K^2 = 3$. We consider W, the quadric hull of the canonical image of T, an étale $(\mathbb{Z}/2)^3$-cover of a minimal surface of general type with $p_g = 0$ and $K^2 = 3$. We construct a good candidate for W and unproject it to an anticanonically embedded $\mathbb{Q}$-Fano 3-fold $V \subset \mathbb{P}(1^7, 2^8)$ that we use to reconstruct T, as a member of $|-2K_V|$. V is an Enriques–Fano 3-fold. The family constructed contains the Burniat surfaces with $K^2 = 3$.

Additionally we construct the universal coverings of the surfaces in our family as follows. We set an action of $\mathbb{Z}/2 \times Q_8$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the Fano cover of V, and consider a complete intersection of 2 hypersurfaces on which the group acts freely.

1. INTRODUCTION

A Burniat surface is the minimal resolution of singularities of a *bidouble* cover, *i.e.*, a finite flat Galois morphism with Galois group $(\mathbb{Z}/2)^2$, of the projective plane branched along the divisors:

$$D_1 = A_1 + A_2 + A_3, \quad D_2 = B_1 + B_2 + B_3, \quad D_3 = C_1 + C_2 + C_3,$$

where A_1, B_1, C_1 form a triangle with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, A_1, A_2, A_3 are lines through $\mathbf{x}_1$, B_1, B_2, B_3 are lines through $\mathbf{x}_2$ and C_1, C_2, C_3 are lines through $\mathbf{x}_3$. (*Cf.* Figure 1.) Burniat surfaces were first constructed by Burniat [Bu], though a substantial part of the initial study of these surfaces was done, about 10 years later by Peters [Pet]. They have an equivalent description known as the Inoue surfaces [Ino], given as the quotient of a divisor in the product of three elliptic curves by a finite group. See [BC1] for an excellent introduction to the subject of Burniat surfaces.

Burniat surfaces are minimal surfaces of general type with $p_g = \dim H^0(\Omega^2) = 0$ and hence with irregularity, $q = \dim H^0(\Omega^1)$, equal to 0. The study of the moduli space of surfaces of general type with these invariants started in 1932 with

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Campedelli's celebrated construction of a surface of general type with $p_g = 0$ and $K^2 = c_1^2 = 2$, as a double cover of the projective plane branched along a curve of degree 10 with 6 infinitely near triple points. Nowadays, this subject is still the object of much attention, with new results on the description of whole components of this moduli space (*e.g.* [AP, CS, MP3, MPR, PY1, PY2]) and on the proof of existence of new ones (*e.g.* [BCG, BCGP, BP, LP, MP2, NP1, PPS]). See [BCP] for a survey on surfaces of general type with $p_g = 0$.

Let S be a Burniat surface. If we assume that the branch divisors D_1, D_2, D_3 in the configuration described earlier, besides satisfying the conditions stated there, are otherwise general, then $K_S^2 = 6$. If, however, we allow an extra m triads of lines to meet (see Figure 1), we get $K_S^2 = 6 - m = 5, 4, 3, 2$, for $1 \leq m \leq 4$. This yields 6

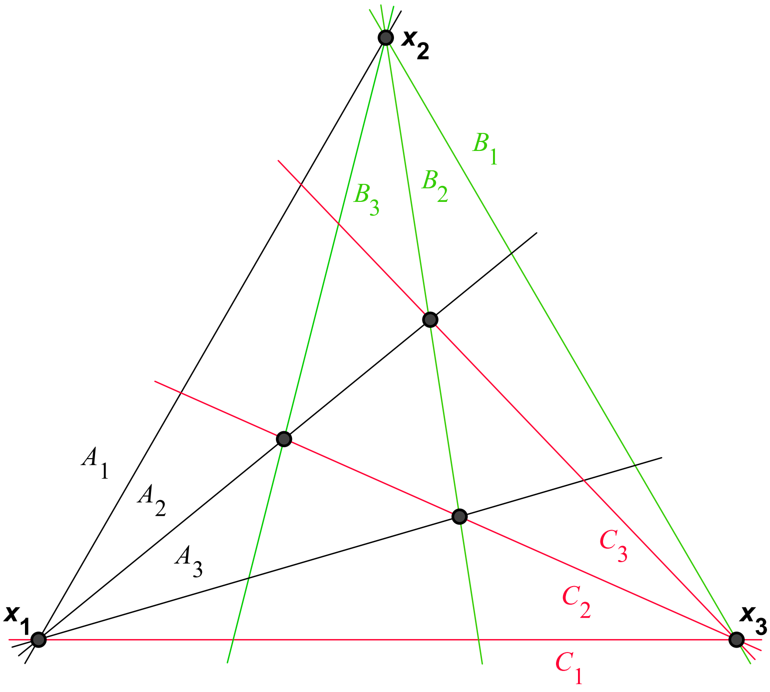

FIGURE 1. Branching divisors for tertiary Burniat surfaces

families (two for $K_S^2 = 4$: the family of nodal type and the family of non nodal type) the dimensions of which are equal to $K_S^2 - 2$, respectively. Following Bauer and Catanese [BC1], call a Burniat surface *primary* if $K_S^2 = 6$, *secondary* if $K_S^2 = 4, 5$, *tertiary* if $K_S^2 = 3$ and *quaternary* if $K_S^2 = 2$. ¿From a certain point of view, what sets apart primary and secondary Burniat surfaces from tertiary and quaternary Burniat surfaces is that the former families have dimensions greater than or equal to the expected dimension $10\chi(\mathcal{O}_S) - 2K_S^2$ of the corresponding moduli spaces, while the latter families have dimensions stricly less than the expected dimension of the corresponding moduli spaces. More precisely, the family of tertiary Burniat surfaces is 1-dimensional, whereas the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 3$ has expected dimension equal to 4 and the family quaternary Burniat surfaces is 0-dimensional, whereas the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 2$ (the Campedelli surfaces) has expected dimension equal to 6.

In 2001, Mendes Lopes and Pardini (*cf.* [MP1]) proved that the 4-dimensional family of primary Burniat surfaces forms a normal, unirational, irreducible connected component of the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 6$. In 2004, Kulikov (*cf.* [K]) proved that the class of the quaternary Burniat surfaces belongs to the component of classical Campedelli surfaces, *i.e.*, $p_g = 0$, $K^2 = 2$ and torsion group $(\mathbb{Z}/2)^3$, which had been completely described (*cf.* [Miy, R]). In a deep recent analysis (*cf.* [BC1, BC2, BC3]), Bauer and Catanese have continued the study of the components of the moduli space of surfaces of general type containing the Burniat surfaces. They gave an alternative proof of Mendes Lopes–Pardini's result on primary Burniat surfaces, they showed that of the 3 families corresponding to secondary Burniat surfaces the one with $K^2 = 5$ and the one with $K^2 = 4$ of non nodal type form irreducible connected components and they have also described the whole connected component containing the Burniat surfaces with $K^2 = 4$ of nodal type, which turns out to have dimension 3.

This article is devoted to a construction of a 4-dimensional family of minimal surfaces, S, of general type with $p_g(S) = 0$ and $K_S^2 = 3$, containing, as a codimension 3 subfamily, the family of tertiary Burniat surfaces. We do this by constructing a 4-dimensional family of surfaces of general type T with $\chi(\mathcal{O}_T) = 8$ and $K_T^2 = 24$, equipped with a free $G = (\mathbb{Z}/2)^3$ action and by taking S to be the quotient T/G. The family of surfaces T is a linear subsystem of $|-2K_V|$, where V is an Enriques–Fano 3-fold in $\mathbb{P}(1^7, 2^8)$ obtained from a complete intersection Fano 3-fold in $\mathbb{P}^6$, using parallel unprojection (*cf.* [NP2]), on which there exists an action of G inducing the action of this group on T. In this respect, we can see V as a *key variety* for this construction; just as weighted projective space acts as key variety in most elementary constructions. This idea is reminiscent of the construction of a numerical Campedelli surface with torsion group $\mathbb{Z}/6$ of [NP1]. Lifting the action of G to the Fano double cover of V we obtain the simple description of our family described in the next theorem, which synthesizes Theorem 2.11, Theorem 3.5, Theorem 3.6 and Theorem 4.5 of this work.

Theorem 1.1. *Consider $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with coordinates (t_{00}, t_{01}), (t_{10}, t_{11}), (t_{20}, t_{21}), (t_{30}, t_{31}) and the group $\tilde{G} < \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ generated by the 3 automorphisms in the following table, where ϵ is a chosen square root of -1:*

	t_{00}	t_{01}	t_{10}	t_{11}	t_{20}	t_{21}	t_{30}	t_{31}
$\tilde{\alpha}_1\tilde{\beta}_2$	$-\epsilon t_{10}$	t_{11}	t_{00}	ϵt_{01}	t_{31}	$-\epsilon t_{30}$	t_{21}	ϵt_{20}
$\tilde{\alpha}_2\tilde{\beta}_3$	$-\epsilon t_{20}$	t_{21}	t_{31}	ϵt_{30}	t_{00}	ϵt_{01}	t_{11}	$-\epsilon t_{10}$
$\tilde{\alpha}_3\tilde{\beta}_1$	$-\epsilon t_{30}$	t_{31}	t_{21}	$-\epsilon t_{20}$	t_{11}	ϵt_{10}	t_{00}	ϵt_{01}

Then $\tilde{G} \cong \mathbb{Z}/2 \times Q_8$*, where* Q_8 *denotes the standard quaternion group. Consider also the* $\tilde{G}$*-invariant hypersurface of multi-degree* $(1,1,1,1)$ *given by*

$$Z_1 := (t_{01}t_{10}t_{20}t_{30} + t_{00}t_{11}t_{21}t_{31} = 0)$$

and the $\tilde{G}$*-invariant surfaces* $\tilde{T}$ *cut out on* Z_1 *by the multi-degree* $(2,2,2,2)$ *hypersurfaces given by*

$$Z_2 = \sum_{i=0}^{3} \nu_i \left(t_{i0}^2 \prod_{j \neq i} t_{j1}^2 + t_{i1}^2 \prod_{j \neq i} t_{j0}^2 \right) - 2\nu_4 \sum_{a+b+c+d \ \mathit{even}} (-1)^{\frac{b+c+d-a}{2}} t_{0a}^2 t_{1b}^2 t_{2c}^2 t_{3d}^2 = 0$$

for general $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$*. Then, if the* ν_i *are general,* $\tilde{G}$ *acts freely on* $\tilde{T}$ *and the quotient* $S = \tilde{T}/\tilde{G}$ *is the canonical model of a surface of general type with* $p_g = 0$*,* $K^2 = 3$ *and* $\pi_1(S) \cong \mathbb{Z}/2 \times Q_8$*. The family obtained in this way describes a* 4*-dimensional locus in the moduli space of the surfaces of general type, containing the tertiary Burniat surfaces, for which* $-\nu_0 = \nu_1 = \nu_2 = \nu_3$*.*

Note that the fundamental group of tertiary Burniat surfaces has already been computed in [BC1], which fixes a mistake in a previous computation in [Pet]. The study of surfaces of general type with $p_g = 0$, $K^2 = 3$ and fundamental group of order 16 is of special interest as, according to a conjecture of M. Reid, this number should be the maximum order of their (algebraic) fundamental groups.

Our construction of the family of surfaces S using unprojection was done in the spring of 2009. At that time we only knew that they are surfaces of general type with $p_g = 0$ and $K^2 = 3$. Later on, the second author had the opportunity to hear some talks by Bauer and Catanese on their work on Burniat surfaces and discuss with them some of their results on this subject, including their work in progress on the deformations of the tertiary Burniat surfaces (now in [BC3]) and the behaviour of the bicanonical map of these surfaces. These discussions lead us to the study of the bicanonical map of our surfaces which then enabled us to find the tertiary Burniat surfaces. Indeed, our construction agrees with theirs; namely, we show that the bicanonical map of S is a bidouble cover of a cubic surface with 3 nodes (*cf.* Proposition 4.1); and, parameterizing this cubic, we show that these surfaces are birational to a bidouble cover of the plane branched along divisors D''_1, D''_2, D''_3 each given as a sum of 2 lines and a conic (*cf.* Remark 4.4). In [BC3], Bauer and Catanese prove, among other things, that the irreducible component of the moduli space of surfaces of general type containing the tertiary Burniat surfaces has dimension 4, and they construct a proper open set of it. As a consequence of their results, our family also forms an open set of the same component. It is possible that our family is not a proper subset, covering the full component, but we do not have a proof of it.

We now explain the motivation for our construction. Let T be a minimal regular surface of general type with $\chi(\mathcal{O}_T) = 8$ and $K_T^2 = 24$. Assume that $T \in |-2K_V|$, where V is a $\mathbb{Q}$-Fano 3-fold with n singular points of type $\frac{1}{2}(1,1,1)$. Then $h^0(-K_V) = p_g = 7$, $-K_V^3 = K_T^2/2 = 12$ and by the orbifold Riemann–Roch formulas (*cf.* [ABR, BS]), $4p_g = K_S^2 + 12 - n$, *i.e.*, $n = 8$. This leads to a candidate 3-fold V anticanonically embedded in $\mathbb{P}(1^7, 2^8)$ that, by the Graded Ring Database [Br], projects to a complete intersection $W_{2,2,2} \subset \mathbb{P}^6$. On the other hand, suppose that T is equipped with a free $G = (\mathbb{Z}/2)^3$ action. By the Lefschetz Holomorphic Fixed Point Formula we know the character of the representation of G on $H^0(nK_T)$. Throughout the paper a, b, c, d vary in $\mathbb{Z}/2 = \{0, 1\}$, and we wil use the notation $0' = 1$ and $1' = 0$. Writing χ_{abc}, for the irreducible representations of G, we get:

$$\begin{aligned} &H^0(K_T) = \textstyle\bigoplus_{(a,b,c) \in G \setminus \{(0,0,0)\}} \chi_{abc}, \quad H^0(2K_T) = \textstyle\bigoplus_{(a,b,c) \in G} \chi_{abc}^{\oplus 4}, \\ &\mathrm{S}^2 H^0(K_T) = \chi_{000}^{\oplus 7} \oplus \chi_{100}^{\oplus 3} \oplus \chi_{010}^{\oplus 3} \oplus \chi_{001}^{\oplus 3} \oplus \chi_{110}^{\oplus 3} \oplus \chi_{101}^{\oplus 3} \oplus \chi_{011}^{\oplus 3} \oplus \chi_{111}^{\oplus 3}. \end{aligned} \tag{1.1}$$

We deduce that the canonical ring of T,

$$R(T, K_T) = \bigoplus_{n \in \mathbb{N}} H^0(nK_T),$$

on which G acts, has 3 invariant quadric relations and needs 7 new generators in degree 2, one for each of the nontrivial rank 1 representations G. This agrees with the properties of V; the anticanonical ring of which, $R(V, -K_V)$, needs 8 generators of degree 2 and has 3 quadric relations between the degree 1 generators, coming from the defining equations of $W_{2,2,2} \subset \mathbb{P}^6$. Note that $R(T, K_T)$ can be obtained from $R(V, -K_V)$ by taking a quotient by a degree 2 regular element.

As in [NP1], the first goal is to construct V from $W_{2,2,2} \subset \mathbb{P}^6$ using parallel unprojection, which is to say, unproject all at once 8 divisors in W satisfying sufficiently general conditions. By the theory of Kustin–Miller unprojection (*cf.* [PR]), to produce unprojection variables of weight 2, we need to unproject divisors, $D_i \subset W$, with canonical class $\mathcal{O}_{D_i}(-3)$. Accordingly, one can obtain $V \subset \mathbb{P}(1^7, 2^8)$ by parallel unprojection from a complete intersection $W_{2,2,2} \subset \mathbb{P}^6$ containing 8 planes satisfying the required generality assumptions (see [NP1, Section 2]). However from the computational point of view it is preferable to make the equations of the divisors as general as possible. Hence, we use the format of equations of [NP2, Section 3]; this means we shall work with a 4-fold complete intersection of 3 quadrics, $X \subset \mathbb{P}^7$ and unproject 8 linear subspaces of X of dimension 3 obtaining $Y \subset \mathbb{P}(1^8, 2^8)$. The geometric construction on the 3-fold level is recovered by taking quasihomogeneous hypersurface sections of degree 1 of both X and Y.

The second goal is to set up an action of $G \cong (\mathbb{Z}/2)^3$ on $\mathbb{P}(1^8, 2^8)$ that leaves Y, V and T invariant and is fixed point free on T. With this in mind we establish a $(\mathbb{Z}/2)^6$ action on $\mathbb{P}(1^8, 2^8)$ which leaves Y and V invariant and for which there exists a subgroup $H \subset (\mathbb{Z}/2)^6$ isomorphic to $(\mathbb{Z}/2)^5$ which leaves T invariant. We then show that H has a subgroup $G \cong (\mathbb{Z}/2)^3$ which acts fixed point freely on T. The upshot is that the quotient group $H/G \cong (\mathbb{Z}/2)^2$ acts on $S := T/G$ and the quotient map coincides with the bicanonical map of S. (*Cf.* Proposition 4.1.)

The paper is divided up as follows. In Section 2 we describe the construction of $Y \subset \mathbb{P}(1^8, 2^8)$ via parallel unprojection of a 4-fold complete intersection of 3 quadrics $X \subset \mathbb{P}^7$ using the format introduced in [NP2]. We obtain a $\mathbb{Q}$-Fano 3-fold

$V \subset Y$ by taking a hypersurface section of degree 1 of V and the surface $T \subset V$ by taking a hypersurface section of degree 2 of V. The bulk of this section is concerned with the study of the geometry of V (with emphasis on its singularities) and setting up of the group action described above. In Section 3, we show that Y is the quotient of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by an involution and we lift the action of G to an action of $\tilde{G} = \mathbb{Z}/2 \times Q_8$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We obtain a description of our surfaces as quotient by a fixed point free action of $\tilde{G}$ on a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which enables the computation of their fundamental group. Finally we show that the family constructed is unirational and has 4 moduli. In Section 4 we carry out a detailed study of the bicanonical map of $S = T/G$. We show that the bicanonical map is a bidouble cover of a singular cubic surface $S_3 \subset \mathbb{P}^3$ and compute the branch loci of this map. Via a birational map $S_3 \dashrightarrow \mathbb{P}^2$ we reinterpret this bidouble cover as a bidouble cover of $\mathbb{P}^2$ and use it to show that the family of surfaces constructed contains the family of tertiary Burniat surfaces.

2. The Construction of S

Consider $\mathbb{P}^7$ with homogeneous coordinates x_{00}, x_{01}, x_{10}, x_{11}, x_{20}, x_{21}, x_{30}, x_{31} and let $X \subset \mathbb{P}^7$ be the 4-fold complete intersection of 3 quadrics given by:

$$x_{00}x_{01} = x_{10}x_{11} = x_{20}x_{21} = x_{30}x_{31}. \tag{2.1}$$

Notice that X contains the 16 linear spaces given by:

$$H_{abcd} = (x_{0a} = x_{1b} = x_{2c} = x_{3d} = 0), \quad a, b, c, d \in \{0, 1\} \tag{2.2}$$

all of which have codimension 1 in X. These 16 linear spaces can be thought of as the vertices of the 4-cube, by identifying their equations (2.2) with the vertex (a, b, c, d). An edge between two vertices means that the intersection of the corresponding linear spaces has dimension ≥ 2 or, equivalently, that the union of the sets of equations of the linear spaces does not contain a regular sequence of length 6.

Since the homogenous coordinate rings of X and of each linear space are Gorenstein graded rings, we can use Kustin–Miller parallel unprojection on a subset of the set of linear spaces in (2.2). Indeed the format of the equations of X was studied in [NP2, Section 3], where a sufficient condition for the existence of the parallel unprojection was given. In our case, a subset of linear spaces can be unprojected if the defining equations of any two linear subspaces in it contain a regular sequence of length 6. Since the 4-cube is a bipartite graph, there are 2 maximal subsets with this property. These subsets yield isomorphic constructions, thus we shall fix one. Let $\mathcal{L}$ denote the subset of $\{0, 1\}^4$ consisting of the 4-tuples with even sum and consider the corresponding subset of linear spaces: $\{H_{abcd} \mid (a, b, c, d) \in \mathcal{L}\}$. Recall that throughout the paper we shall be using the following shorthand notation: $0' = 1$ and $1' = 0$.

Remark 2.1. Notice that $H_{abcd} \cap H_{a'b'c'd'} = \emptyset$. Any other pair of distinct elements in $\{H_{abcd} \mid (a, b, c, d) \in \mathcal{L}\}$ intersect along a line. These 24 lines form the singular locus of X.

According to [NP2, Lemma 3.2] we can perform the parallel unprojection of these 8 linear spaces in X, to obtain a projectively Gorenstein subscheme of a weighted projective space, $Y \subset \mathbb{P}(1^8, 2^8)$.

Definition 2.2. We denote the weight 2 variables of the ambient weighted projective space by y_{abcd}, for $(a, b, c, d) \in \mathcal{L}$. These are the unprojection variables. Reading from [NP2], $Y \subset \mathbb{P}(1^8, 2^8)$ is $\operatorname{Proj}(R_{\text{un}})$, where R_{un} is the quotient of the polynomial ring $\mathbb{C}[x_{00}, x_{01}, \dots, x_{31}, y_{abcd} \mid (a, b, c, d) \in \mathcal{L}]$ by an ideal, J, generated by the following homogenous polynomials: the original 3 quadrics — given by the difference of two terms in (2.1) — 32 cubics, given by

$$\begin{aligned} &y_{abcd}x_{0a} - x_{1b'}x_{2c'}x_{3d'}, \quad && y_{abcd}x_{1b} - x_{0a'}x_{2c'}x_{3d'}, \\ &y_{abcd}x_{2c} - x_{0a'}x_{1b'}x_{3d'}, \quad && y_{abcd}x_{3d} - x_{0a'}x_{1b'}x_{2c'}, \end{aligned} \tag{2.3}$$

for every $(a, b, c, d) \in \mathcal{L}$; and 28 quartics, given by

$$y_{a_0a_1a_2a_3}y_{b_0b_1b_2b_3} - \frac{x_{0a_0'}x_{1a_1'}x_{2a_2'}x_{3a_3'}}{x_{ia_i'}x_{ia_i}} \cdot \frac{x_{0b_0'}x_{1b_1'}x_{2b_2'}x_{3b_3'}}{x_{jb_j'}x_{jb_j}} \tag{2.4}$$

for every distinct $(a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) \in \mathcal{L}$, where, given (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) in $\mathcal{L}$, i and j are such that $a_i \neq b_i$ and $a_j \neq b_j$, so that the fractional expression of (2.4) is always a polynomial.

Remark 2.3. The unprojection procedure can be interpreted as follows. Consider, for each $(a, b, c, d) \in \mathcal{L}$ the rational section of $\mathcal{O}_X(2)$

$$\varphi_{abcd} := \frac{x_{1b'}x_{2c'}x_{3d'}}{x_{0a}} = \frac{x_{0a'}x_{2c'}x_{3d'}}{x_{1b}} = \frac{x_{0a'}x_{1b'}x_{3d'}}{x_{2c}} = \frac{x_{0a'}x_{1b'}x_{2c'}}{x_{3d}}, \tag{2.5}$$

where the equalities follow from (2.1). The divisor of the poles of φ_{abcd} is exactly H_{abcd}. Parallel unprojection of H_{abcd}, for $(a, b, c, d) \in \mathcal{L}$ identifies each φ_{abcd} with a new degree 2 (unprojection) variable y_{abcd} adjoined to the coordinate ring of the ambient projective space of X. Together with the variables x_{ia}, these new variables form the coordinate ring of the ambient weighted projective space of the unprojected variety Y. The cubic equations (2.3) follow by removal of denominators in the equalities (2.5) and the quartic equations (2.4) by taking suitable products of two expressions for two given rational functions in a way that the denominators cancel. This is always possible given the way $\mathcal{L}$ was chosen and, modulo (2.1), it is independent of choice. For example, if we multiply $\varphi_{0011} = \frac{x_{01}x_{11}x_{30}}{x_{21}}$ with $\varphi_{0000} = \frac{x_{01}x_{11}x_{21}}{x_{30}}$ we get the equation $y_{0011}y_{0000} = x_{01}^2x_{11}^2$; likewise, using $\varphi_{1111} = \frac{x_{10}x_{20}x_{30}}{x_{01}}$ and the previous expression for φ_{0000} one obtains the equation $y_{1111}y_{0000} = x_{10}x_{11}x_{20}x_{21}$. Unlike the case of unprojection from a hypersurface ring, this parallel unprojection ideal, J, needs the initial quadrics (2.1) as generators as they are (clearly) not in the ideal generated by the cubics and quartics.

Notation 2.4. We denote by $\varphi \colon X \dashrightarrow Y$ the unprojection map, *i.e.*, the rational map defined using the 8 rational functions described in Remark 2.3:

$$\varphi(x_{00}, x_{01}, \dots, x_{31}) = (x_{00}, x_{01}, \dots, x_{31}, \varphi_{0000}(x_{ia}), \dots, \varphi_{1111}(x_{ia})).$$

Let $\pi \colon \mathbb{P}(1^8, 2^8) \dashrightarrow \mathbb{P}^7$ denote the projection map, *i.e.*, the rational map obtained by forgetting the degree 2 variables.

Remark 2.5. The unprojection map $\varphi\colon X \dashrightarrow Y$ is a birational map between X and Y, with inverse $\pi_{|Y}\colon Y \dashrightarrow X$. Indeed, φ induces an isomorphism

$$X \setminus \left(\bigcup\nolimits_{abcd\in\mathcal{L}} H_{abcd}\right) \to Y \setminus \left(\bigcup\nolimits_{abcd\in\mathcal{L}} \mathcal{H}_{abcd}\right), \tag{2.6}$$

where $\mathcal{H}_{abcd}$ is the subscheme of Y given by $x_{0a} = x_{1b} = x_{2c} = x_{3d} = 0$.

Notation 2.6. Firstly we make notation for the coordinate points of $\mathbb{P}^7$ and $\mathbb{P}(1^8, 2^8)$. Given $0 \le i \le 3$ and $a \in \{0, 1\}$ we denote by $\mathbf{x}_{ia}$ the point of $\mathbb{P}^7$, or of $\mathbb{P}(1^8, 2^8)$, depending on the context, having all but the coordinate x_{ia} equal to zero. Similarly, given $(a, b, c, d) \in \mathcal{L}$, we denote by $\mathbf{y}_{abcd} \in \mathbb{P}(1^8, 2^8)$ the point defined in an analogous way. Note that the 8 points $\mathbf{y}_{abcd}$ are the intersection of Y with the singular locus of the ambient space, and also the centers of the projection $\pi_{|Y}$. Secondly we establish notation for a distinguish set of surfaces in $\mathbb{P}(1^8, 2^8)$. There are 24 quartic polynomials in (2.4) involving the product of 2 squares. Such is the case with $y_{0011}y_{0000} - x_{01}^2x_{11}^2$. This polynomial defines a subscheme, $\mathcal{S}^{01}_{11}$, of dimension 2 of the 3-dimensional projective space $\mathbb{P}(1^2, 2^2)$ with variables x_{01}, x_{11}, y_{0011}, y_{0000} that we can regard as a subscheme $\mathcal{S}^{01}_{11} \subset \mathbb{P}(1^8, 2^8)$, by setting all but the coordinates $x_{01}, x_{11}, y_{0011}, y_{0000}$ equal to 0. Similarly, given $0 \le i < j \le 3$ and $a, b \in \{0, 1\}$ we denote by $\mathcal{S}^{ij}_{ab}$ the subscheme of $\mathbb{P}(1^2, 2^2) \subset \mathbb{P}(1^8, 2^8)$ defined by the quartic polynomial of (2.4) involving $x_{ia}^2x_{jb}^2$. These are 24 surfaces contained in Y.

Lemma 2.7. *Set-theoretically,* $\mathcal{H}_{a'b'c'd'} = \{\mathbf{y}_{abcd}\}\cup\mathcal{S}^{01}_{ab}\cup\mathcal{S}^{02}_{ac}\cup\mathcal{S}^{03}_{ad}\cup\mathcal{S}^{12}_{bc}\cup\mathcal{S}^{13}_{bd}\cup\mathcal{S}^{23}_{cd}$. *In particular* $\mathcal{H}_{abcd}$ *is* 2*-dimensional, for all* $(a, b, c, d) \in \mathcal{L}$.

Proof. We prove the lemma for $(a, b, c, d) = (0, 0, 0, 0)$. The proof for the remaining $(a, b, c, d) \in \mathcal{L}$ is similar. Comparing the definitions of $\mathcal{H}_{abcd}$ in Remark 2.5 and of $\mathcal{S}^{ij}_{ab}$ and $\mathbf{y}_{abcd}$ of Notation 2.6 it follows that

$$\mathcal{S}^{01}_{11} \cup \mathcal{S}^{02}_{11} \cup \mathcal{S}^{03}_{11} \cup \mathcal{S}^{12}_{11} \cup \mathcal{S}^{13}_{11} \cup \mathcal{S}^{23}_{11} \cup \{\mathbf{y}_{1111}\} \subset \mathcal{H}_{0000}.$$

Conversely, let $\mathbf{x} \in \mathcal{H}_{0000}$. From the cubic equations (2.3) involving y_{0000}, we see that there exist distinct $i, j \in \{0, 1, 2, 3\}$ such that $x_{i1} = x_{j1} = 0$. Assume that $i = 0$ and $j = 1$. If $y_{abcd} = 0$, for all $(a, b, c, d) \in \mathcal{L} \setminus \{(0, 0, 0, 0), (1, 1, 0, 0)\}$, then $y_{0000}y_{1100} - x_{21}^2x_{31}^2 = 0$ is the only equation of Y not made trivial. In this situation $\mathbf{x} \in \mathcal{S}^{23}_{11}$. Suppose that $y_{abcd} \neq 0$ for some $(a, b, c, d) \in \mathcal{L} \setminus \{(0, 0, 0, 0), (1, 1, 0, 0)\}$. Then, from the quartic equations (2.4) involving y_{abcd} we see that all other weight 2 variables are zero and, using the cubic equations (2.3) involving y_{abcd}, that $x_{2c} = x_{3d} = 0$. Note that necessarily $(c, d) \neq (0, 0)$. Now, if $(c, d) = (1, 0)$ then all variables but y_{ab10} and x_{31} vanish. In this case, either $(a, b) = (0, 1)$ and $\mathbf{x} \in \mathcal{S}^{03}_{11}$, or $(a, b) = (1, 0)$ and $\mathbf{x} \in \mathcal{S}^{13}_{11}$. Similarly, if $(c, d) = (0, 1)$, $\mathbf{x} \in \mathcal{S}^{02}_{11} \cup \mathcal{S}^{12}_{11}$. Finally, if $(c, d) = (1, 1)$ then, $\mathbf{x} = \mathbf{y}_{1111}$ or $\mathbf{x} = \mathbf{y}_{0011}$, and we conclude by observing that $\mathbf{y}_{0011} \in \mathcal{S}^{01}_{11}$. The same reasoning applies for any other distinct $i, j \in \{0, 1, 2, 3\}$. □

Proposition 2.8. Y *is a reduced and irreducible normal* 4*-dimensional subscheme of* $\mathbb{P}(1^8, 2^8)$*. Moreover* $K_Y = \mathcal{O}_Y(-2)$ *and* $\deg Y = \deg X + 4 = 12$.

Proof. Let R denote the coordinate ring of X. The fact that $\dim Y = 4$ is a consequence of the fact that $\dim R_{\mathrm{un}} = \dim R = 4$, coming from the general theory of Kustin–Miller unprojection. However it is also a consequence of the isomorphism (2.6) and Lemma 2.7. R_{un} is obtained as an unprojection of R, that has canonical module equal to $R(-2)$. Hence R_{un} is Gorenstein and has a canonical module equal to $R_{\text{un}}(-2)$, *cf.* [NP2]. In view of Remark 2.1, isomorphism (2.6) and Lemma 2.7, $\operatorname{codim}\operatorname{Sing} Y \geq 2$. Since R_{un} is Cohen–Macaulay we deduce that R_{un} is a normal domain, *cf.* [E, Theorem 18.15]. Hence Y is a reduced and irreducible normal subscheme of $\mathbb{P}(1^8, 2^8)$. That $K_Y = \mathcal{O}_Y(-2)$ follows from the computation of the canonical module of R_{un}. By [NP2, Proposition 3.4], $\deg Y = \deg X + 4 = 12$. □

We can now define the key variety V. This variety is obtained intersecting Y with the hypersurface given by $x_{00} + x_{01} = 0$. The reason for this choice of degree 1 polynomial will be clear from the action of $G \cong (\mathbb{Z}/2)^{\oplus 3}$ on V that we describe below. We will regard V as a subvariety of $\mathbb{P}(1^8, 2^8)$ defined by the ideal $J + (x_{00} + x_{01})$, *i.e.*, the ideal generated by $x_{00} + x_{01}$ and the polynomials in (2.1), (2.3) and (2.4). Since $x_{00} + x_{01}$ is a regular element of R_{un} and this ring is Cohen–Macaulay we deduce that V is a 3-fold of degree 12. Clearly, V is the parallel unprojection of the 8 planes $\Pi_{abcd} := H_{abcd} \cap (x_{00} + x_{11} = 0)$ in the 3-fold $W = X \cap (x_{00} + x_{01} = 0)$. The following diagram shows the construction so far.

(2.7)
$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d, dashed] & Y \subset \mathbb{P}(1^8, 2^8) \arrow[d, dashed, "\pi_{|Y}"] \\
W \arrow[r, hook] & X \subset \mathbb{P}^7
\end{tikzcd}$$

Proposition 2.9. *The singular locus of* $V = Y \cap (x_{00} + x_{01} = 0)$ *consists of* 14 *points,* 8 *quotient singularities of type* $\frac{1}{2}(1,1,1)$ *at the points* $\mathbf{y}_{abcd}$ *and* 6 *isolated singularities locally analytically isomorphic to the vertex of a cone over the Del Pezzo surface* $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^8$ *at the points* $\mathbf{x}_{ia} \in \mathbb{P}(1^8, 2^8)$*, for* $i > 0$*.*

Proof. Consider $W = X \cap (x_{00} + x_{01} = 0)$. The variety W is smooth away from $\cup_{abcd \in \mathcal{L}} H_{abcd}$. Since π is an isomorphism away from $\cup_{abcd \in \mathcal{L}} \mathcal{H}_{abcd}$ (*cf.* Remark 2.5), we deduce that

$$\operatorname{Sing}(V) \subset V \cap (\cup_{abcd \in \mathcal{L}} \mathcal{H}_{abcd}) = V \cap (\cup \mathcal{S}_{ab}^{ij}). \tag{2.8}$$

We start by analyzing the points $\mathbf{x}$ of $\operatorname{Sing}(V)$ in the locus $\{\mathbf{y}_{abcd} \mid (a,b,c,d) \in \mathcal{L}\}$. We assume, without loss of generality, that $\mathbf{x} = \mathbf{y}_{0000}$. Consider the affine piece of V given by $y_{0000} = 1$. Then, using the quartic equations (2.4) we can eliminate all of the remaining y_{abcd}, using the cubic equations (2.3) we can eliminate $x_{00}, x_{10}, x_{20}, x_{30}$ and using $x_{00} + x_{01}$ we can eliminate x_{01}. The coordinates x_{11}, x_{21}, x_{31}, y_{0000} map an analytic neighborhood of $\mathbf{y}_{0000} \in V$ isomorphically onto a neighborhood of the point $(0,0,0,1) \in \mathbb{P}(1^3, 2)$, which is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

Suppose now that $\mathbf{x} \in \operatorname{Sing}(V) \setminus \{\mathbf{y}_{abcd} \mid (a,b,c,d) \in \mathcal{L}\}$. Let $V_{\text{a}} \subset \mathbb{A}^{16}$ denote the affine cone of V. Among the equations of V_{a}, besides $x_{00} + x_{01} = 0$, we find the 7 quartic equations $y_{abcd}y_{0000} - \cdots = 0$, plus

$$y_{0000}x_{00} - x_{11}x_{21}x_{31} = 0, \quad y_{0000}x_{10} - x_{01}x_{21}x_{31} = 0,$$
$$y_{0000}x_{20} - x_{01}x_{11}x_{31} = 0, \quad y_{0000}x_{30} - x_{01}x_{11}x_{21} = 0.$$

Let us take the 12×12 minor of the Jacobian matrix of the ideal defining V_{a} of the gradients of these 12 polynomials with respect to the variables x_{01}, y_{abcd} for $(a,b,c,d) \in \mathcal{L} \setminus \{(0,0,0,0)\}$ and $x_{00}, x_{10}, x_{20}, x_{30}$. This minor is equal to $\pm y_{0000}^{11}$, where the sign depends on the order we give to the equations and to the variables. Similarly we can find minors of the form $\pm y_{abcd}^{11}$, for all $(a,b,c,d) \in \mathcal{L}$. Hence if $\mathbf{x} \in \operatorname{Sing}(V) \setminus \{\mathbf{y}_{abcd} \mid (a,b,c,d) \in \mathcal{L}\}$ then $y_{abcd} = 0$, for all $(a,b,c,d) \in \mathcal{L}$. From (2.8) and Lemma 2.7, we deduce $\mathbf{x} \in \{\mathbf{x}_{10}, \mathbf{x}_{11}, \mathbf{x}_{20}, \mathbf{x}_{21}, \mathbf{x}_{30}, \mathbf{x}_{11}\}$. We assume, without loss of generality, that $\mathbf{x} = \mathbf{x}_{10}$. Consider the affine piece of Y given by $x_{10} = 1$. Here, we can use the cubic equations (2.3) to eliminate all variables of the form y_{a0cd} and one of the quadrics (2.1) to eliminate x_{11}. After eliminating these 5 variables, we see that this affine piece of Y is isomorphic to the subvariety of $\mathbb{A}^9$ defined by the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} y_{1100} & x_{31} & x_{21} & x_{00} \\ & y_{0110} & x_{01} & x_{20} \\ & & y_{0101} & x_{30} \\ \text{sym} & & & y_{1111} \end{pmatrix},$$

with $\mathbf{x}_{10}$ being identified with the origin of $\mathbb{A}^9$. Hence $\mathbf{x}_{10}$ is a singular point of Y locally isomorphic to the cone over the 2-Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$. Since $V = Y \cap (x_{00} + x_{01} = 0)$ we conclude that V is locally, near $\mathbf{x}_{10}$, analytically isomorphic to a cone over the Del Pezzo surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^8$. Similarly for all other points in $\{\mathbf{x}_{10}, \mathbf{x}_{11}, \mathbf{x}_{20}, \mathbf{x}_{21}, \mathbf{x}_{30}, \mathbf{x}_{31}\}$. $\square$

Corollary 2.10. *V is a reduced and irreducible normal 3-dimensional subscheme of $\mathbb{P}(1^8, 2^8)$. Moreover $K_V = \mathcal{O}_V(-1)$ and $\deg(V) = 12$.*

Proof. The proof is similar to that of Proposition 2.8. $\square$

The surface T, on which we will set up a group action of $G \cong (\mathbb{Z}/2)^3$ will be a suitable hypersurface section of V of degree 2, and therefore a canonical surface. In particular the group action is induced by action of G on the ambient weighted projective space. What we do next is to set an action of the larger group $(\mathbb{Z}/2)^6$ on the ambient space, which leaves V invariant. Following that, we single out a subgroup $G \cong (\mathbb{Z}/2)^3$ of $(\mathbb{Z}/2)^6$ inducing on $H^0(\mathcal{O}_V(1))$ the regular representation of G minus the trivial rank 1 representation. Finally, we choose the surface $T \in |\mathcal{O}_V(2)|$ in such a way that G leaves it invariant and that the induced representation of G on $H^0(\mathcal{O}_T(2)) = H^0(K_T)$ is the sum of 4 copies of the regular representation.

Let α_1, α_2, α_3, β_1, β_2, β_3 be generators of $(\mathbb{Z}/2)^6$. Let them act on the space $\langle x_{ij} \rangle$ in the following way: α_i exchanges x_{00} with x_{01} and exchanges x_{i0} with x_{i1}, fixing all the remaining variables; β_i takes x_{i0} to $-x_{i0}$ and x_{i1} to $-x_{i1}$, fixing all the remaining variables. Since the actions of two generators commute, we obtain an action of $(\mathbb{Z}/2)^6$ on $\mathbb{P}^7$. Clearly, by inspection of (2.1), X is invariant under this action. The identification of the variables y_{abcd} with the rational functions on X of (2.5) induces an extension of this action to $\mathbb{P}(1^8, 2^8)$ in way that Y (and, indeed, V) also become invariant. Since

$$\varphi_{abcd} = \frac{x_{1b'}x_{2c'}x_{3d'}}{x_{0a}} \xrightarrow{\alpha_1} \frac{x_{1b}x_{2c'}x_{3d'}}{x_{0a'}} = \varphi_{a'b'cd}, \tag{2.9}$$

etc., it suffices to set $\alpha_1(y_{abcbd}) = y_{a'b'cd}$, $\alpha_2(y_{abcd}) = y_{a'bc'd}$, $\alpha_3(y_{abcd}) = y_{a'bcd'}$ and $\beta_i(y_{abcd}) = -y_{abcd}$, for all $1 \leq i \leq 3$. We summarize this in Table 1.

Table 1. The $(\mathbb{Z}/2)^6$-action.

α_1	$x_{00} \leftrightarrow x_{01}$	$x_{10} \leftrightarrow x_{11}$	$y_{abcd} \leftrightarrow y_{a'b'cd}$
α_2	$x_{00} \leftrightarrow x_{01}$	$x_{20} \leftrightarrow x_{21}$	$y_{abcd} \leftrightarrow y_{a'bc'd}$
α_3	$x_{00} \leftrightarrow x_{01}$	$x_{30} \leftrightarrow x_{31}$	$y_{abcd} \leftrightarrow y_{a'bcd'}$
β_i	$x_{i0} \rightarrow -x_{i0}$	$x_{i1} \rightarrow -x_{i1}$	$y_{abcd} \rightarrow -y_{abcd}$

Consider the subgroup $G \subset (\mathbb{Z}/2)^6$ given by

$$G = \langle \alpha_1\beta_2, \alpha_2\beta_3, \alpha_3\beta_1 \rangle \cong (\mathbb{Z}/2)^3. \tag{2.10}$$

It is easy to see that the representation of G on $H^0(\mathcal{O}_V(1))$ is the regular representation minus the trivial rank 1 representation; indeed the representation of G on $\langle x_{00}, x_{01}, \ldots, x_{31} \rangle$ is the regular representation and $x_{00}+x_{01}$ generates the invariant eigenspace. Likewise, given a character $\epsilon \in \mathrm{Hom}((\mathbb{Z}/2)^3, \mathbb{C})$, it is not hard to see that the polynomial

$$\sum_{abcd \in \mathcal{L}} \epsilon(b,c,d) y_{abcd} \tag{2.11}$$

is an eigenvector for the action of G on the space $\langle y_{abcd} \mid (a,b,c,d) \in \mathcal{L} \rangle$ and that the 8 polynomials obtained in this way generate distinct eigenspaces of the action. The expression for the trivial eigenvector, obtained from (2.11) using the character given by $\epsilon(b,c,d) = (-1)^{b+c+d}$, for all $(b,c,d) \in (\mathbb{Z}/2)^3$, is given by:

$$\sum_{abcd \in \mathcal{L}} (-1)^{b+c+d} y_{abcd} = \sum_{abcd \in \mathcal{L}} (-1)^a y_{abcd}. \tag{2.12}$$

The representation theory of G on the cohomology of T dictates the eigenspace of $H^0(\mathcal{O}_V(2))$ from which to take the equation of $T \in |\mathcal{O}_V(2)|$. According to (1.1) and the discussion above, the equation for T belongs to the invariant eigenspace of $H^0(\mathcal{O}_V(2))$. Consider the following invariant quadratic forms in the x_{ia} variables:

$$s_i = \frac{x_{i0}^2 + x_{i1}^2}{2} \quad \text{and} \quad t_i = x_{i0}x_{i1} \quad \text{for} \quad i = 0,1,2,3. \tag{2.13}$$

using $x_{00} + x_{01} = 0$ and (2.1) we obtain

$$t_i = t_0 = -s_0 \tag{2.14}$$

on V and W. Hence s_0, s_1, s_2, s_3 form a basis for the invariant subspace of the second symmetric power of $H^0(\mathcal{O}_V(1))$. From this and (2.12) we see that a general element of the invariant eigenspace of $H^0(\mathcal{O}_V(2))$ is given by:

$$q = l + \nu_4 \sum_{abcd \in \mathcal{L}} (-1)^a y_{abcd}, \quad \text{where} \quad l = \nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3 \tag{2.15}$$

and ν_0, ν_1, ν_2, ν_3, ν_4 are general complex parameters. Let $\mathcal{N} \cong \mathbb{P}^4$ be the linear system of surfaces given by

$$\mathcal{N} = \left\{ T = V \cap (q = 0) \mid (\nu_0, \nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{P}^4 \right\}. \tag{2.16}$$

Then G acts on every $T \in \mathcal{N}$, and we can take the quotient $S = T/G$.

Theorem 2.11. *A general element $T \in \mathcal{N}$ is smooth surface of general type with ample canonical divisor and with $p_g(T) = 7$, $q(T) = 0$ and $K_T^2 = 24$. Furthermore the canonical map of T is a birational morphism onto a complete intersection of three quadrics and a cubic in $\mathbb{P}^6$. For a general surface $T \in \mathcal{N}$, the action of G is free and therefore $S := T/G$ is a surface of general type with ample canonical divisor and with $p_g(S) = 0$ and $K_S^2 = 3$.*

Proof. The base locus of $\mathcal{N}$ is contained in the locus given by $(s_0 = s_1 = s_2 = s_3 = 0)$. Using (2.13) and (2.14), we get $x_{i0}x_{i1} = 0$ for all $i = 0, 1, 2, 3$; and since $2s_i = x_{i0}^2 + x_{i1}^2$ we deduce $x_{i0} = x_{i1} = 0$ for all $i = 0, 1, 2, 3$. Therefore

$$(s_0 = s_1 = s_2 = s_3 = 0) \cap V = \{\mathbf{y}_{abcd} \mid (a, b, c, d) \in \mathcal{L}\},$$

which for general $\nu_0, \nu_1, \nu_2, \nu_3$ does not intersect T. By Bertini's Theorem, $\mathrm{Sing}(T)$ is contained in the union of the base locus of $\mathcal{N}$ and $\mathrm{Sing}(V)$. For a general choice of $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4$, the surface T does not meet $\mathrm{Sing}(V)$, (*cf.* Proposition 2.9), and as we showed, $\mathcal{N}$ is base point free. Hence T is nonsingular. Since the coordinate ring of T is the quotient of R_{un} by a regular sequence, it is a Gorenstein graded ring and, in particular, Cohen-Macaulay. By [E, Theorem 18.15] the coordinate ring of T is a domain and, accordingly, T is reduced and irreducible. By adjunction, $K_T = \mathcal{O}_T(1)$ which is ample, and the projectively Gorensteinness of T yields $q = \dim H^1(K_T) = 0$ and $p_g(T) = 7$. Finally $K_T^2 = \deg(T) = 2\deg(V) = 24$.

The canonical map φ_{K_T} of T equals $\pi_{|T}$, the map given by the sections $x_{00} = x_{01}$, x_{10}, x_{11}, x_{20}, x_{21}, x_{30}, x_{31}, *cf.* Notation 2.4. Since the locus of common zeros of these sections is contained in the locus $(s_0 = s_1 = s_2 = s_3 = 0)$ we deduce that φ_{K_T} is a morphism. Moreover since K_T is ample, φ_{K_T} is finite. Since $\pi_{|V}$ is birational, and $T \in \mathcal{N}$ is a general element of a movable linear system, φ_{K_T} is also birational. Then the canonical image $\varphi_{K_T}(T)$ is a nondegenerate surface of degree $K_T^2 = 24$ in the hyperplane $\mathbb{P}^6 := (x_{00} + x_{01} = 0) \subset \mathbb{P}^7$, contained in the locus defined by (2.1). By elimination, we find a new cubic hypersurface through $\varphi_{K_T}(T)$. ¿From $q = 0 \iff \nu_4 \sum_{abcd \in \mathcal{L}} (-1)^a y_{abcd} = -l$, substituting y_{abcd} with $\frac{x_{1b'}x_{2c'}x_{3d'}}{x_{0a}}$ and using $x_{01} = -x_{00}$ we get

$$\nu_4 \textstyle\sum_{abcd \in \mathcal{L}} (-1)^a \frac{x_{1b'}x_{2c'}x_{3d'}}{x_{0a}} = l \iff \nu_4 \textstyle\sum_{bcd \in \{0,1\}^3} \frac{x_{1b'}x_{2c'}x_{3d'}}{x_{00}} = l,$$

which yields the irreducible cubic equation:

$$\nu_4(x_{10}+x_{11})(x_{20}+x_{21})(x_{30}+x_{31}) = x_{00}l. \tag{2.17}$$

Therefore $\varphi_{K_T}(T)$ is a surface of degree 24 contained in the intersection of the hyperplane $(x_{00}+x_{01}=0)$, the quadrics (2.1) and the cubic defined by (2.17). Since these polynomials form a regular sequence, we deduce that $\varphi_{K_T}(T)$ coincides with the complete intersection of 3 quadrics and 1 cubic that, choosing x_{00}, x_{10}, $x_{11},\dots,x_{30},x_{31}$ as basis for $H^0(K_T)$, are obtained substituting x_{01} for $-x_{00}$ in (2.1) and (2.17).

Let us now show that the action of G on T is free. By symmetry it is enough to check that the 3 elements $\alpha_1\beta_2$, $\alpha_1\alpha_2\beta_2\beta_3$ and $\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$ act on T without fixed points. In the weighted projective space $\mathbb{P}(1^8,2^8)$ the fixed locus of an involution splits into three spaces; the $(+,+)$ part (*i.e.*, positive on the x variables and positive on the y variables), the $(-,+)$ part and the $(0,-)$ part (*i.e.*, negative on the y variables with all the x variables 0); since the last space cuts out the empty set on T, we will repeatedly ignore it. Denote these spaces by $\mathrm{Fix}_{(+,+)}$ and $\mathrm{Fix}_{(-,+)}$. Then, referring to Table 1, we see that $\mathrm{Fix}_{(+,+)}(\alpha_1\beta_2)$ is equal to:

$$(x_{00}-x_{01}=x_{10}-x_{11}=x_{20}=x_{21}=y_{abcd}+y_{a'b'cd}=0,\forall_{abcd\in\mathcal{L}}).$$

¿From (2.1) we get $x_{00}x_{01}=x_{10}x_{11}=0$ and hence $x_{00}=x_{01}=x_{10}=x_{11}=0$. Thus all coordinates x_{ia} vanish except for, possibly, x_{30} or x_{31}. ¿From the quartic relation $y_{abcd}y_{a'b'cd}=x_{2c'}^2x_{3d'}^2=0$, *cf.* (2.4), and $y_{abcd}+y_{a'b'cd}=0$ we deduce that $y_{abcd}=0$ for all $(a,b,c,d)\in\mathcal{L}$. Using $q=0$ we obtain $x_{30}=x_{31}=0$. Hence T does not meet $\mathrm{Fix}_{(+,+)}(\alpha_1\beta_2)$.

Next we consider the loci $\mathrm{Fix}_{(-,+)}(\alpha_1\beta_2)$, $\mathrm{Fix}_{(+,+)}(\alpha_1\alpha_2\beta_2\beta_3)$, $\mathrm{Fix}_{(-,+)}(\alpha_1\alpha_2\beta_2\beta_3)$ and $\mathrm{Fix}_{(+,+)}(\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3)$ which are given by:

$$(x_{00}+x_{01}=x_{10}+x_{11}=x_{30}=x_{31}=y_{abcd}+y_{a'b'cd}=0,\forall_{abcd\in\mathcal{L}}),$$

$$(x_{10}-x_{11}=x_{20}+x_{21}=x_{30}=x_{31}=y_{abcd}-y_{ab'c'd}=0,\forall_{abcd\in\mathcal{L}}),$$

$$(x_{00}=x_{01}=x_{10}+x_{11}=x_{20}-x_{21}=y_{abcd}-y_{ab'c'd}=0,\forall_{abcd\in\mathcal{L}}),$$

$$(x_{00}-x_{01}=x_{10}+x_{11}=x_{20}+x_{21}=x_{30}+x_{31}=y_{abcd}+y_{a'b'c'd'}=0,\forall_{abcd\in\mathcal{L}}),$$

respectively. Arguing as before (remembering, for the last locus, that $x_{00}+x_{01}=0$ holds) we see that none of them meets T.

Finally $\mathrm{Fix}_{(-,+)}(\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3)$ is given by:

$$(x_{00}+x_{01}=x_{10}-x_{11}=x_{20}-x_{21}=x_{30}-x_{31}=y_{abcd}+y_{a'b'c'd'}=0,\forall_{abcd\in\mathcal{L}}).$$

Using (2.1) we get $x_{j0}^2=-x_{00}^2$. Hence $s_j=-s_0$, for all $j=1,2,3$. ¿From the quartic equations (2.4) we get $-y_{abcd}^2=y_{abcd}y_{a'b'c'd'}=x_{20}x_{21}x_{30}x_{31}=x_{00}^4$. Taking square roots of this equation, substituting in $q=0$ and using the generality of ν_0, ν_1, ν_2, ν_3, ν_4 we deduce that $x_{00}=0$; and hence $x_{01}=x_{j0}=x_{j1}=0$ for all $j=1,2,3$ and $y_{abcd}=0$ for all $(a,b,c,d)\in\mathcal{L}$. Therefore $T\cap\mathrm{Fix}_{(-,+)}(\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3)=\emptyset$.

Since the action of G on T is free, $S=T/G$ is a nonsingular surface of general type with $p_g(S)=0$ and $K_S^2=3$. Since K_T is ample, we deduce that K_S is ample. □

Remark 2.12. Theorem 2.11 shows that for every $T \in \mathcal{N}$ such that

- T has at most canonical singularities,
- the action of G on T is free,

the quotient $S = T/G$ is the canonical model of a surface of general type with $p_g(S) = 0$ and $K^2 = 3$: this provides a 4-dimensional family of these surfaces.

Remark 2.13. By analysis of the proof of Theorem 2.11, we see that if, for a given $T \in \mathcal{N}$, the action of G has any fixed points on T then either $\nu_1\nu_2\nu_3 = 0$ or there exists δ in a finite set of (integer) multiples of i such that $\nu_0 - \nu_1 - \nu_2 - \nu_3 + \delta\nu_4 = 0$. We shall use this observation later on.

3. A DOUBLE COVER

Consider the Fano 4-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates (t_{00}, t_{01}), (t_{10}, t_{11}), (t_{20}, t_{21}), (t_{30}, t_{31}), and let $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}(1^8, 2^8)$ be the map given by:

$$\sigma^\sharp(x_{0a}) = t_{0a'}t_{1a}t_{2a}t_{3a}, \quad \sigma^\sharp(x_{1a}) = t_{0a}t_{1a'}t_{2a}t_{3a},$$
$$\sigma^\sharp(x_{2a}) = t_{0a}t_{1a}t_{2a'}t_{3a}, \quad \sigma^\sharp(x_{3a}) = t_{0a}t_{1a}t_{2a}t_{3a'},$$
$$\sigma^\sharp(y_{abcd}) = t_{0a'}^2 t_{1b'}^2 t_{2c'}^2 t_{3d'}^2, \text{ if } a = b = c = d \text{ and } \sigma^\sharp(y_{abcd}) = t_{0a}^2 t_{1b}^2 t_{2c}^2 t_{3d}^2 \text{ otherwise.}$$

It is straightforward to check that $\sigma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = Y \subset \mathbb{P}(1^8, 2^8)$.

Proposition 3.1. *The map $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to Y$ is finite of degree 2 branched exactly at the set $\{\mathbf{x}_{ia}, \mathbf{y}_{abcd} : 0 \le i \le 3, (a, b, c, d) \in \mathcal{L}\}$.*

Proof. Let $U_{ia} \subset Y$ be the open subset of Y given by $x_{ia} \neq 0$. First note that $\sigma^{-1}(\mathbf{y}_{abcd})$ consists of a point, more precisely one of the coordinate points of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, the family $\{U_{ia}\}$, with $0 \le i \le 3$ and $a \in \{0, 1\}$ is an open affine cover of $Y \setminus \{\mathbf{y}_{abcd}\}$. Consider $\sigma_| \colon \sigma^{-1}(U_{00}) \to U_{00}$. The open set $\sigma^{-1}(U_{00})$ is simply $\mathbb{C}^4$ with coordinates

$$\frac{t_{00}}{t_{01}}, \frac{t_{11}}{t_{10}}, \frac{t_{21}}{t_{20}}, \frac{t_{31}}{t_{30}}.$$

The coordinate ring of U_{00}, which we denote by $\mathbb{C}[U_{00}]$, is generated by the regular functions:

$$\frac{x_{ia}}{x_{00}}, \quad \frac{y_{abcd}}{x_{00}^2}, \quad \text{with } 0 \le i \le 3 \text{ and } (a, b, c, d) \in \mathcal{L}.$$

Computing the image by $\sigma_|^\sharp$ of each of the generators of $\mathbb{C}[U_{00}]$, we get the generators of the ideal $(\frac{t_{00}}{t_{01}}, \frac{t_{11}}{t_{10}}, \frac{t_{21}}{t_{20}}, \frac{t_{31}}{t_{30}})^2$. Hence $\sigma_| \colon \sigma^{-1}(U_{00}) \to U_{00}$ is finite of degree 2. The same computation on each U_{ia} yields the same result, showing that σ is a double cover. Additionally, the involution $s \in \operatorname{Aut}\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right)$ given by

$$s(t_{ia}) = (-1)^a t_{ia}, \quad \text{for } 0 \le i \le 3 \text{ and } a \in \{0, 1\} \tag{3.1}$$

satisfies $\sigma \circ s = \sigma$. Note that s has exactly 16 fixed points, the coordinate points of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In particular, σ branches exactly at their images, *i.e.* the points in the set $\{\mathbf{x}_{ia}, \mathbf{y}_{abcd} : 0 \le i \le 3, (a, b, c, d) \in \mathcal{L}\}$. □

Remark 3.2. We can deduce from Proposition 3.1 that $\operatorname{Sing} Y$ is the set of 16 points $\{\mathbf{x}_{ia}, \mathbf{y}_{abcd} : 0 \leq i \leq 3, (a, b, c, d) \in \mathcal{L}\}$, which are quotient singularities of type $\frac{1}{2}(1, 1, 1, 1)$. This agrees with Proposition 2.9.

Remark 3.3. The restriction of σ to the Fano 3-fold

$$Z_1 = (t_{01}t_{10}t_{20}t_{30} + t_{00}t_{11}t_{21}t_{31} = 0) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \tag{3.2}$$

is a double cover of V, branched on the 14 singularities of V. The 3-fold Z_1 is a (special) member of $|\mathcal{O}(1, 1, 1, 1)|^-$, the linear system of effective divisors on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1, 1, 1, 1)$ anti-invariant respect to the involution s. A general member of $|\mathcal{O}(1, 1, 1, 1)|^-$ is the canonical double cover of an Enriques–Fano 3-fold with only terminal singularities. These 3-folds were classified by Bayle and Sano [Ba, S]. The image of a general member of $|\mathcal{O}(1, 1, 1, 1)|^-$ under σ falls in case 10 of Sano's list. Indeed the whole construction in this section has been inspired by that case.

Recall that $(\mathbb{Z}/2)^6$ acts on Y as given in Table 1.

Table 2. Automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. (For the last 4, since the action is diagonal we list only the eigenvalues. Here ϵ is a square-root of -1.)

	t_{00}	t_{01}	t_{10}	t_{11}	t_{20}	t_{21}	t_{30}	t_{31}
$\tilde{\alpha}_1$	t_{10}	t_{11}	t_{00}	t_{01}	t_{31}	t_{30}	t_{21}	t_{20}
$\tilde{\alpha}_2$	t_{20}	t_{21}	t_{31}	t_{30}	t_{00}	t_{01}	t_{11}	t_{10}
$\tilde{\alpha}_3$	t_{30}	t_{31}	t_{21}	t_{20}	t_{11}	t_{10}	t_{00}	t_{01}
$\tilde{\beta}_1$	-ϵ	1	1	$-\epsilon$	1	ϵ	1	ϵ
$\tilde{\beta}_2$	-ϵ	1	1	ϵ	1	$-\epsilon$	1	ϵ
$\tilde{\beta}_3$	-ϵ	1	1	ϵ	1	ϵ	1	$-\epsilon$
s	1	-1	1	-1	1	-1	1	-1

In Table 2, we distinguish a set of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, one of which, s, has already been defined in (3.1) and the remaining ones are meant to lift the actions of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$. One can check by direct computation that $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3$ lift the action of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, i.e., that $\sigma \circ \tilde{\alpha}_i = \alpha_1 \circ \sigma$ and $\sigma \circ \tilde{\beta}_i = \beta_1 \circ \sigma$, for $i = 1, 2, 3$. On the other hand, there are a number of small checks that are straightforward. It is clear that s commutes with every other automorphism listed in Table 2; it is also clear that $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ are automorphisms of order 2 commuting with each other, that $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3$ commute with each other and that $\tilde{\beta}_1^2 = \tilde{\beta}_2^2 = \tilde{\beta}_3^2 = s$. Finally, a less straightforward (but still elementary) computation shows that $\tilde{\alpha}_i \tilde{\beta}_j = s^{\delta_{ij}} \tilde{\beta}_j \tilde{\alpha}_i$, where δ_{ij} is Kronecker's delta.

These identities are useful in the proof of the next proposition, where we characterize the group $\tilde{G}$ generated by the automorphisms that lift the generators of $G = \langle \alpha_1\beta_2, \alpha_2\beta_3, \alpha_3\beta_1 \rangle \simeq (\mathbb{Z}/2)^3$.

Lemma 3.4. *$\tilde{G} := \langle \tilde{\alpha}_1\tilde{\beta}_2, \tilde{\alpha}_2\tilde{\beta}_3, \tilde{\alpha}_3\tilde{\beta}_1 \rangle$ is isomorphic to $\mathbb{Z}/2 \times Q_8$, where Q_8 is the classical quaternion group.*

Proof. Since $\deg \sigma = 2$, $|\tilde{G}|$ equals either $2|G|$, if $s \in \tilde{G}$, or $|G|$, if $s \notin \tilde{G}$. Since $(\tilde{\alpha_1}\tilde{\beta_2})^2 = s$, we get $|\tilde{G}| = 2\,|G| = 16$. Consider the standard presentation of Q_8 given by

$$\langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$$

and, for clarity, let us use multiplicative notation for $\mathbb{Z}/2 = \{1, -1\}$. Set:

$$\mu(1,-1) = s, \quad \mu(-1,1) = \tilde{\alpha}_1\tilde{\beta}_2\tilde{\alpha}_2\tilde{\beta}_3\tilde{\alpha}_3\tilde{\beta}_1,$$
$$\mu(1,i) = \tilde{\alpha}_2\tilde{\beta}_3\tilde{\alpha}_3\tilde{\beta}_1, \quad \mu(1,j) = \tilde{\alpha}_3\tilde{\beta}_1\tilde{\alpha}_1\tilde{\beta}_2, \quad \mu(1,k) = \tilde{\alpha}_1\tilde{\beta}_2\tilde{\alpha}_2\tilde{\beta}_3.$$

Using the identities stated earlier, one can check easily that these definitions respect all the relations of $(\mathbb{Z}/2) \times Q_8$ and therefore determine a group homomorphism $\mu\colon (\mathbb{Z}/2) \times Q_8 \to \tilde{G}$. Since:

$$\mu(-1,-i) = \mu(-1,1)\mu(1,i)^{-1} = \tilde{\alpha}_1\tilde{\beta}_2\tilde{\alpha}_2\tilde{\beta}_3\tilde{\alpha}_3\tilde{\beta}_1\tilde{\beta}_1^{-1}\tilde{\alpha}_3\tilde{\beta}_3^{-1}\tilde{\alpha}_2 = \tilde{\alpha}_1\tilde{\beta}_2,$$
$$\mu(-1,-j) = (\tilde{\alpha}_1\tilde{\beta}_2)\tilde{\alpha}_2\tilde{\beta}_3\tilde{\alpha}_3\tilde{\beta}_1(\tilde{\beta}_2^{-1}\tilde{\alpha}_1)\tilde{\beta}_1^{-1}\tilde{\alpha}_3 = \tilde{\alpha}_2\tilde{\beta}_3\tilde{\alpha}_3\tilde{\beta}_1\tilde{\beta}_1^{-1}\tilde{\alpha}_3 = \tilde{\alpha}_2\tilde{\beta}_3,$$
$$\mu(-1,-k) = \tilde{\alpha}_1\tilde{\beta}_2(\tilde{\alpha}_2\tilde{\beta}_3)\tilde{\alpha}_3\tilde{\beta}_1(\tilde{\beta}_3^{-1}\tilde{\alpha}_2)\tilde{\beta}_2^{-1}\tilde{\alpha}_1 = s\tilde{\alpha}_1\tilde{\beta}_2\tilde{\alpha}_3\tilde{\beta}_1\tilde{\beta}_2^{-1}\tilde{\alpha}_1 = \tilde{\alpha}_3\tilde{\beta}_1,$$

we deduce that μ is surjective, which, as $|\tilde{G}| = |(\mathbb{Z}/2) \times Q_8|$, implies that μ is an isomorphism. ☐

We can now give a good description of the family of surfaces T/G, for general T in the linear system $\mathcal{N}$.

Theorem 3.5. *Let $T \in \mathcal{N}$ be a surface with at most canonical singularities for which the action of G on it is free. Then $\pi_1(T/G) \cong \mathbb{Z}/2 \times Q_8$ and the universal cover of T is a complete intersection of the two hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, Z_1 and Z_2, of multi-degrees $(1,1,1,1)$ and $(2,2,2,2)$, respectively, given by:*

$$Z_1 = (t_{01}t_{10}t_{20}t_{30} + t_{00}t_{11}t_{21}t_{31} = 0) \text{ and}$$

$$Z_2 = \sum_{i=0}^{3} \nu_i \left(t_{i0}^2 \prod_{j\neq i} t_{j1}^2 + t_{i1}^2 \prod_{j\neq i} t_{j0}^2 \right) - 2\nu_4 \sum_{abcd\in\mathcal{L}} (-1)^{\frac{b+c+d-a}{2}} t_{0a}^2 t_{1b}^2 t_{2c}^2 t_{3d}^2 = 0.$$

Proof. We note that T does not contain any of the 16 points in the set

$$\{\mathbf{x}_{ia}, \mathbf{y}_{abcd} \mid 0 \le i \le 3, (a,b,c,d) \in \mathcal{L}\}.$$

Indeed T is a Cartier divisor in V, which contains 14 of these points that, by Proposition 2.9, are singular points of V with Zariski tangent space of dimension 5 or 8. In particular, if T contains one of these points, the Zariski tangent space of T at this point has at least dimension 4, whereas every canonical singularity of a surface has Zariski tangent space of dimension 3. Since T is the complete intersection of two divisors (V and a quadric section, given by $x_{00} + x_{01} = 0$ and

(2.15), respectively) in Y, the surface $\tilde{T} := \sigma^{-1}(T)$ is the complete intersection of their pull-back to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which one easily sees are the hypersurfaces Z_1 and Z_2, respectively, of the statement of this theorem. By the Leftschetz hyperplane section theorem, $\pi_1(\tilde{T}) = 0$. Now as the composition $\tilde{T} \overset{\sigma|}{\to} T \to S$ is étale, we conclude that $\tilde{T}$ is the universal cover of S. In particular, $\pi_1(S)$ is isomorphic to the group of automorphisms of the cover, which coincides with the group of automorphisms of $\tilde{T}$ lifting the action of G. This is $\tilde{G}$, which, by Lemma 3.4, is isomorphic to $\mathbb{Z}/2 \times Q_8$. □

We conclude this section by studying the locus of the moduli space of the surfaces of general type described by the surfaces S.

Theorem 3.6. *Let $\mathcal{U}$ be the dense open set of $\mathcal{N} \cong \mathbb{P}^4$ consisting of the surfaces T with at most canonical singularities on which G acts freely. Then, the map associating to each point of $\mathcal{U}$ the class of the surface S/G, in the Gieseker moduli space of surfaces of general type with $\chi = 1$ and $K^2 = 3$, is finite. In particular, its image is 4-dimensional and unirational.*

Proof. Let $S_1 := T_1/G$, $S_2 := T_2/G$ be surfaces with $T_1, T_2 \in \mathcal{U}$. Assume that $S_1 \cong S_2$. By Theorem 3.5, $\pi_1(S_1) \cong \pi_1(S_2) \cong (\mathbb{Z}/2) \times Q_8$. Since the Abelianization of $(\mathbb{Z}/2) \times Q_8$ is $(\mathbb{Z}/2)^3$, each S_i has exactly one $(\mathbb{Z}/2)^3$-cover up to isomorphism. Therefore from $S_1 \cong S_2$, it follows that $T_1 \cong T_2$. This isomorphism induces an isomorphism of the canonical rings of T_1 and T_2. Choose an automorphism Φ of $\mathbb{P}(1^8, 2^8)$ that lifts the isomorphism between $\operatorname{Proj} R(T_1, K_{T_1})$ and $\operatorname{Proj} R(T_2, K_{T_2})$. Note that Φ is not unique, as the image by $\Phi^\sharp$ of each generator of the underlying polynomial ring is determined only modulo the ideal of T_2 and therefore, in particular, $\Phi^\sharp(x_{ia})$ is determined only up to $x_{00} + x_{10}$. In what follows we show that $\Phi_{|(x_{00}+x_{01}=0)}$ belongs to a finite set.

The restriction of the isomorphism $\Phi^\sharp$ to the variables of degree 1 yields an automorphism of $\mathbb{P}^7$, which we denote by $\tilde{\Phi}$, mapping the canonical image of T_1 to the canonical image of T_2. In particular $\tilde{\Phi}$ preserves the hyperplane $(x_{00} + x_{01} = 0)$, which is the linear span of both surfaces, and $W \subset \mathbb{P}^7$, given by (2.1), which is their quadric hull. Recall that, by Remark 2.3, for every $(a, b, c, d) \in \mathcal{L}$, H_{abcd} is the divisor of poles of the rational function $\varphi^\sharp(y_{abcd}) = \varphi_{abcd}$ on W. Let us consider the 8 planes given by $\Pi_{abcd} := H_{abcd} \cap (x_{00} + x_{01} = 0) \subset W$. Then $\tilde{\Phi}^{-1}(\Pi_{abcd})$ is a plane, it is contained in W, and it is the intersection of $(x_{00} + x_{01} = 0)$ with the divisor of the poles of $\varphi^\sharp \Phi^\sharp(y_{abcd})$. Since $\Phi^\sharp(y_{abcd}) \in \mathrm{S}^2 \langle x_{ia} \rangle \oplus \langle y_{abcd} \rangle$, we deduce that $\tilde{\Phi}^{-1}$ permutes the 8 planes Π_{abcd}. The first consequence is that $\tilde{\Phi}^{-1}$ preserves the linear span of these 8 planes, $(x_{00} = x_{10} = 0)$. Now, as we are only interested on $\Phi_{|(x_{00}+x_{01}=0)}$, we may modify Φ so that $\Phi^\sharp(x_{0a}) = \lambda x_{0a}$, for $a = 0, 1$ and for some $\lambda \in \mathbb{C}^*$. Then, rescaling $\Phi^\sharp$, and thus still without changing $\Phi_{|(x_{00}+x_{01}=0)}$, we may finally assume that $\Phi^\sharp(x_{0a}) = x_{0a}$. Another consequence of the fact that $\tilde{\Phi}^{-1}$ permutes the 8 planes Π_{abcd} is that there exists $\tau \in \mathfrak{S}_3$, $(a, b, c, d) \in \mathcal{L}$, with $a = 0$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$ such that $\Phi^\sharp(x_{10}) = \lambda_1 x_{\tau(1)b}$,

$\Phi^\sharp(x_{20}) = \lambda_2 x_{\tau(2)c}$ and $\Phi^\sharp(x_{30}) = \lambda_3 x_{\tau(3)b}$. Since $\Phi^\sharp$ must also preserve (2.1) we deduce that $\Phi^\sharp(x_{11}) = \lambda_1^{-1} x_{\tau(1)b'}$, $\Phi^\sharp(x_{21}) = \lambda_2^{-1} x_{\tau(2)c'}$ and $\Phi^\sharp(x_{31}) = \lambda_3^{-1} x_{\tau(3)b'}$.

Consider the action of $\mathfrak{S}_3$ on $\mathbb{P}(1^8, 2^8)$ given, for every $\tau \in \mathfrak{S}_3$ by,

$$\tau^\sharp(x_{0a}) = x_{0a}, \quad \tau^\sharp(x_{ia}) = x_{\tau(i)a}, \quad \tau^\sharp(y_{a_0a_1a_2a_3}) = y_{a_0 a_{\tau^{-1}(1)} a_{\tau^{-1}(2)} a_{\tau^{-1}(3)}},$$

for all $1 \leq i \leq 3$ and $(a,b,c,d) \in \mathcal{L}$. We note that given $\tau \in \mathfrak{S}_3$, we have $\tau\alpha_i\tau^{-1} = \alpha_{\tau(i)}$ and $\tau\beta_i\tau^{-1} = \beta_{\tau(i)}$, where, recall, α_i and β_i generate $(\mathbb{Z}/2)^6$ and act on $\mathbb{P}(1^8, 2^8)$ as given in Table 1. These actions generate a finite group Λ of automorphisms of $\mathbb{P}(1^8, 2^8)$ preserving V and Y which is a semidirect product, $\Lambda \cong (\mathbb{Z}/2)^6 \rtimes \mathfrak{S}_3$. Accordingly, going back to Φ, there exists $\Psi \in \Lambda$ and constants $\lambda_{ia} \in \mathbb{C}^*$ such that

$$(\Phi \circ \Psi)^\sharp(x_{ia}) = (\Psi^\sharp \circ \Phi^\sharp)(x_{ia}) = \lambda_{ia} x_{ia}$$

with $\lambda_{00} = \lambda_{01} = 1$ and $\lambda_{i1} = \lambda_{i0}^{-1}$. Notice that from the cubic relations (2.3) we get $(\Phi \circ \Psi)^\sharp(y_{abcd}) = \lambda_{1b'}\lambda_{2c'}\lambda_{3d'} y_{abcd}$. Since for both $T_1, T_2 \in \mathcal{U} \subset \mathcal{N}$ we must have $\nu_4 \neq 0$ (for otherwise T_1 or T_2 would be too singular), from the equation of the quadric section (2.15), we deduce that the products $\lambda_{1b'}\lambda_{2c'}\lambda_{3d'}$, for $(a,b,c,d) \in \mathcal{L}$ are all equal. This can only happen if $\lambda_{i1} = \lambda_{i0}$ for $i \in \{1,2,3\}$. Hence $\lambda_{ia} \in \{\pm 1\}$, for $i = 1,2,3$ and thus there are only finitely many possibilities for $\Phi_{|(x_{00}+x_{01})=0} \circ \Psi$. Since Ψ belongs to the finite group Λ, we deduce that there are, as well, only finitely many possibilities for $\Phi_{|(x_{00}+x_{01})=0}$. □

4. The bicanonical map of S

The main goal of this section is to compare the surfaces we have constructed with the other constructions existing in literature. To reach this goal we study the bicanonical map of S, which is interesting in its own right. We show that, as in the Burniat case, the bicanonical map of S is a bidouble cover of a cubic surface in $\mathbb{P}^3$ with 3 nodes, which is rational. This induces a birational description of these surfaces as bidouble covers of the plane. We compute the branch divisors, and then identify the Burniat surfaces. Not surprisingly, the branch divisors corresponding to a general surface in our family (cf. Figure 4) correspond exactly to the one used in the recent paper [BC3] to define the *extended* tertiary Burniat surfaces.

Consider the action of $(\mathbb{Z}/2)^6$ on V as given in Table 1. For a subgroup of $(\mathbb{Z}/2)^6$ to act on T it must preserve the equation $q = 0$. An element of $(\mathbb{Z}/2)^6$, written as $\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\beta_1^{b_1}\beta_2^{b_2}\beta_3^{b_3}$, sends $q = 0$ to a scalar multiple of it if and only if the integer $a_1 + a_2 + a_3 + b_1 + b_2 + b_3$ is even. Let H be the subgroup of $(\mathbb{Z}/2)^6$ given by

$$H = \left\{ \alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\beta_1^{b_1}\beta_2^{b_2}\beta_3^{b_3} \in (\mathbb{Z}/2)^6 \mid a_1 + a_2 + a_3 + b_1 + b_2 + b_3 \text{ is even} \right\}. \tag{4.1}$$

The group G defined in (2.10) is obviously a subgroup of H. Hence the quotient $\Gamma := H/G \cong (\mathbb{Z}/2)^2$ acts on $S = T/G$. Denote by $\gamma\colon S \to S/\Gamma$ the quotient morphism. In the next proposition we show that γ is the bicanonical map of S.

Proposition 4.1. *Let $T \in \mathcal{N}$ be a surface with at most canonical singularities and such that the action of G on it is fixed-point free. Consider $S = T/G$. Then,* *the bicanonical map of S is a bidouble cover of the cubic surface $S_3 \subset \mathbb{P}^3$ given by* $8\nu_4^2(s_1 - s_0)(s_2 - s_0)(s_3 - s_0) - s_0(\nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3)^2 = 0$.

Proof. The bicanonical system of S is generated by the 4 invariants quadratic forms s_0, s_1, s_2, s_3. We showed in the proof of Theorem 2.11 that $s_0 = s_1 = s_2 = s_3 = 0$ cuts out the empty set on T; therefore $|2K_S|$ has no fixed part and no base points. Since S is a minimal surface of general type with $p_g = 0$ and $K^2 \geq 2$, by [X], the bicanonical system is not composed with a pencil. Hence the image of φ_{2K_S} is a surface. To find its equation, we square both sides of (2.17):

$$\nu_4^2(x_{10} + x_{11})^2(x_{20} + x_{21})^2(x_{30} + x_{31})^2 = x_{00}^2 l^2$$

and use $(x_{i0} + x_{i1})^2 = 2(s_i + t_i) = 2(s_i - s_0)$, for $i = 1, 2, 3$, and $s_0 = x_{00}^2$, *cf.* (2.13), (2.14). Substituting, we get $8\nu_4^2(s_1 - s_0)(s_2 - s_0)(s_3 - s_0) - s_0 l^2 = 0$. For a general choice of $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4$ this cubic is irreducible, hence the cubic surface $S_3 \subset \mathbb{P}^3$ it defines coincides with $\varphi_{2K_S}(S)$. S has no (-2)-curves, as by construction K_S is ample, thus φ_{2K_S} is a finite morphism of degree 4. Since s_0, s_1, s_2, s_3 are invariant for the action of H on T, φ_{2K_S} factors through γ, which is also a finite morphism of degree 4. Hence, since S_3 is normal (*cf.* Remark 4.2), $S/\Gamma \cong S_3$ and, up to isomorphism, $\varphi_{2K_S} = \gamma$. $\square$

Remark 4.2. For general $\nu_0, \dots, \nu_4$, the cubic $S_3 \subset \mathbb{P}^3$ has 3 ordinary double points:

$$\begin{aligned} n_1 &= (s_2 - s_0 = s_3 - s_0 = \nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3 = 0), \\ n_2 &= (s_1 - s_0 = s_3 - s_0 = \nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3 = 0), \\ n_3 &= (s_1 - s_0 = s_2 - s_0 = \nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3 = 0); \end{aligned} \tag{4.2}$$

and these are the only singularities of S_3.

Let us denote by $\theta_i \in \Gamma = H/G$ the class of $\alpha_i\beta_i$, *i.e.*,

$$\theta_i = [\alpha_i\beta_i] = \{\alpha_i\beta_i g \mid g \in G\}. \tag{4.3}$$

By Proposition 4.1, φ_{2K_S} is the quotient by the action of $\Gamma = \{1, \theta_1, \theta_2, \theta_3\}$. To study this map, by the general theory of the bidouble covers (see [C]), we study its branch locus. A ramification point of φ_{2K_S} is the image of a point $\mathbf{x} \in S$ fixed by some θ_i, *i.e.*, for which $I_{\mathbf{x}} \neq \{1\}$, where $I_{\mathbf{x}} = \{g \in \Gamma \mid g\mathbf{x} = \mathbf{x}\}$ is the inertia group of $\mathbf{x}$. When S is smooth, there are 3 possibilities for $I_{\mathbf{x}}$:

(a) $I_{\mathbf{x}} = \langle\theta_i\rangle$ and $\mathbf{x}$ is an isolated fixed point of θ_i. Then, in suitable local coordinates, θ_i acts by $(z_1, z_2) \mapsto (-z_1, -z_2)$ and $\varphi_{2K_S}(\mathbf{x})$ is a node.

(b) $I_{\mathbf{x}} = \langle\theta_i\rangle$ and $\mathbf{x}$ is not an isolated fixed point of θ_i. Then, in local coordinates, θ_i acts by $(z_1, z_2) \mapsto (-z_1, z_2)$ and the locus of all such points is a smooth curve $R_i \subset S$.

(c) $I_{\mathbf{x}} = \Gamma$. Then $\mathbf{x}$ belongs to exactly two R_i, intersecting transversally in $\mathbf{x}$.

Let $D_i := \varphi_{2K_S}(R_i)$ and denote by Δ_i the image of the set of isolated fixed points of θ_i the inertia group of which is not the whole of Γ — as in type (a), above. Then each Δ_i is a set of nodes of S_3. The bidouble cover is determined by D_1, D_2, D_3 and $\Delta_1, \Delta_2, \Delta_3$.

To describe D_i we introduce some notation. The intersection of S_3 with the plane $s_0 + s_i = 0$ splits as the union of a line with a conic. Denote these by L_i and C_i, respectively. In other words, set

$$\begin{gathered} L_i = (s_0 = s_i = 0) \text{ and} \\ C_i = (s_0 + s_i = 16\nu_4^2(s_{i+1} - s_0)(s_{i+2} - s_0) + l^2 = 0), \end{gathered} \tag{4.4}$$

taking the indices in $\{1, 2, 3\}$, modulo 3.

Proposition 4.3. *Let D_i, Δ_i, for $i = 1, 2, 3$, be the branch loci of the map $\varphi_{2K_S} \colon S \to S_3 \subset \mathbb{P}^3$. Then $\Delta_i = \{n_i\}$ and $D_i = C_{i+1} + L_{i-1}$, taking indices in $\{1, 2, 3\}$, modulo* 3.

Proof. By cyclic symmetry, it is enough to compute Δ_1 and D_1, *i.e.*, the images of the points of S that are fixed by θ_1 but not fixed by θ_2, θ_3. On the other hand, the fixed points of θ_1 are the images on S of the points of T fixed by an element of $[\alpha_1\beta_1]$. Recall that the elements of $\theta_1 = [\alpha_1\beta_1]$ are $\alpha_1\beta_1$, $\beta_1\beta_2$, $\alpha_1\alpha_2\beta_1\beta_3$, $\alpha_1\alpha_3$, $\alpha_2\beta_1\beta_2\beta_3$, $\alpha_3\beta_2$, $\alpha_1\alpha_2\alpha_3\beta_3$ and $\alpha_2\alpha_3\beta_2\beta_3$.

$\mathrm{Fix}_{(+,+)}(\alpha_2\beta_1\beta_2\beta_3)$ and $\mathrm{Fix}_{(-,+)}(\alpha_2\beta_1\beta_2\beta_3)$ are given by:

$$\begin{gathered} (x_{00} - x_{01} = x_{10} = x_{11} = x_{20} + x_{21} = x_{30} = x_{31} = y_{abcd} + y_{a'bc'd} = 0, \forall_{abcd \in \mathcal{L}}), \\ (x_{00} + x_{01} = x_{20} - x_{21} = y_{abcd} + y_{a'bc'd} = 0, \forall_{abcd \in \mathcal{L}}), \end{gathered}$$

respectively. We have $\mathrm{Fix}_{(+,+)}(\alpha_2\beta_1\beta_2\beta_3) \cap T = \emptyset$. This can be seen either directly on T or by noticing that its image in S_3 must have $s_0 = s_1 = s_3 = 0$ and, by (2.1), $s_2 = 0$. In $\mathrm{Fix}_{(-,+)}(\alpha_2\beta_1\beta_2\beta_3) \cap T$ we have $x_{00} = -x_{01}$ and $x_{20} = x_{21}$, which by (2.1) imply that $s_0 + s_2 = 0$. We deduce that the image of $\mathrm{Fix}_{(-,+)}(\alpha_2\beta_1\beta_2\beta_3) \cap T$ in S_3 is contained in $L_2 \cup C_2$. Suppose that $s_0 = s_2 = 0$. Then

$$x_{00} = x_{01} = x_{20} = x_{21} = x_{10}x_{11} = x_{30}x_{31} = 0.$$

Assume that $x_{10} = x_{30} = 0$. Then using (2.4), we get $y_{abcd}^2 = -y_{abcd}y_{a'bc'd} = 0$, for all $(a, b, c, d) \in \mathcal{L} \setminus \{(0, 0, 0, 0), (1, 0, 1, 0)\}$. Hence we are left with the 2 equations:

$$-y_{0000}^2 = x_{11}^2 x_{31}^2 = 4s_1s_3 \quad \text{and} \quad 2\nu_4 y_{0000} + l = 0,$$

given by (2.4) and $q = 0$. Eliminating y_{0000}, we get the equation of C_2 with $s_0 = s_2 = 0$. This is independent of the choices we made. We deduce that the image of $\mathrm{Fix}_{(-,+)}(\alpha_2\beta_1\beta_2\beta_3) \cap T$ is contained in C_2. To see that the image of this locus coincides with C_2 it suffices to check that it is 1-dimensional. The equations $x_{00} + x_{01} = 0$ and $x_{20} - x_{21} = 0$ define in W a 2-dimensional subscheme (in fact, $x_{00} + x_{01}$ is an equation of W). It is clear that this subscheme is not contained in the exceptional locus of $\varphi \colon W \dashrightarrow V$. Denote by Z its strict transform in V. Assume $x_{00}, x_{01} \neq 0$. Then

$$y_{abcd}x_{0a} = x_{1b'}x_{2c'}x_{3d'} = x_{1b'}x_{2c}x_{3d'} = -y_{a'bc'd}x_{0a'} \implies y_{abcd} = -y_{a'bc'd}.$$

Hence on the open set $x_{00} \neq 0$ of $Z \subset V$, the equations $y_{abcd} + y_{a'bc'd} = 0$ are redundant. Hence $\dim Z = 2$. Since we obtain T from V by taking a hypersurface section $(q = 0)$ we deduce that $\mathrm{Fix}_{(-,+)}(\alpha_2\beta_1\beta_2\beta_3) \cap T$ is 1-dimensional. We conclude that the fixed points of $\alpha_2\beta_1\beta_2\beta_3$ do not contribute to Δ_1 and that their contribution to D_1 is C_2.

$\mathrm{Fix}_{(+,+)}(\beta_1\beta_2)$ and $\mathrm{Fix}_{(-,+)}(\beta_1\beta_2)$ are given by:

$$(x_{10} = x_{11} = x_{20} = x_{21} = 0) \quad \text{and} \quad (x_{00} = x_{01} = x_{30} = x_{31} = 0), \tag{4.5}$$

respectively. The image of $\mathrm{Fix}_{(-,+)}(\beta_1\beta_2) \cap T$ equals $L_3 = (s_0 = s_3 = 0)$: it is clearly contained in L_3 and the equality follows since $\mathrm{Fix}_{(-,+)}(\beta_1\beta_2) \supset \mathcal{S}_{11}^{12}$ and $\mathcal{S}_{11}^{12} \cap T$ is positive dimensional. Hence $L_3 \subset D_1$. Notice that by symmetry of the indices we have just shown that $L_1 \subset D_2$ and $L_2 \subset D_3$. For the image of $\mathrm{Fix}_{(+,+)}(\beta_1\beta_2) \cap T$, in S_3 we get $s_1 = s_2 = 0$ and then $s_0 = 0$, which means that $\mathrm{Fix}_{(+,+)}(\beta_1\beta_2) \cap T$ consists of the preimages of the point $L_1 \cap L_2$; these are points in $R_2 \cap R_3$ — type (c) above. We conclude that the fixed points of $\beta_1\beta_2$ do not contribute to Δ_1 and that their contribution to D_1 is L_3.

$\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3)$ and $\mathrm{Fix}_{(-,+)}(\alpha_2\alpha_3\beta_2\beta_3)$ are given by:

$$(x_{20} + x_{21} = x_{30} + x_{31} = y_{abcd} - y_{abc'd'} = 0, \forall_{abcd \in \mathcal{L}}),$$
$$(x_{00} = x_{01} = x_{10} = x_{11} = x_{20} - x_{21} = x_{30} - x_{31} = y_{abcd} - y_{abc'd'} = 0, \forall_{abcd \in \mathcal{L}}),$$

respectively. The locus $\mathrm{Fix}_{(-,+)}(\alpha_2\alpha_3\beta_2\beta_3) \cap T$ is clearly empty. For the image of the locus $\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3) \cap T$ we get $s_0 - s_2 = 0$ and $s_0 - s_3 = 0$ and then from the equation of S_3, $s_0 l^2 = 0$. If $s_0 = 0$ then $s_2 = s_3 = 0$ and then in $\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3)$ we get $x_{00} = x_{01} = x_{20} = x_{21} = x_{30} = x_{31} = x_{10}x_{11} = 0$. From this we deduce that all y_{abcd} are zero, which together with $q = 0$ forces all variables to be zero. Hence $s_0 \neq 0$ and we must have $l = 0$. To show that the image of $\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3) \cap T$ coincides with n_1, as in (4.2), it suffices to show that this locus is nonempty. The equations $x_{20} + x_{21} = x_{30} + x_{31} = 0$ define in W a subscheme of dimension 1 which is not contained in the exceptional locus of $\varphi_{|W} \colon W \dashrightarrow V$, hence in V they define a positive-dimensional subscheme $Z' \subset V$. If $x_{00}, x_{01} \neq 0$ then,

$$y_{abcd}x_{0a} = x_{1b'}x_{2c'}x_{3d'} = x_{1b'}x_{2c}x_{3d} = y_{abc'd'}x_{0a} \implies y_{abcd} = y_{abc'd'},$$

which means that in the corresponding (nonempty) open set of Z' the equations $y_{abcd} = y_{abc'd'}$ of $\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3)$ are redundant. Hence $\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3) \cap V$ is positive-dimensional. Since T is obtained from V by taking a hypersurface section, we deduce that $\mathrm{Fix}_{(+,+)}(\alpha_2\alpha_3\beta_2\beta_3) \cap T$ is nonempty. We conclude that the fixed points of $\alpha_2\alpha_3\beta_2\beta_3$ do not contribute to D_1 and that their contribution to Δ_1 is $\{n_1\}$.

The arguments we have used so far can be used to show that the 5 remaining elements of $[\alpha_1\beta_1]$ contribute neither to Δ_1 nor to D_1. □

The cubic surface with 3 nodes S_3 contains exactly 12 lines, as represented in Figure 2 (courtesy of [LvS]). The lines L_1, L_2, L_3 form the black triangle in the picture, spanning the plane $(s_0 = 0)$. Denote by N_{ij} the line through n_i and n_j:

$$N_{ij} = (s_0 - s_k = l = 0), \tag{4.6}$$

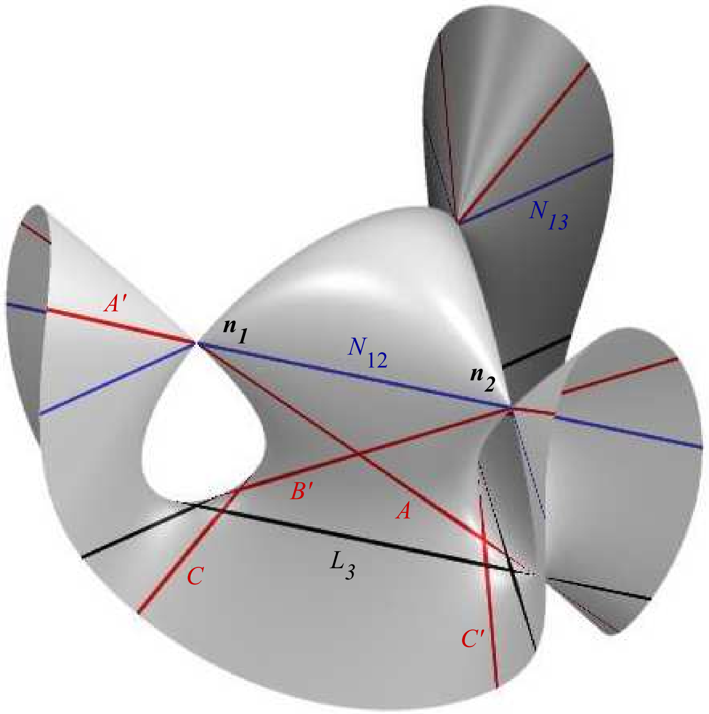

Figure 2. Lines on S_3

where $\{i, j, k\} = \{1, 2, 3\}$. Then $\{N_{23}, N_{13}, N_{12}\}$ form another triangle, the blue triangle, contained in the plane $(l = 0)$. Each L_i intersects exactly one of the N_{ij}: L_1 intersects N_{23}, L_2 intersects N_{13} and L_3 intersects N_{12}. These intersection points are not in the region visible in Figure 2. Consider the plane $\mathcal{P}_1$ through L_1 and n_1, given by

$$(\nu_0 + \nu_2 + \nu_3)s_0 + \nu_1 s_1 = 0. \tag{4.7}$$

Intersecting it with S_3 and eliminating s_1 we get:

$$s_0(\mu(s_2 - s_0)(s_3 - s_0) + \nu_1(\nu_2(s_2 - s_0) + \nu_3(s_3 - s_0))^2) = 0$$

where $\mu = 8\nu_4^2(\nu_0 + \nu_1 + \nu_2 + \nu_3)$. Hence $S_3 \cap \mathcal{P}_1$ splits as the union of 3 lines $L_1 \cup A \cup A'$. Similarly, $S_3 \cap \mathcal{P}_2 = L_2 \cup B \cup B'$ and $S_3 \cap \mathcal{P}_3 = L_3 \cup C \cup C'$, where $\mathcal{P}_2$ is the plane through L_2 and n_2 and $\mathcal{P}_3$ is the plane through L_3 and n_3. We have labeled these last 6 lines as in Figure 2, so that A, B, C are pairwise disjoint and the same holds for A', B', C'.

Let $\zeta \colon \Sigma \to S_3$ be the the blow-up of the 3 nodes, and let E_i denote the exceptional divisor of n_i. With abuse of notation, let us denote by A, A', B, B', C, C', L_i, N_{ij} the strict transforms in Σ of the namesake lines. Similarly we do not change the notation for the strict transforms in Σ of $C_i \subset S_3$. Denote by H_Σ be the pull-back of an hyperplane section. Since $K_\Sigma = -H_\Sigma$, the strict transform of every line in S_3 is a (-1)-curve. The curves N_{12}, N_{13}, N_{23}, A', B', C' are pairwise disjoint rational curves with self-intersection -1; by Castelnuovo's criterion we can contract them to a smooth rational surface with $K^2 = 9$. Therefore, the contraction of these curves yields a morphism $\xi\colon \Sigma \to \mathbb{P}^2$. Again, with abuse of notation, we shall continue using the same notation for a curve in S_3, its strict transform in Σ, and, when it is not contracted to a point, its image in $\mathbb{P}^2$. Let us denote by r_{12}, r_{13}, r_{23} and by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, the points of $\mathbb{P}^2$ to which ξ contracts N_{12}, N_{13}, N_{23} and A', B', C', respectively. In $\mathbb{P}^2$ we get the configuration of curves of Figure 3. We leave to the reader the straightforward check that L_1, L_2, L_3, E_1, E_2, E_3, A, B,

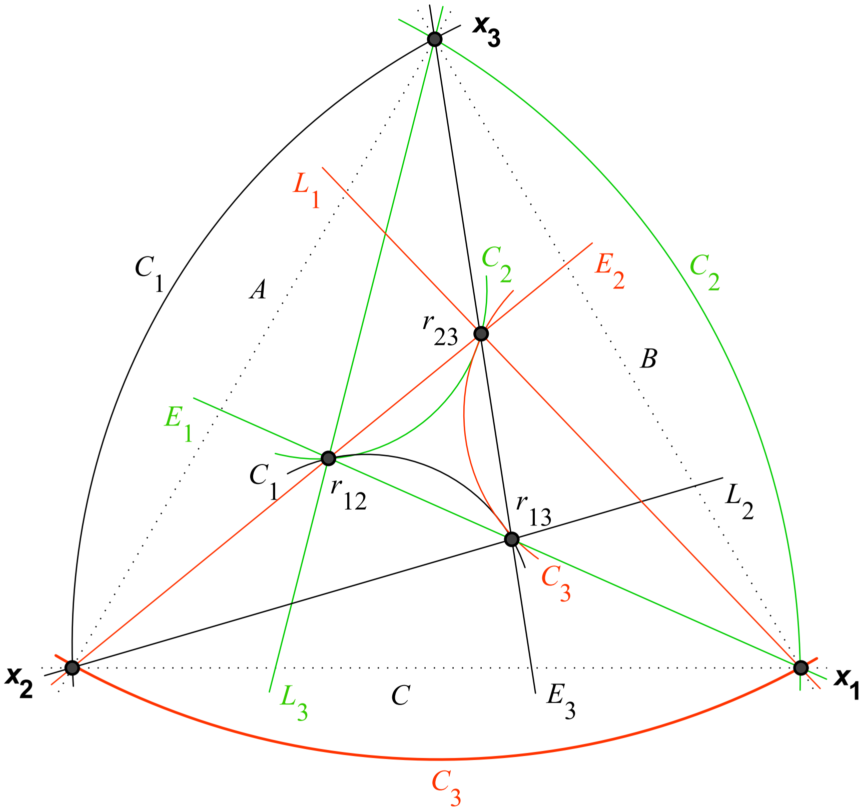

FIGURE 3. The branch divisors of $\gamma''\colon S'' \to \mathbb{P}^2$

C in $\mathbb{P}^2$ are in the configuration of Figure 3. As for C_2, using the equations of C_2 and N_{12}, (4.4) and (4.6), we see that it meets N_{12} in S_3. Similarly C_2 meets N_{23}. Hence in the plane C_2 contains the points r_{12}, r_{23}. To see that C_2 contains the points $\mathbf{x}_1$ and $\mathbf{x}_3$ it is enough to show that, in S_3, C_2 meets the lines A, A' and the lines C, C'. Indeed, as $C_2 = H_{S_3} - L_2$ and $A + A' = H_{S_3} - L_1$ in S_3, we have $C_2(A+A') = (H_{S_3} - L_2)(H_{S_3} - L_1) = H_{S_3}^2 - H_{S_3}L_1 - H_{S_3}L_2 + L_1L_2 = 2$. Likewise one shows that $C_2(C + C') = 2$. Additionally, $C_2(B + B') = 0$, hence C_2 does not contain $\mathbf{x}_2$. The conics C_1 and C_3 have similar properties, obtained by cyclic permutation of the indices $\{1,2,3\}$.

Remark 4.4. Consider the following commutative diagram:

$$\begin{tikzcd}
S \arrow[d, "\gamma"'] & S' \arrow[l, "\hat{\zeta}"'] \arrow[r, "\hat{\xi}"] \arrow[d, "\gamma'"'] & S'' \arrow[d, "\gamma''"] \\
S_3 & \Sigma \arrow[l, "\zeta"] \arrow[r, "\xi"'] & \mathbb{P}^2
\end{tikzcd} \tag{4.8}$$

where S', $\hat{\zeta}$, γ' is the fiber product of the corresponding morphisms: $\zeta\colon \Sigma \to S_3$ and $\varphi_{K_S} = \gamma\colon S \to S_3$ and S'' is obtained as the closure in $\mathbb{P}^2 \times S'$ of the fiber product of the restriction of the map $\xi^{-1}\colon \mathbb{P}^2 \to \Sigma$ to the complement of the its indeterminacy locus and the morphism $\gamma'\colon S' \to \Sigma$. Notice that S'' can be obtained from S' by contracting the preimages of N_{12}, N_{13}, N_{23}, A', B', C'. All horizontal maps in (4.8) are birational morphisms and all vertical maps are bidouble covers; moreover, γ' and γ'' are the quotient morphisms of the induced actions of Γ on S' and S'', respectively. Let us denote by D'_i and D''_i the images via γ' and γ'', on Σ and $\mathbb{P}^2$, respectively, of the fix locus of the action of Γ, on S' and S'', respectively. According to Proposition 4.3, each of the non-trivial involutions of Γ, θ_i, fix each of the 4 pre-images of the node n_i. Since these points are smooth points of S and an involution on a surface fixing a smooth point has to fix all tangent directions to that point, we deduce that the exceptional lines of the blow up of the points of S in the pre-image of n_i are ramification divisors of γ' and accordingly E_i is a branch divisor of γ'' associated with θ_i. In conclusion, we have

$$D''_1 = E_1 + C_2 + L_3 \qquad D''_2 = E_2 + C_3 + L_1 \qquad D''_3 = E_3 + C_1 + L_2. \tag{4.9}$$

In the Figure 3 we have depicted the divisor D''_1 in green and in red the divisor D''_2. Note that S'' is singular and $\hat{\xi}$ is a resolution of its singularities.

Theorem 4.5. *A general surface in the* 1*-dimensional linear subsystem*

$$\mathcal{B} = \left\{T = V \cap (q=0) \mid (\nu_0,\nu_1,\nu_2,\nu_3,\nu_4) = (-\nu,\nu,\nu,\nu,\nu_4),\ \textit{for } (\nu,\nu_4) \in \mathbb{P}^1\right\} \subset \mathcal{N}$$

is a surface with 24 *isolated rational double points as only singularities and* G *acts freely on it. The quotient* T/G *is the canonical model of a tertiary Burniat surface and, conversely, every tertiary Burniat surface arises in this way.*

Proof. Analyzing the base locus of $\mathcal{B}$, like in the proof of Theorem 2.11, we get $\operatorname{Sing}(T) \subset (l = \sum_{abcd\in\mathcal{L}}(-1)^a y_{abcd} = 0)$, where $l = \nu(-s_0+s_1+s_2+s_3)$. Let $T_1, T_2 \in \mathcal{B}$ be given in V by $s_0 - s_1 - s_2 - s_3 = 0$ and $\sum_{abcd\in\mathcal{L}}(-1)^a y_{abcd} = 0$.

Fix coordinates x_1, x_2, x_3 on $\mathbb{C}^3$. Consider the open set $\Omega_{00} = V \setminus \{x_{00} = 0\}$ and the map $\xi_1\colon \Omega_{00} \to \mathbb{C}^3$ given by

$$\xi_1(\mathbf{x},\mathbf{y}) = \left(\frac{x_{10}}{x_{00}}, \frac{x_{20}}{x_{00}}, \frac{x_{30}}{x_{00}}\right). \tag{4.10}$$

Since $x_{00}x_{01} \neq 0$ and (2.1) hold, $\xi_1(\Omega_{00})$ is contained in $\mathbb{C}^3 \setminus \{x_1x_2x_3 = 0\}$ and the map $\xi_2 \colon \mathbb{C}^3 \setminus \{x_1x_2x_3 = 0\} \to \Omega_{00}$ given by

$$\xi_2(x_1, x_2, x_3) = (1, -1, x_1, -1/x_1, x_2, -1/x_2, x_3, -1/x_3, \dots), \tag{4.11}$$

(where, to be more precise, $\xi_2^\sharp(y_{abcd}) = x_{1b'}x_{2c'}x_{3d'}$, for all $(a, b, c, d) \in \mathcal{L}$), is the inverse of ξ_1. Let $\mathcal{B}_{\Omega_{00}}$ denote the pencil $\{T \cap \Omega_{00} : T \in \mathcal{B}\}$ on Ω_{00}. The pencil $\xi_2^*\mathcal{B}_{\Omega_{00}}$ is spanned by $\xi_1(T_1 \cap \Omega_{00})$ and $\xi_1(T_2 \cap \Omega_{00})$. We claim that these are two surfaces in $\mathbb{C}^3 \setminus \{x_1x_2x_3 = 0\}$ given by

$$\begin{aligned} F_1 &:= 1 - \tfrac{1}{2} \textstyle\sum_{i>0} \left(x_i^2 + 1/x_i^2\right) = 0, \\ F_2 &:= (x_1 - 1/x_1)(x_2 - 1/x_2)(x_3 - 1/x_3) = 0, \end{aligned} \tag{4.12}$$

respectively. To see this, notice that the equation $F_1 = 0$ is obtained from (4.11) by straightforward substitution into the equation of T_1 in V; as for F_2, it suffices to notice that, on Ω_{00},

$$\sum_{abcd\in\mathcal{L}} (-1)^a y_{abcd} = \sum_{bcd} x_{1b'}x_{2c'}x_{3d'} = (x_{10} + x_{11})(x_{20} + x_{21})(x_{30} + x_{31}),$$

since $x_{00} = 1$, and to use $x_{i0}x_{i1} = -1$, for all $i = 1, 2, 3$, which one deduces from the quadric equations (2.1).

We show next that a general member of $\xi_2^*\mathcal{B}_{\Omega_{00}}$ is smooth outside a (fixed) set of 24 rational double points. Since $\frac{\partial F_1}{\partial x_i} = 1/x_i^3 - x_i$, the singularities of $\xi_1(T_1 \cap \Omega_{00})$ lie in the set defined by $x_1^4 = x_2^4 = x_3^4 = 1$. These equations define 64 points. However only the 24 points of the set

$$\mathfrak{D} = \{(\pm\epsilon, \pm1, \pm1), (\pm1, \pm\epsilon, \pm1), (\pm1, \pm1, \pm\epsilon)\}, \tag{4.13}$$

where, again, ϵ is a square root of -1, actually belong to $\xi_1(T_1 \cap \Omega_{00})$. As $\frac{\partial^2 F_1}{\partial x_i \partial x_j} = 0$, for $i \neq j$, and $\frac{\partial^2 F_1}{\partial x_i^2} = -3/x_i^4 - 1$, we see that the determinant of the Hessian matrix is nonzero at the points of $\mathfrak{D}$, showing that they are indeed ordinary double points of $\xi_1(T_1 \cap \Omega_{00})$. For every point of $\mathfrak{D}$, two factors of F_2 vanish, thus it is clear that $\xi_1(T_2 \cap \Omega_{00})$ is also singular at the points of $\mathfrak{D}$. This shows that a general member of $\xi_2^*\mathcal{B}_{\Omega_{00}}$ has a rational double point at each point of $\mathfrak{D}$. Since $\xi_1(T_1 \cap \Omega_{00})$ is smooth away from $\mathfrak{D}$ it follows that a general member of $\xi_2^*\mathcal{B}_{\Omega_{00}}$ is also smooth away from $\mathfrak{D}$.

We proceed to show that a general $T \in \mathcal{B}$ is smooth along $T \cap (x_{00} = 0) = T \setminus \Omega_{00}$. If $x_{00} = 0$, then $x_{01} = -x_{00} = 0$ and, from (2.1), $x_{10}x_{11} = x_{20}x_{21} = x_{30}x_{31} = 0$. Let Ω_{ab}^{ij} denote the open set of V given by $x_{ia}x_{jb} \neq 0$. Since a general $T \in \mathcal{B}$ has $\operatorname{Sing}(T) \subset (l = 0)$, if all the variables $x_{00}, x_{01}, \dots, x_{30}, x_{31}$ but one vanish at a point of $\operatorname{Sing}(T) \cap (x_{00} = 0)$, then from $l = 0$ we deduce the remaining one must vanish also. ¿From this, using (2.4) and the fact that $\operatorname{Sing}(T)$ is contained in $T_2 = (\sum_{abcd\in\mathcal{L}} (-1)^a y_{abcd} = 0)$, we deduce that all variables must vanish, which is impossible. Hence, for a general $T \in \mathcal{B}$, there exist i, j, a, b with $j > i > 0$ such that $\operatorname{Sing}(T) \cap (x_{00} = 0) \subset \Omega_{ab}^{ij}$. Since the role that $x_{10}, x_{11}, x_{20}, x_{21}, x_{30}, x_{31}$ play in the equations of V and $T \in \mathcal{B}$ is symmetric, we may reduce to showing that a general member $T \in \mathcal{B}$ is smooth along $T \cap (x_{00} = 0) \cap \Omega_{11}^{23}$. Similarly to what we did earlier, we consider a map $\zeta_1\colon \Omega_{11}^{23} \to \mathbb{P}(1^3,2)$ given by $\zeta_1(\mathbf{x},\mathbf{y}) = (x_{00}, x_{21}, x_{31}, y_{0000})$. This map has image the (affine) open set defined by $x_{21}x_{31}y_{0000} \neq 0$. This is a consequence of the quartic relation $y_{0000}y_{1100} - x_{21}^2x_{31}^2$ in (2.4) which holds in V. As before, to show that an inverse $\zeta_2\colon \mathbb{P}(1^3,2) \setminus (x_{21}x_{31}y_{0000} \neq 0) \to \Omega_{11}^{23}$ to ζ_1 exists, it is enough to express every variable on Ω_{11}^{23} has a rational function of $x_{00}, x_{21}, x_{31}, y_{0000}$. Using the equations of V, i.e., (2.1), (2.3), (2.4) and $x_{00}+x_{01} = 0$, on Ω_{11}^{23} we have:

$$\begin{aligned} & x_{01} = -x_{00}, \quad y_{1100} = \frac{x_{21}^2x_{31}^2}{y_{0000}}, \\ & x_{20} = \frac{x_{00}x_{01}}{x_{21}} = -\frac{x_{00}^2}{x_{21}}, \quad x_{30} = \frac{x_{00}x_{01}}{x_{31}} = -\frac{x_{00}^2}{x_{31}}, \\ & x_{10} = \frac{x_{01}x_{21}x_{31}}{y_{0000}} = -\frac{x_{00}x_{21}x_{31}}{y_{0000}}, \quad x_{11} = \frac{x_{00}x_{21}x_{31}}{y_{1100}} = \frac{x_{00}y_{0000}}{x_{21}x_{31}}. \end{aligned} \tag{4.14}$$

which are all rational functions of $x_{00}, x_{21}, x_{31}, y_{0000}$. Moreover

$$y_{abc1} = \frac{x_{0a'}x_{1b'}x_{2c'}}{x_{31}} \quad \text{and} \quad y_{ab1d} = \frac{x_{0a'}x_{1b'}x_{3d'}}{x_{21}}, \tag{4.15}$$

which, using (4.14), can be seen to be also rational functions of $x_{00}, x_{21}, x_{31}, y_{0000}$.

Consider $\mathcal{B}_{\Omega_{11}^{23}}$ the pencil $\{T \cap \Omega_{11}^{23} : T \in \mathcal{B}\}$. Next we show that a general member of $\zeta_2^*\mathcal{B}_{\Omega_{11}^{23}}$ is smooth along $x_{00} = 0$. It suffices to show that $\zeta_1(T_1 \cap \Omega_{11}^{23})$ is smooth along $x_{00} = 0$. Additionally, since $\zeta_1(T_1 \cap \Omega_{11}^{23})$ does not meet the singular point of $\mathbb{P}(1^3,2)$ we can reduce to showing quasi-smoothness, or, more precisely, non-vanishing of the Jacobian matrix of the polynomial F_3, obtained from the equation of $\zeta_1(T_1 \cap \Omega_{11}^{23})$ by setting $y_{0000} = 1$, at the points of $\zeta_1(T_1 \cap \Omega_{11}^{23} \cap (x_{00} = 0))$. ¿From (4.14) we deduce $F_3 = x_{00}^2x_{21}^2x_{31}^2 - x_{00}^2x_{21}^4x_{31}^4 - x_{00}^2 - x_{00}^4x_{31}^2 - x_{21}^4x_{31}^2 - x_{00}^4x_{21}^2 - x_{21}^2x_{31}^4$ using, to ease notation, x_{00}, x_{21}, x_{31} as coordinates for the corresponding affine piece of $\mathbb{P}(1^3,2)$. Hence, at $x_{00} = 0$,

$$\frac{\partial F_3}{\partial x_{21}} = -4x_{21}^3x_{31}^2 - 2x_{21}x_{31}^4 \quad \text{and} \quad \frac{\partial F_3}{\partial x_{31}} = -2x_{21}^4x_{31} - 4x_{21}^2x_{31}^3$$

which have no common zeros for $x_{21}x_{31} \neq 0$.

We have shown that a general member of $\mathcal{B}$ is a smooth away from a set of 24 rational double points given by $\xi_2(\mathfrak{D})$ where $\mathfrak{D}$ is the set of points (4.13), in other words, the set of points given in local coordinates by (4.13). Notice that by Remark 2.13, the group G acts freely on a general member of $\mathcal{B}$. To show that $S := T/G$ for a general T in $\mathcal{B}$ is the canonical model of a Burniat surface we analyze (4.8) for this case in detail. We start by observing that if $(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4) = (-\nu, \nu, \nu, \nu, \nu_4)$ then the plane $\mathcal{P}_1$ defined in (4.7) is nothing other that the plane $(s_0 + s_1 = 0)$ and hence the conic C_1 splits as $A \cup A'$. Likewise, C_2 splits up as $B \cup B'$ and C_3 splits up as $C \cup C'$. Recall that $n_1 \in A \cap A'$, $n_2 \in B \cap B'$ and $n_3 \in C \cap C'$. Also, the nodes become $n_1 = (1,-1,1,1)$, $n_2 = (1,1,-1,1)$ and $n_3 = (1,1,1,-1)$. Their pre-image in T coincides with the 24 ordinary nodes of T. Indeed we see that 8 points of T, written in local coordinates of $\Omega_{00} \simeq \mathbb{C}^3 \setminus (x_1x_2x_3 = 0)$ as $(\pm\epsilon, \pm 1, \pm 1)$, map to n_1; the 8 points $(\pm 1, \pm\epsilon, \pm 1)$ map to n_2 and the 8 points $(\pm 1, \pm 1, \pm\epsilon)$ map to n_3.

Since G acts freely on T, each of these sets of 8 points maps to a single point in $S := T/G$ which is a node of S and is fixed by every element of Γ. Denote these 3 nodes of S by $\hat{n}_1, \hat{n}_2, \hat{n}_3$. Using the notation of Proposition 4.3, we claim that

$$\begin{aligned} \Delta_1 &= \{n_1, n_3\}, \quad & D_1 &= B + B' + L_3, \\ \Delta_2 &= \{n_2, n_1\}, \quad & D_2 &= C + C' + L_1, \\ \Delta_3 &= \{n_3, n_2\}, \quad & D_3 &= A + A' + L_2. \end{aligned}$$

We compute Δ_1 and D_1 by analyzing the fixed loci on T of the elements of:

$$[\alpha_1\beta_1] = \{\alpha_1\beta_1, \beta_1\beta_2, \alpha_1\alpha_2\beta_1\beta_3, \alpha_1\alpha_3, \alpha_2\beta_1\beta_2\beta_3, \alpha_3\beta_2, \alpha_1\alpha_2\alpha_3\beta_3, \alpha_2\alpha_3\beta_2\beta_3\}.$$

The computation of Δ_2, D_2, Δ_3 and D_3 follows by symmetry. Recall from the proof of Proposition 4.3, that $\operatorname{Fix}(\beta_1\beta_2) \cap T$ maps to L_3 on S_3, $\operatorname{Fix}(\alpha_2\beta_1\beta_2\beta_3) \cap T$ maps to C_2 which is now $B \cup B'$ and that $\operatorname{Fix}(\alpha_2\alpha_3\beta_2\beta_3) \cap T$ maps to n_1. With the assumptions of that proposition, all of the other elements of $[\alpha_1\beta_1]$ have empty fixed locus on T. In case of $T \in \mathcal{B}$ this is no longer true. Indeed all but $\alpha_1\alpha_2\alpha_3\beta_3$ have empty fixed locus on T. To see this, recall that $\operatorname{Fix}_{(+,+)}(\alpha_1\alpha_2\alpha_3\beta_3)$ and $\operatorname{Fix}_{(-,+)}(\alpha_1\alpha_2\alpha_3\beta_3)$ are given by:

$$\begin{aligned} &(x_{00} - x_{01} = x_{10} - x_{11} = x_{20} - x_{21} = x_{30} + x_{31} = y_{abcd} + y_{a'bc'd} = 0, \forall_{abcd \in \mathcal{L}}), \\ &(x_{00} + x_{01} = x_{10} + x_{11} = x_{20} + x_{21} = x_{30} - x_{31} = y_{abcd} - y_{abc'd'} = 0, \forall_{abcd \in \mathcal{L}}), \end{aligned}$$

respectively. It is easy to see that $\operatorname{Fix}_{(+,+)}(\alpha_1\alpha_2\alpha_3\beta_3) \cap T$ is empty. However the locus $\operatorname{Fix}_{(-,+)}(\alpha_1\alpha_2\alpha_3\beta_3)$ now contains the set of points $\{(\pm 1, \pm 1, \pm \epsilon)\}$, given in local coordinates in Ω_{00}. To see why this happens only for $T \in \mathcal{B}$ notice that the equations of $\operatorname{Fix}_{(-,+)}(\alpha_1\alpha_2\alpha_3\beta_3)$ imply that $s_1 = s_0$, $s_2 = s_0$ and $s_3 = -s_0$. Using the equation of S_3 this would give either $s_1 - s_0 = s_2 - s_0 = s_3 + s_0 = s_0 = 0$ which defines the emptyset set (in any case) or $s_1 - s_0 = s_2 - s_0 = s_3 + s_0 = l = 0$, which, in the case of T general in $\mathcal{N}$, defines the empty set but in the case of $T \in \mathcal{B}$ actually defines $\{n_3\}$, since $l = s_0 - s_1 - s_2 - s_3 \in (s_1 - s_0, s_2 - s_0, s_3 + s_0)$. Since n_1 and n_3 are the only isolated fixed points of θ_1 (notice that $n_2 \in B \cap B'$) we have $\Delta_1 = \{n_1, n_3\}$ and $D_1 = B + B' + L_3$.

We claim that the branch loci of $\gamma' \colon S' \to \Sigma$ are:

$$\begin{aligned} D'_1 &= E_3 + B + B' + L_3, \\ D'_2 &= E_2 + C + C' + L_1, \\ D'_3 &= E_1 + A + A' + L_2. \end{aligned}$$

Again by symmetry it is enough to compute D'_1. To do this we must analyze the action of θ_1 on the tangent cone at $\hat{n}_1, \hat{n}_2$ and $\hat{n}_3$, showing that it fixes every tangent direction in the tangent cone at $\hat{n}_3$ and that it does not act in this way on the tangent cones at the nodes $\hat{n}_1$ and $\hat{n}_2$. This will mean that the action of θ_1 on S' will fix pointwise E_3 and will not fix pointwise E_1 and E_2. (Recall that S' can be obtained by blowing up $\hat{n}_1, \hat{n}_2, \hat{n}_3$.) To analyze the action of θ_1 on each of the nodes we will study the action of the corresponding involutions of $[\alpha_1\beta_1]$ on the local model given by $\Omega_{00} \subset V$. But first, recall that an involution acting on the local equation of an ordinary double point $x^2+y^2+z^2=0$ is one of the involutions given by $(x,y,z)\mapsto(\pm x,\pm y,\pm z)$. Of these, only one, the involution given by $(x,y,z)\mapsto(-x,-y,-z)$, fixes a single point. This is also the only one which fixes every tangent direction to the node. Now, as we showed earlier, the involutions in $[\alpha_1\beta_1]$ which fix in T points in the pre-image of $\hat{n}_1,\hat{n}_2,\hat{n}_3$ are $\alpha_2\alpha_3\beta_2\beta_3$, $\alpha_2\beta_1\beta_2\beta_3$ and $\alpha_1\alpha_2\alpha_3\beta_3$. We have seen that $\alpha_2\beta_1\beta_2\beta_3$ fixes a positive dimensional locus containing the pre-images of $\hat{n}_2$, hence, locally at each pre-image, this involution cannot fix every tangent direction to it and therefore E_2 is not in D'_1. The involution $\alpha_2\alpha_3\beta_2\beta_3$, whose fixed locus on T maps to $\{\hat{n}_1\}$, can be written in the local model $\mathbb{C}^3\setminus(x_1x_2x_3=0)\simeq\Omega_{00}$ as:

$$(x_1,x_2,x_3)=\left(\frac{x_{10}}{x_{00}},\frac{x_{20}}{x_{00}},\frac{x_{30}}{x_{00}}\right)\mapsto\left(\frac{x_{10}}{x_{00}},-\frac{x_{21}}{x_{00}},-\frac{x_{31}}{x_{00}}\right)=\left(x_1,\frac{1}{x_2},\frac{1}{x_3}\right).$$

(Recall that $x_{00}+x_{01}=0$ and $x_{00}x_{01}=x_{i0}x_{i1}\implies x_{00}/x_{i0}=x_{i1}/x_{01}$.) We see that $(\pm\epsilon,\pm1,\pm1)$ are fixed, which is not surprising as these are the local coordinates for the pre-images of $\hat{n}_1$. However we see also that the fixed loci of this involution in the ambient $\mathbb{C}^3\setminus(x_1x_2x_3=0)$ is a set of four lines, going through the 8 points $(\pm\epsilon,\pm1,\pm1)$. Hence this involution does not fix all of the tangent directions at any of these points. This implies that θ_1 does not fix all of the tangent directions of $\hat{n}_1$ and thus E_1 is also not in D'_1. Finally, writing $\alpha_1\alpha_2\alpha_3\beta_3$ in the local model:

$$(x_1,x_2,x_3)=\left(\frac{x_{10}}{x_{00}},\frac{x_{20}}{x_{00}},\frac{x_{30}}{x_{00}}\right)\mapsto\left(\frac{x_{11}}{x_{01}},\frac{x_{21}}{x_{01}},-\frac{x_{31}}{x_{01}}\right)=\left(\frac{1}{x_1},\frac{1}{x_2},-\frac{1}{x_3}\right),$$

we see that this involution fixes exactly the set of points given in local coordinates by $\{(\pm1,\pm1,\pm\epsilon)\}$. This coincides with the pre-image of $\{\hat{n_3}\}$. Moreover, since it fixes only finitely many points, it must fix every tangent direction at each of these points. We conclude that θ_3 fixes every tangent direction of $\hat{n}_3$ and thus $D'_1=E_3+B+B'+L_3$.

It is now easy to compute the ramification divisors of $\gamma''\colon S''\to\mathbb{P}^2$. Since S'' can be obtained by contracting the pre-images of $N_{23},N_{13},N_{12},A',B',C'$ it is clear that

$$\begin{aligned}D''_1&=E_3+B+L_3,\\ D''_2&=E_2+C+L_1,\\ D''_3&=E_1+A+L_2.\end{aligned}\tag{4.16}$$

With the help of Figure 3 we see that these are exactly the branch loci for a tertiary Burniat surface.

The space of tertiary Burniat surfaces is parameterized by $\lambda\in\mathbb{C}^*\setminus\{1\}$ as follows. In Figure 1, we may always choose coordinates (u_0,u_1,u_2) such that $\mathbf{x}_1=(1,0,0)$, $\mathbf{x}_2=(0,1,0)$, $\mathbf{x}_3=(0,0,1)$ and the further 3 marked points are respectively $(1,1,1)$, $(1,1,\lambda)$ and $(\lambda,1,\lambda)$. The bicanonical image of the Burniat surface is the image of $\mathbb{P}^2$ in $\mathbb{P}^3$ by the linear system of cubics through the 6 marked points. If we choose, as basis for this system, the cubics:

$$\begin{aligned}
s_0 &= -\tfrac{1}{2}(u_0 - \lambda u_1)(u_1 - u_2)(u_2 - \lambda u_0)\\
s_1 &= (1-\lambda)u_0(u_1 - u_2)(\lambda u_1 - u_2) - s_0\\
s_2 &= (1-\lambda)u_1(u_2 - u_0)(u_2 - \lambda u_0) - s_0\\
s_3 &= (1-\lambda)u_2(u_0 - u_1)(u_0 - \lambda u_1) - s_0
\end{aligned}$$

then, one can check that

$$(\lambda+1)^2(s_1 - s_0)(s_2 - s_0)(s_3 - s_0) = -2\lambda s_0(s_3 + s_2 + s_1 - s_0)^2$$

and we easily conclude that the tertiary Burniat surface under consideration is isomorphic to $S = T/G$ with $T \in \mathcal{N}$ given by $-\nu_0 = \nu_1 = \nu_2 = \nu_3 = \sqrt{-\lambda}$ and $\nu_4 = 4(\lambda + 1)$. ☐

Remark 4.6. We can see the branch divisors (4.16) as the branch divisors of the bidouble cover of $\mathbb{P}^2$ obtained normalizing the bidouble cover of $\mathbb{P}^2$ whose branch divisors are obtained by degenerating the branch divisors (4.9) of $\gamma'' \colon S'' \to \mathbb{P}^2$, for a general $T \in \mathcal{N}$. Each of the conics in the branch divisors (4.9) is restricted to go through 4 points (see also Figure 3). We can degenerate these divisors by imposing the condition that each conic should go through a 5th point, which forces it to split as the union of two lines. For example, if we impose that C_2 goes through the intersection of E_2 and B it will split as $E_2 \cup B$. If we proceed in a similar fashion with the 2 remaining conics we obtain the following 3 divisors:

$$\tilde{D}_1 = E_1 + E_2 + B + L_3 \quad \tilde{D}_2 = E_2 + E_3 + C + L_1 \quad \tilde{D}_3 = E_3 + E_1 + A + L_2.$$

The surface obtained as a bidouble cover of $\mathbb{P}^2$ with these branch divisors is not normal, and, as explained in [C], its normalization is the bidouble cover with branch divisors (4.16). This process can be described using Figure 3. After one degenerates C_i, the line E_i acquires a second color, say, for example, after degenerating C_2, E_2 becomes red and green. To normalize the bidouble cover we simply set the color of E_i to the third color, so that E_2 becomes black and so on.

References

[ABR] Altınok, S.; Brown, G.; Reid, M. *Fano 3-folds, K3 surfaces and graded rings.* Topology and geometry: commemorating SISTAG, 25–53, Contemp. Math., **314**, Amer. Math. Soc., Providence, RI, 2002.

[AP] Alexeev, V.; Pardini, R. *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces.* arxiv: 0901.4431

[BC1] Bauer, I.; Catanese, F. *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces.* arXiv:0909.3699

[BC2] Bauer, I.; Catanese, F. *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space.* Invent. Math. **180** (2010), no. 3, 559–588

[BC3] Bauer, I.; Catanese, F. *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces.* arXiv:1012.3770

[BCG] Bauer, I.; Catanese, F.; Grunewald, F. *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves.* Pure Appl. Math. Q. **4** (2008), no. 2, part 1, 547?586.

[BCGP] Bauer, I.; Catanese, F.; Grunewald, F.; Pignatelli, R. *Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups.* arXiv:0809.3420v3. To appear on Amer. J. Math.

[BCP] Bauer, I.; Catanese, F.; Pignatelli, R. *Surfaces of general type with geometric genus zero: a survey.* arXiv:1004.2583. To appear on the Proceedings of the conference *Complex and Differential Geometry* held in Hannover, September 14 - September 18, 2009

[BP] Bauer, I.; Pignatelli, R. *The classification of minimal product-quotient surfaces with* $p_g = 0$. arXiv:1006.3209v1

[Ba] Bayle, L. *Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d'Enriques.* J. Reine Angew. Math. **449** (1994), 9–63.

[Br] Brown, G. *Graded ring database homepage* Online searchable database, available from `http://grdb.lboro.ac.uk/`

[BS] Brown, G.; Suzuki, K. *Fano 3-folds with divisible anticanonical class.* Manuscripta Math. **123** (2007), no. 1, 37–51.

[Bu] Burniat, P. *Sur les surfaces de genre* $P_{12} > 1$. Ann. Mat. Pura Appl. (4) **71** (1966), 1–24.

[CS] Cartwright, D. I., Steger, T. *Enumeration of the 50 fake projective planes.* C. R. Math. Acad. Sci. Paris **348** (2010), no. 1-2, 11–13.

[C] Catanese, F. *Singular bidouble covers and the construction of interesting algebraic surfaces.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97–120, Contemp. Math., **241**, Amer. Math. Soc., Providence, RI, 1999.

[E] Eisenbud, D. *Commutative algebra, with a view toward algebraic geometry.* Graduate Texts in Mathematics, **150**. Springer–Verlag, 1995.

[Ino] Inoue, M. *Some new surfaces of general type.* Tokyo J. Math. **17** (1994), no. 2, 295–319.

[KZ] Keum, J.; Zhang, D.-Q. *Fundamental groups of open* $K3$ *surfaces, Enriques surfaces and Fano* 3*-folds.* J. Pure Appl. Algebra **170** (2002), no. 1, 67–91.

[K] Kulikov, V., S. *Old examples and a new example of surfaces of general type with* $p_g = 0$. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 5, 123170; translation in Izv. Math. 68 no. 5, (2004), 9651008.

[LvS] Labs, O., van Straten, D. *The Cubic Surface Homepage.* University of Mainz (2001) *www.Cubics.AlgebraicSurface.net*

[LP] Lee, Y.; Park, J. *A simply connected surface of general type with* $p_g = 0$ *and* $K^2 = 2$. Invent. Math. **170** (2007), no. 3, 483–505.

[MP1] Mendes Lopes, M; Pardini, R. *A connected component of the moduli space of surfaces with* $p_g = 0$. Topology **40** (5) (2001), 977-991.

[MP2] Mendes Lopes, M.; Pardini, R. *A new family of surfaces with* $p_g = 0$ *and* $K^2 = 3$. Ann. Scient. Ec. Norm. Sup. 4^{e} série, **37** (2004), 507-531.

[MP3] Mendes Lopes, M.; Pardini, R. *Numerical Campedelli surfaces with fundamental group of order* 9. J. Eur. Math. Soc. (JEMS) **10** (2008), no. 2, 457–476.

[MPR] Mendes Lopes, M.; Pardini, R.; Reid, M. *Campedelli surfaces with fundamental group of order* 8. Geom. Dedicata **139** (2009), 49–55.

[Miy] Miyaoka, Y. *On numerical Campedelli surfaces.* Complex analysis and algebraic geometry, pp. 113–118. Iwanami Shoten, Tokyo, 1977.

[NP1] Neves, J.; Papadakis, S. A. *A construction of numerical Campedelli surfaces with torsion* $\mathbb{Z}/6$. Trans. Amer. Math. Soc. **361** (2009), no. 9, 4999–5021.

[NP2] Neves, J.; Papadakis, S. A. *Parallel Kustin-Miller unprojection with an application to Calabi-Yau geometry.* arXiv:0903.1335v2

[PPS] Park, H.; Park, J.; Shin, D. *A simply connected surface of general type with* $p_g = 0$ *and* $K^2 = 3$. Geom. Topol. **13** (2009), no. 2, 743–767.

[PR] Papadakis, S.; Reid, M. *Kustin–Miller unprojection without complexes.* J. Algebraic Geometry **13** (2004), 563-577

[Pet] Peters, C. A. M. *On certain examples of surfaces with $p_g = 0$ due to Burniat.* Nagoya Math. J. **66** (1977), 109–119.

[PY1] Prasad, G.; Yeung, S-K. *Fake projective planes.* Invent. Math. **168** (2007), no. 2, 321–370.

[PY2] Prasad, G.; Yeung, S-K. *Arithmetic fake projective spaces and arithmetic fake Grassmannians.* Amer. J. Math. **131** (2009), no. 2, 379–407.

[R] Reid, M. *Surfaces with $p_g = 0$, $K^2 = 2$.*
Preprint available at: `http://www.warwick.ac.uk/~masda/surf/`.

[S] Sano, T *On classifications of non-Gorenstein $\mathbb{Q}$-Fano 3-folds of Fano index 1* J. Math. Soc. Japan **47** (1995), no. 2, 369–380.

[X] Xiao, G. *Finitude de l'application bicanonique des surfaces de type général.* Bull. Soc. Math. France **113** (1985), no. 1, 23–51.

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