ON SURFACES WITH A CANONICAL PENCIL

ROBERTO PIGNATELLI

Abstract. We classify the minimal surfaces of general type with $K^2 \leq 4\chi - 8$ whose canonical map is composed with a pencil, up to a bounded family.

More precisely we prove that there is exactly one irreducible family for each value of $\chi \gg 0$, $4\chi - 10 \leq K^2 \leq 4\chi - 8$. All these surfaces are complete intersections in a toric 4−fold. They can also be obtained as bidouble covers of Hirzebruch surfaces.

Introduction

In the celebrated paper [B], A. Beauville studies the surfaces of general type whose canonical map is not birational; here we denote by canonical map the map induced by $H^0(\Omega_S^2)$. Beauville's approach is "up to a bounded family": he shows that if we assume $\chi(\mathcal{O})$ big enough, which is equivalent to disregarding a finite number of components of the moduli scheme of the surfaces of general type, the situation simplifies considerably.

Beauville distinguishes two cases, since (if $\chi(\mathcal{O}) \geq 3$) the canonical image has dimension 1 or 2. We are interested in the first case, the case of the surfaces *whose canonical map is composed with a pencil*. In fact, by Stein factorisation, then the canonical map factors through a rational map with connected fibres onto a smooth curve, which is said to be a *canonical pencil*.

In this case [B] proves

Theorem 0.1 (Beauville). *If S is a minimal surface of general type whose canonical map is composed with a pencil and $\chi(\mathcal{O}_S) > 20$ then the pencil is free and the general fibre has genus $2 \leq g \leq 5$.*

Beauville first proves that the inequality $K^2 \geq 2(g-1)(\chi-2)$ holds, and then the theorem follows by the Miyaoka–Yau inequality $K^2 \leq 9\chi$. This argument motivated many authors to improve the lower bound for K^2; when $g = 2$ [X] proves

Theorem 0.2 (Xiao Gang). *If S is a minimal surface of general type whose canonical map is composed with a free pencil of curves of genus 2 then $K^2 \geq 4p_g - 6 \geq 4\chi - 10$.*

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The inequality is sharp, since there are examples of regular surfaces as above for all values of the pair of invariants (K^2, p_g) *in the range* $p_g \geq 2$, $4p_g - 6 \leq K^2 \leq 4p_g$.

If $g \geq 3$ stronger inequalities hold (see [S1], [S2], results summarised in the Theorem 4.10; see also [Z], [YM]). Therefore, if K^2 is sufficiently near to $4\chi - 10$ and χ is sufficiently big, then the genus of the canonical pencil has to be 2.

A natural question arise: is it possible, up to a bounded family, to classify the minimal surfaces of general type whose canonical map is composed with a pencil and with $K^2 - (4\chi - 10)$ "small enough"? To the best of our knowledge, nobody has studied this problem up to now.

In this note we answer this question. The main result is the following

Theorem 0.3. *Let* S *be a minimal surface of general type with* $K_S^2 \leq 4\chi(\mathcal{O}_S) - 8$ *and canonical map composed with a pencil. Assume moreover* $\chi(\mathcal{O}_S) \gg 0$.

Then the canonical pencil is a rational pencil of genus 2 *curves,* $q(S) = 0$ *and* S *is birational to a complete intersection* $X = Q \cap G$ *in the toric* $4-$*fold* $\mathbb{P} := \mathrm{Proj}(\mathrm{Sym}\, V)$ *where*

$$V := \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(\chi)x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - 2\chi + 8)y \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - \chi + 7)z$$

and the grading of $\mathrm{Sym}\, V$ *is given by* $\deg x_i = 1,\ \deg y = 2,\ \deg z = 3.$

X *has only canonical singularities and* Q, G *are of the form*

$$Q := \left\{x_0^2 + q_x x_1^2 + q_y y = 0\right\},$$

$$G := \left\{ z^2 + \sum_{\substack{i,j,k \geq 0 \\ i+j+2k=6}} G_{ijk} x_0^i x_1^j y^k = 0 \right\},$$

where q_x, q_y *and* G_{ijk} *are homogeneous polynomial on* $\mathbb{P}^1$.

Conversely, for any pair of positive integers (K^2, χ)*, the minimal resolution* S *of the singularities of a complete intersection* X *as above is a minimal surface of general type with* $K_S^2 = K^2$, $\chi(\mathcal{O}_S) = \chi$, $q(S) = 0$ *and the canonical map of* S *is the composition* $S \xrightarrow{f} \mathbb{P}^1 \overset{Ver}{\hookrightarrow} \mathbb{P}^{\chi-2}$ *where* f *is the genus* 2 *pencil induced by the projection* $\mathbb{P} \to \mathbb{P}^1$ *and* Ver *is the* $(\chi - 2)-$*Veronese embedding.*

In particular, if (K^2, χ) *are integers with* $\chi >> 0$ *and* $4\chi - 10 \leq K^2 \leq 4\chi - 8$*, then the subscheme of the moduli space of the minimal surfaces of general type given by the surfaces* S *with* $K_S^2 = K^2$, $\chi(\mathcal{O}_S) = \chi$ *whose canonical map is composed with a pencil is irreducible of dimension* $12\chi - 2K^2 - 15$.

The construction in Theorem 0.3 gives also surfaces with canonical map composed with a pencil and $K^2 > 4\chi - 8$; at least for every pair (K^2, χ) with $\chi \geq 3$, $4\chi - 10 \leq K^2 \leq 4\chi - 6$. Anyway, when

$K^2 > 4\chi - 8$ we can't guarantee that these are all the surfaces with a canonical pencil. Indeed, using an alternative description of these surfaces as bidouble covers, we show that if $K^2 = 4\chi - 6$ then this family is always properly contained in a larger irreducible family of surfaces with a canonical pencil.

We discuss the strategy of the proof.

By the above mentioned results we can assume that the pencil is free, has rational base and fibres of genus $g = 2$. We classify these, for $\chi \gg 0$ and $K^2 \leq 4\chi + 8$, computing their relative canonical algebra $\mathcal{R}$.

The main difficulty is in the fact that these genus 2 fibration have big Horikawa number $H := K^2 - 2\chi + 6$ (cf. [H], [R]): it grows asymptotically as 2χ. When studying the relative canonical algebra $\mathcal{R}$ of genus 2 fibrations over surfaces with fixed invariants (K^2, χ), big H corresponds to many *a priori* possibilities for its second graded piece $\mathcal{R}_2$, which is the key computation. We will see that, if the fibration is canonical, $K^2 \leq 4\chi + 8$ and $\chi \gg 1$, then $\mathcal{R}_2$ is determined up to isomorphisms, while this is not true for small χ, when some *exceptional* family appear.

To prove it, we consider the relative canonical model $X := \operatorname{Proj} \mathcal{R}$ and the fixed part of $|K_X|$, and we let $\mathfrak{h}$ be the union of its horizontal components, the components not contracted by the pencil. We consider moreover the involution i on X induced by the hyperelliptic involution of every fibre, and we denote by C the quotient X/i and by $\gamma\colon X \to C$ the resulting double cover. C contains two interesting effective divisors: the divisorial part Δ of the branch locus of γ and the curve $\mathfrak{s} := \gamma(\mathfrak{h})$ which is a section of the map $C \to \mathbb{P}^1$ induced by the fibration. Xiao Gang noticed that $\mathfrak{s}$ cannot be contained in Δ, and he used it to prove his inequality $K^2 \geq 4p_g - 6$, which is equivalent to $\mathfrak{s} \cdot \Delta \geq 0$.

The above geometrical construction can be translated in algebra as follows: $C = X/i$ is the relative Proj of a sub-algebra $\mathcal{A} \subset \mathcal{R}$ (cf. [CP]), and the inclusion $\mathfrak{s} \subset C$ determines a sheaf $\mathcal{S}$, quotient of $\mathcal{A}$ by some sheaf of ideals. The equation of Δ is a map from a line bundle, say $\mathcal{L}$, to $\mathcal{A}$ and more precisely to its homogeneous component $\mathcal{A}_6$, which depends from the multiplication map $\sigma_2\colon \operatorname{Sym}^2 \mathcal{R}_1 \to \mathcal{R}_2$ in a rather complicated way. On the contrary the relation between $\mathcal{S}$ and σ_2 is much simpler. The condition $\mathfrak{s} \not\subset \mathcal{A}$ means that the composition map $\mathcal{L} \to \mathcal{A}_6 \to \mathcal{S}_6$ is nonzero; $\operatorname{Hom}(\mathcal{L}, \mathcal{S}_6) \neq 0$ implies (not surprisingly) Xiao's inequality but unfortunately it is not strong enough to determine σ_2.

We consider then the "intermediate" subscheme $2\mathfrak{s}$. As $\mathfrak{s} \subset 2\mathfrak{s} \subset \mathcal{A}$, $2\mathfrak{s}$ determines a new sheaf of algebras $\mathcal{S}'$, a quotient of $\mathcal{A}$ which dominates $\mathcal{S}$. $\mathcal{S}'$ is still reasonably easy to compute and $\operatorname{Hom}(\mathcal{L}, \mathcal{S}'_6) \neq 0$. Studying this condition, stronger than the previous one, we will be able to compute $\mathcal{R}_2$, σ_2 and then $\mathcal{R}$.

The paper is structured as follows.

In Section 1 we study the surfaces constructed in Theorem 0.3 as complete intersections in a toric 4−fold, showing that the family is not empty for every value of the pair (χ, K^2) in the range $\chi \geq 3$, $4\chi - 10 \leq K^2 \leq 4\chi - 6$. Then we check that these surfaces have the properties stated in Theorem 0.3. Moreover we note that, if χ is big enough, these surfaces are bidouble covers of a Hirzebruch surface $\mathbb{F}_{|4\chi-8-K^2|}$ and use it to show that, when $K^2 = 4\chi - 6$ this family is properly contained in a larger irreducible family of surfaces with a canonical pencil.

In Section 2 we recall some results from the theory of genus 2 fibrations, and we introduce and compute the algebra of a section of the related conic bundle C. In Section 3 we study the algebra of $\mathfrak{s}$, and prove a version of Xiao's inequality. In Section 4 we consider the algebra of $2\mathfrak{s}$ and use this algebra (and a lifting lemma) to prove Theorem 3.1 which determines exactly for which pair (K^2, χ) there are surfaces with canonical map composed with a free genus 2 pencil not in the families described in Theorem 0.3. Theorem 0.3 follows then easily.

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1. Some surfaces with a canonical pencil

In this section we study the surfaces constructed in the Theorem 0.3. For later convenience we introduce the integers $p_g := \chi - 1$ and $\Theta := K^2 - 4\chi + 10$, so that

$$V = \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(p_g+1)x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(2p_g+\Theta)y \oplus \mathcal{O}_{\mathbb{P}^1}(3p_g+\Theta)z.$$

$\mathbb{P}$ is the 4−fold $\operatorname{Proj}(\operatorname{Sym} V)$ where the grading of $\operatorname{Sym} V$, sheaf of graded algebras over $\mathbb{P}^1$, is given by $\deg x_i = 1$, $\deg y = 2$, $\deg z = 3$. So the first graded pieces of $\operatorname{Sym} V$ are $(\operatorname{Sym} V)_1 = \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(p_g+1)x_1$, $(\operatorname{Sym} V)_2 = \mathcal{O}_{\mathbb{P}^1}(2)x_0^2 \oplus \mathcal{O}_{\mathbb{P}^1}(p_g+2)x_0x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(2p_g+2)x_1^2 \oplus \mathcal{O}_{\mathbb{P}^1}(2p_g+\Theta)y$, and so on.

$\mathbb{P}$ is a bundle over $\mathbb{P}^1$ whose fibres are weighted projective spaces $\mathbb{P}(1:1:2:3)$. In $\mathbb{P}$ we have the hypersurfaces $Q = \{x_0^2 + q_x x_1^2 + q_y y = 0\}$, $G = \left\{z^2 + \sum_{i+j+2k=6} G_{ijk} x_0^i x_1^j y^k = 0\right\}$ where the q_i and the G_{ijk} are homogeneous polynomial on $\mathbb{P}^1$ of the right degree to have Q and G well defined:

$$\begin{aligned}\deg q_x &= 2p_g\\ \deg q_y &= 2p_g - 2 + \Theta\\ \deg G_{ijk} &= -ip_g + (k-2)\Theta + (6-2k)\end{aligned}$$

$X := Q \cap G$ is a surface and the fibres of the surjective morphism $f\colon X \to \mathbb{P}^1$ are complete intersections of type $(2,6)$ in $\mathbb{P}(1:1:2:3)$; if X is not too singular then f is a genus 2 fibration.

Proposition 1.1. *If $0 \le \Theta \le 4$ then the general X is smooth.*

Proof. The equation of Q do not involve the variable z, so Q is a cone over $C := Q \cap \mathbb{P}'$, where $\mathbb{P}' := \{z = 0\}$ is a $\mathbb{P}(1:1:2)-$bundle over $\mathbb{P}^1$. Since $p_g \ge 2$ and $\Theta \ge 0$, q_x and q_y have positive degree and therefore by Bertini the general C is *quasi-smooth* and more precisely has only singularities of type A_1 (nodes) at the intersection with the section $\{x_0 = x_1 = 0\}$: $2p_g - 2 + \Theta$ nodes dominating the zeroes of q_y.

By the equation of G, X is a double cover of C branched at these nodes and along the curve $\Delta := C \cap \bar{\Delta}$ where $\bar{\Delta} := \{\sum G_{ijk} x_0^i x_1^j y^k = 0\}$ is a surface in $\mathbb{P}'$. To prove the smoothness of the general X we need that for general choice of the G_{ijk}, Δ is smooth and does not contain the nodes.

We study the fixed locus of $|\Delta|$. Note that $|\bar{\Delta}|$ has a big fixed locus: in fact if $p_g \gg 0$ all G_{ijk} with $i > 0$ vanish (having negative degree) and therefore the curve $\{x_1 = y = 0\}$ is contained in $\mathrm{Fix}(|\bar{\Delta}|)$.

If $0 \le \Theta \le 3$, since $\deg G_{003} = \Theta$ and $\deg G_{060} = 6 - 2\Theta$, the coefficients of both the monomials y^3 and x_1^6 can be chosen nonzero and therefore $\mathrm{Fix}(|\bar{\Delta}|)$ is contained in the curve $\{x_1 = y = 0\}$. Since this curve does not intersect C (due to the term x_0^2 in the equation of Q), $|\Delta|$ has no base points and its general element is smooth and irreducible.

If $\Theta = 4$, $G_{060} = 0$ and therefore, if $p_g \gg 0$, the equation of G is divisible by y: $\{y = 0\} \cap C$ is a smooth fixed component of $|\Delta|$. Since $\deg G_{041} = 0$, the residual linear system has no base points (as in the previous case) and its general element does not intersect the fixed component, so Δ is disconnected but still smooth.

□

Proposition 1.2. *If $\Theta \le 2$ and $p_g > 6 - 2\Theta$ the subscheme of the moduli space of surfaces of general type given by these surfaces has dimension $4p_g + 9 - 2\Theta$.*

Proof. The proof is identical to the analogous one in [P, Proposition 5.2].

□

One can use the same method to compute the dimension of the other families, which can be a little bit bigger. Indeed, if $\Theta \le 2$ and $p_g > 6 - 2\Theta$, then no term including the variable x_0 appears in the equation of G: when these terms appear, they give a slightly larger number of moduli.

If $p_g > 2$ these families are not irreducible components of the moduli space, see the forthcoming Remark 1.3. We do not know the codimension of these families in the irreducible component of the moduli space containing them.

The canonical system and the relative canonical algebra. By adjunction, denoting by H the class of $|\mathcal{O}_{\mathbb{P}}(1)|$, the linear system of the relative hyperplanes defined by maps $\mathcal{O}_{\mathbb{P}^1} \hookrightarrow (\operatorname{Sym} V)_1$, and by F the pull back of a point of $\mathbb{P}^1$

$$\begin{aligned} K_{X|\mathbb{P}^1} &= (K_{\mathbb{P}|\mathbb{P}^1} + Q + G)_{|X} \\ &= ((6p_g + 2\Theta + 2)F - 7H - 2F + 2H - (6p_g + 2\Theta)F + 6H)_{|X} \\ &= H_{|X}. \end{aligned}$$

In fact the relative canonical algebra of the fibration is the quotient of $\operatorname{Sym} V$ by the ideal $\mathcal{I}$ generated by the equations of Q and G.

In particular $K_X = (H - 2F)_{|X}$, the restriction map $H^0(\mathbb{P}, H - 2F) \to H^0(X, K_X)$ is an isomorphism and $H^0(X, K_X)$ is cut by the hypersurfaces $\{hx_1 = 0\}$, with h varying in $H^0(\mathcal{O}_{\mathbb{P}^1}(p_g - 1))$: the fixed part of the canonical system is cut by $\{x_1 = 0\}$ and the canonical map is the composition of f with the $(p_g - 1)-$uple embedding $\mathbb{P}^1 \to \mathbb{P}^{p_g - 1}$.

Then $p_g(X) = p_g$ and

$$\begin{aligned} K_X^2 &= (H - 2F)^2 \cdot Q \cdot G \\ &= 12H^4 - 4(3p_g + \Theta + 15)H^3F \\ &= 2\left(1 + p_g + 1 + \frac{2p_g + \Theta}{2} + \frac{3p_g + \Theta}{3}\right) - 4\frac{3p_g + \Theta + 15}{6} \\ &= 4p_g - 6 + \Theta. \end{aligned}$$

Remark 1.3. M. Reid has shown me how to deform the above surfaces with $p_g > 2$ to surfaces which still have the genus 2 pencil but whose canonical map is no longer composed with it.

The idea is to deform $\mathcal{R}_1 = \mathcal{O}(1) \oplus \mathcal{O}(p_g + 1)$ to any "more balanced" bundle, as, *e.g.*, $\mathcal{O}(2) \oplus \mathcal{O}(p_g)$, so that $f_* K_S$ becomes generated. More precisely we deform $\mathbb{P}$ by deforming V to $\mathcal{O}(2) \oplus \mathcal{O}(p_g) \oplus \mathcal{O}(2p_g + \Theta) \oplus \mathcal{O}(3p_g + \Theta)$ and then deform accordingly Q and G.

Description as bidouble covers. For $p_g > 6$, x_0 does not appear in the equation of G. In this case X is invariant by the $(\mathbb{Z}/2\mathbb{Z})^2$ action on $\mathbb{P}$ defined by $(x_0 : x_1 : y : z) \mapsto (\pm x_0 : x_1 : y : \pm z)$.

The quotient $X/(\mathbb{Z}/2\mathbb{Z})^2$ is the $\mathbb{P}(1:2)$ bundle over $\mathbb{P}^1$ given by the variables x_1 and y, which is the Hirzebruch surface

$$\mathbb{F}_{|2-\Theta|} = \operatorname{Proj}(\operatorname{Sym}(\mathcal{O}(2p_g + 2)x_1^2 \oplus \mathcal{O}(2p_g + \Theta)y)).$$

As in [C], a bidouble cover of a Hirzebruch surface is determined by the branch divisors D_1, D_2, D_3 of the 3 intermediate double covers. In our case $D_1 = \{q_x x_1^2 + q_y y = 0\}$, $D_2 = \{\sum G_{0jk} x_1^j y^k = 0\}$, $D_3 = \{x_1 = 0\}$.

Writing Γ_∞ for the only negative section of the ruling $|\Gamma|$ of $\mathbb{F}_r$, and $|\Gamma_1|$, $|\Gamma_2|$ for the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, the D_i are general in the following linear system

Θ	base	$\|D_1\|$	$\|D_2\|$	$\|D_3\|$
0	$\mathbb{F}_2$	$\|\Gamma_\infty + 2p_g\Gamma\|$	$\|3\Gamma_\infty + 6\Gamma\|$	Γ_∞
1	$\mathbb{F}_1$	$\|\Gamma_\infty + 2p_g\Gamma\|$	$\|3\Gamma_\infty + 4\Gamma\|$	Γ_∞
2	$\mathbb{P}^1 \times \mathbb{P}^1$	$\|\Gamma_1 + 2p_g\Gamma_2\|$	$\|3\Gamma_1 + 2\Gamma_2\|$	$\|\Gamma_1\|$
3	$\mathbb{F}_1$	$\|\Gamma_\infty + (2p_g + 1)\Gamma\|$	$\|3\Gamma_\infty + 3\Gamma\|$	$\|\Gamma_\infty + \Gamma\|$
4	$\mathbb{F}_2$	$\|\Gamma_\infty + (2p_g + 2)\Gamma\|$	$\Gamma_\infty + \|2\Gamma_\infty + 4\Gamma\|$	$\|\Gamma_\infty + 2\Gamma\|$

This table shows in particular that the surfaces we have constructed with $\Theta = 2$ are the same as those constructed by Catanese in [C, Example 2], (the case $a = 2$, $k = 0$ in the notation there). Catanese in the same Example gave also surfaces with $\Theta = 4$ or 6 as bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$: it is easy to see that our examples with $\Theta = 4$ are a degeneration of Catanese's examples, obtained by deforming $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{F}_2$.

Proposition 1.4. *There are surfaces whose canonical map is composed with a genus 2 pencil which are bidouble covers of a Hirzebruch surface for each value of $p_g \geq 2$, $\Theta := K^2 - 4p_g + 6 \in \{0, 1, 2, 3, 4, 5, 6\}$.*

Proof. The above table proves the statement for $\Theta \leq 4$, and Catanese's examples above mentioned the case $\Theta = 6$. For $\Theta = 5$ it is enough to take a bidouble cover of $\mathbb{F}_1$ with branching divisors $D_1 \in |\Gamma_\infty + (2p_g + 2)\Gamma|$, $D_2 \in \Gamma_\infty + |2(\Gamma_\infty + \Gamma)|$, $D_3 \in |\Gamma_\infty + 2\Gamma|$. □

The existence of surfaces with these values of the invariants were already know to Xiao Gang who constructed in [X] one example for each value of p_g, K^2 in the above range.

2. Genus 2 fibrations, related conic bundles and their sections

If a surface S has a fibration onto a curve $f\colon S \to B$ with fibres of genus $g \geq 2$, then S is birational to $X := \mathrm{Proj}(\mathcal{R}(f))$, where $\mathcal{R}(f)$ (or $\mathcal{R}$ for short) is the relative canonical algebra

$$\mathcal{R} := \bigoplus_{d \geq 0} f_* \mathcal{O}_S(dK_{S|B}).$$

X, the *relative canonical model* of f, is obtained from S by contracting all *vertical* $(-2)-$curves. To simplify the arguments, it is convenient to replace S by X, considering (with a mild abuse of notation) the fibration $f\colon X \to B$.

The hyperelliptic involution of the fibres extends to an involution i of X, inducing a quotient surface $C := X/i$. The fibration f factors as $\pi \circ \gamma$ where $\pi\colon C \to B$ is a conic bundle and $\gamma\colon X \to C$ is a finite double cover branched at a finite sets of points $P \subset C$ and along a curve $\Delta \subset C$. Obviously

$$P \cap \Delta = \emptyset. \tag{1}$$

As $X = \mathrm{Proj}(\mathcal{R})$, $C = \mathrm{Proj}(\mathcal{A})$ for a subalgebra $\mathcal{A} \subset \mathcal{R}$ which has been studied in [CP] (see also [R]).

Lemma 2.1 ([CP]). *$\mathcal{A}$, as sheaf of algebras, is generated by $\mathcal{A}_1 = \mathcal{R}_1$ and $\mathcal{A}_2 = \mathcal{R}_2$ and related by the multiplication map*

$$\sigma_2\colon \mathrm{Sym}^2 \mathcal{A}_1 \hookrightarrow \mathcal{A}_2$$

whose cokernel is the structure sheaf of an effective divisor τ on B of degree $K_X^2 - 2\chi + 6$.

Proof. This has been proved in [CP]. More precisely generators and relations of $\mathcal{A}$ are computed by Lemma 4.4 and Remark 4.5 and the cokernel of σ_2 is computed in Lemma 4.1. For the degree of τ see the formulae in Theorem 4.13.

□

In the next remark we recall the description of the stalks of $\mathcal{A}$ and $\mathcal{R}$ at any point of the base curve B which will be used in the forthcoming sections.

Remark 2.2. If $p \in B$ does not belong to $\mathrm{supp}\,\tau$, then the stalk of $\mathcal{R}$ at p is isomorphic to

$$\mathcal{O}_{B,p}[x_0, x_1, z]/(z^2 - f_6(x_0, x_1; t));$$

else, letting r be the multiplicity of p in $\mathrm{supp}\,\tau$, the stalk of $\mathcal{R}$ at p is isomorphic to

$$\mathcal{O}_{B,p}[x_0, x_1, y, z]/(t^r y - f_2(x_0, x_1; t), z^2 - f_6(x_0, x_1, y; t)). \tag{2}$$

Here f_d are weighted homogeneous polynomials of degree d, where the variables x_i have weight 1 and y has weight 2. The involution acts as $z \mapsto -z$ (z has weight 3) fixing all the other variables. $\mathcal{A}$ is the subalgebra generated by x_0, x_1 and, in the second case, y.

From this description follows that π maps P bijectively to $\mathrm{supp}\,\tau$ and, equivalently f maps bijectively the set of the isolated fixed points of i to $\mathrm{supp}\,\tau$; in fact, if $p \in \mathrm{supp}\,\tau$ there is an isolated fixed point contained in $f^{-1}(p)$ which is, in the local coordinates induced by the above description of the stalk of $\mathcal{R}$ at p, the point $Q_0 := ((x_0 : x_1 : y : z); t) = ((0 : 0 : 1 : 0); 0)$.

Assume, by sake of simplicity, that $r = 1$ and $f_2(x_0, x_1; 0)$ is not a square. Then $f^{-1}(p)$ is union of two elliptic curves intersecting transversally in Q_0, each of them mapped by γ to a rational curve: the two rational curves intersect in $P_0 = \gamma(Q_0) \in P$, which is a node of C.

By Remark 2.2 follows, if we denote by $\mathcal{R}'_3$ the direct summand of $\mathcal{R}_3$ of rank 1 locally generated by z,

Proposition 2.3 ([CP])**.** $\mathcal{R} \cong \mathcal{A} \oplus (\mathcal{A} \otimes \mathcal{R}'_3)[-3]$ *as graded* $\mathcal{A}-$*module. The ring structure of* $\mathcal{R}$ *gives a multiplication map* $\delta\colon (\mathcal{R}'_3)^2 \to \mathcal{A}_6$ *inducing the divisor* $\Delta \subset C$*. Moreover* $\mathcal{R}'_3 \cong \det \mathcal{A}_1 \otimes \mathcal{O}_B(\tau)$.

Proof. See [CP], in particular Proposition 4.8. Note that $\mathcal{R}'_3$ is V_3^+ in the notation of [CP].

□

Definition 2.4. Let $f\colon S \to B$ be a genus 2 fibration, and consider the associated conic bundle $\pi\colon C = \operatorname{Proj}(\mathcal{A}) \to B$.

To any section s of π we associate a sheaf of graded algebras $\mathcal{S}$, which is the quotient of $\mathcal{A}$ by the ideal sheaf of the elements vanishing along s.

Note that by definition each homogeneous piece $\mathcal{S}_d$ of $\mathcal{S}$ is of rank 1 and torsion free, so a line bundle.

Proposition 2.5. *There exists an effective divisor* $\tau' < \tau$ *such that* $\forall d \geq 1$, $\mathcal{S}_d \cong \mathcal{S}_1^d\left(\left\lfloor \frac{d}{2} \right\rfloor \tau'\right)$.

Proof. The statement is empty for $d = 1$, so we start with $d = 2$. By Lemma 2.1 we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Sym}^2 \mathcal{A}_1 & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \mathcal{O}_\tau & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{S}_1^2 & \longrightarrow & \mathcal{S}_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It follows that there exists a surjection from $\mathcal{O}_\tau$ to the cokernel of the map $\mathcal{S}_1^2 \hookrightarrow \mathcal{S}_2$, which is then isomorphic to $\mathcal{O}_{\tau'}$ for some $\tau' < \tau$. In particular $\mathcal{S}_2 \cong \mathcal{S}_1^2(\tau')$.

To conclude we show that $\forall k \geq 2$, $\mathcal{S}_{2k} \cong \mathcal{S}_2^k$ and $\mathcal{S}_{2k-1} \cong \mathcal{S}_2^{k-1} \otimes \mathcal{S}_1$.

The composition of the maps $\operatorname{Sym}^k \mathcal{A}_2 \to \mathcal{A}_{2k} \to \mathcal{S}_{2k}$ is surjective (since both maps are surjective) and factors through the multiplication map $\mathcal{S}_2^k \to \mathcal{S}_{2k}$, which is therefore surjective too; the injectivity follows since it is a map between line bundles. A similar argument works for the map $\mathcal{S}_2^{k-1} \otimes \mathcal{S}_1 \to \mathcal{S}_{2k-1}$.

□

Recall that by Remark 2.2, $\pi_{|P}$ maps P bijectively onto $\operatorname{supp}\tau$. In the next remark we show that, roughly speaking, $\tau' < \tau$ marks those points of P which belong to s.

Remark 2.6. Let p be a point of τ with multiplicity $r > 0$; we write the stalks of $\mathcal{R}$ and $\mathcal{A}$ at p as in Remark 2.2 by asking further that x_1 generates the kernel of the map $\mathcal{A}_1 \to \mathcal{S}_1$.

$\mathcal{S}_2$ is a line bundle and the surjection $\mathcal{A}_2 \to \mathcal{S}_2$ factors through $(\mathcal{A}/x_1)_2$, which has rank 1 but possibly torsion. More precisely, if $p \in \operatorname{supp}\tau$ then the stalk $(\mathcal{A}/x_1)_{2,p} \cong (\mathcal{O}_{B,p}[x_0,y]/(t^r y - f_2(x_0,0;t))_2$ has torsion when t divides $f_2(x_0,0;t)$. It follows that

$$\mathcal{S}_{2,p} \cong (\mathcal{O}_{B,p}[x_0,y]/((t^r y - f_2(x_0,0;t))/t^{r''})_2$$

where r'' is the maximal power of t which divides $t^r y - f_2(x_0,0;t)$.

The stalk at p of $\mathcal{S}_1$ is generated by the class of x_0. Therefore

$$\mathcal{O}_{\tau'} \cong \operatorname{coker}(\mathcal{S}_1^2 \to \mathcal{S}_2) \cong \bigoplus_{p\in\operatorname{supp}\tau} (\mathcal{O}_{B,p}[y]/(t^{r-r''}y))_2;$$

the multiplicity of p in τ' is $r' := r - r''$.

Note that p belongs to τ' if and only if $r \neq r''$ *i.e.* the section passes through P_0. Therefore, if τ is reduced, then $\tau' < \tau$ marks the points of P contained in s. Moreover

$$\tau' = 0 \Leftrightarrow P \cap s = \emptyset. \tag{3}$$

3. The section of the conic bundle induced by Fix(K)

We want to prove the following

Theorem 3.1. *Let S be a minimal surface of general type with $K_S^2 \leq 4\chi - 8$, having a free canonical pencil of genus 2 curves. Then S is regular and the pencil has rational base.*

If moreover we assume either $\Theta := K^2 - 4\chi + 10 = 0$ or $p_g > \Theta + 4$ then $\mathcal{R} \cong (\operatorname{Sym} V)/\mathcal{I}$, with

$$V := \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(\chi)x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - 2\chi + 8)y \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - \chi + 7)z,$$

the grading of $\operatorname{Sym} V$ is given by $\deg x_i = 1$, $\deg y = 2$, $\deg z = 3$, and $\mathcal{I}$ is the sheaf of ideals generated by $x_0^2 + q_x x_1^2 + q_y y$, $z^2 + \sum_{\substack{i,j,k\geq 0\\ i+j+2k=6}} G_{ijk} x_0^i x_1^j y^k = 0$, where q_x, q_y and G_{ijk} are homogeneous polynomial on $\mathbb{P}^1$.

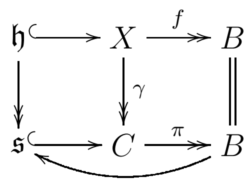
The proof of Theorem 3.1 requires some preparation, and we delay it to the end of the next section.

If $K^2 \leq 4\chi - 8$ Xiao Gang's inequality ([X]) $K^2 \geq 4p_g - 6$ forces $q(S) = 0$, so the base curve is $\mathbb{P}^1$, which is the first part of Theorem 3.1. The difficult part is the computation of $\mathcal{R}$.

Lemma 3.2. $\mathcal{R}_1 \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(p_g + 1)$

Proof. S has genus 2 pencil and we are assuming K_S composed with f. *i.e.* $|K| = \Phi + |f^*L|$ so $f_*\mathcal{O}_S(K_S) = f_*\mathcal{O}_S(\Phi) \otimes \mathcal{O}_{\mathbb{P}^1}(L)$. By $p_g = h^0(K) = h^0(L)$, $f_*\mathcal{O}_S(K_S) \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(p_g - 1)$ with $a < 0$.
Finally $q(S) = 0$ implies $h^1(f_*\mathcal{O}_S(K_S)) = 0$ and therefore $a = -1$. □

Let $\mathfrak{h}$ be the horizontal fixed part of $|K_X|$, the union of the components of $\mathrm{Fix}(|K_X|)$ which are not contracted by f. Since, on a general fibre, $\mathfrak{h}$ cuts a canonical divisor, the pull back of a point of the canonical image, $\mathfrak{h} = \gamma^*(\mathfrak{s})$ for some section $\mathfrak{s}$ of the conic bundle $\pi \colon C \to B$, to which we can apply the results of the previous section.



Since $\mathfrak{h}$ does not move in X, $\mathfrak{s}$ can't move in C and $\mathrm{Hom}(\mathcal{A}_1, \mathcal{S}_1) = \mathbb{C}$. Therefore by Lemma 3.2

Remark 3.3. If $\mathcal{S}$ is the sheaf of algebras of the section $\mathfrak{s} \subset C$, $\mathcal{S}_1 \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Then, by Proposition 2.5, $\mathcal{S}$ is determined up to isomorphisms by a divisor $\tau' < \tau$.

Lemma 3.4. $\tau' \neq 0$ *and* $\mathfrak{s} \not\subset \Delta$.

Proof. Assume by contradiction $\mathfrak{s} \cap P = \emptyset$. Then, in a neighbourhood of $\mathfrak{h}$, $\mathcal{O}_X(K_X)$ is the pull back of $\mathcal{O}_{C \setminus P}(K_C + \delta)$ where $\mathcal{O}_{C\setminus P}(2\delta) \cong \mathcal{O}_{C\setminus P}(\Delta)$. Therefore $\mathfrak{h}K_X = 2\mathfrak{s}(K_C + \delta)$.
Since $K_{X|\mathbb{P}^1}$ is the relative $\mathcal{O}(1)$ of $\mathrm{Proj}(\mathcal{R})$, $|K_X|$ is given by maps $\omega_{\mathbb{P}^1}^{-1} \to \mathcal{R}_1$, $|K_C + \delta|$ by maps $\omega_{\mathbb{P}^1}^{-1} \to \mathcal{A}_1$, and $(K_C + \delta)_{|\mathfrak{s}}$ by maps $\omega_{\mathbb{P}^1}^{-1} \to \mathcal{S}_1$.
So $\mathfrak{s}(K_C + \delta) = \deg(K_C + \delta)_{|\mathfrak{s}} = \deg \mathcal{S}_1 - \deg \omega_{\mathbb{P}^1}^{-1} = 1 - 2 = -1$. It follows that $K_X\mathfrak{h} = -2$ contradicting the assumption K_X nef.
Therefore $\mathfrak{s} \cap P \neq \emptyset$, which by (3) implies $\tau' \neq 0$, and, since $\mathfrak{s}$ is irreducible, by (1) follows $\mathfrak{s} \not\subset \Delta$. □

Remark 3.5. Since $\mathfrak{s} \not\subset \Delta$, then the composition of maps $(\mathcal{R}'_3)^2 \xrightarrow{\delta} \mathcal{A}_6 \to \mathcal{S}_6$ is nonzero. Therefore $\mathrm{Hom}((\mathcal{R}'_3)^2, \mathcal{S}_6) \neq 0$.

It follows a version of Xiao's inequality

Corollary 3.6 (Xiao's inequality)**.** $K^2 \geq 4\chi - 10 + 3\deg(\tau - \tau')$. *In particular, if* $K^2 \leq 4\chi - 8$, *then* $\tau = \tau'$

Proof.

$$3(2 + \deg \tau') = \deg \mathcal{S}_6 \geq \deg(\mathcal{R}'_3)^2 = 2(\deg \tau + \chi + 1)$$

and the inequality follows by the formula (see [H, Theorem 3], [CP, Theorem 4.13]) $K^2 = 2\chi + \deg \tau - 6$.

□

Remark 3.7. As mentioned in the introduction our proof is very similar to Xiao's proof; what we have shown is that $\deg \mathcal{S}_6 - \deg(\mathcal{R}'_3)^2 = \Delta\mathfrak{s}$ is nonnegative.

In the next section we will see how, considering the non reduced divisor $2\mathfrak{s}$, this approach gives formulae which cannot be obtained by intersection arguments.

Remark 3.8. Maybe $\tau = \tau'$ holds more generally, and not only when $K^2 \leq 4\chi - 8$.

In fact, as pointed out to the author by Polizzi, $P \subset \gamma(\mathrm{Fix}(|K_X|))$, since each isolated fixed point of the involution on X is a base point of the canonical system of the fibre through it, and therefore it is contained in the fixed locus of $|K_X|$.

Anyway, the condition $\tau = \tau'$ is slightly stronger; if we assume τ reduced (by sake of simplicity) then $\tau = \tau' \Leftrightarrow P \subset \mathfrak{s}$, and in general $\mathfrak{s} \subsetneq \gamma(\mathrm{Fix}(|K_X|))$.

Assumption: From now on we assume $K^2 \leq 4\chi - 8$, and therefore $\tau = \tau'$.

By Lemma 2.1 and Lemma 3.2

$$\mathcal{A}_1 \cong \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(p_g + 1)x_1 \tag{4}$$

labelling for our convenience its summands with the variables x_0, x_1. By Remark 3.3, x_1 belongs to the kernel of the map $\mathcal{A} \to \mathcal{S}$. In fact

Lemma 3.9. $\mathcal{S} \cong \mathcal{A}/x_1$.

Proof. Since x_1 belongs to the kernel of the map $\mathcal{A} \to \mathcal{S}$ we have a surjection $\mathcal{A}/x_1 \to \mathcal{S}$, and we conclude by showing that each graded piece of $\mathcal{A}/x_1$ is a line bundle.

Looking at the stalks of $\mathcal{A}$ as described in Remark 2.2 this fails if and only if there is a point $p \in \operatorname{supp} \tau$ such that t divides $f_2(x_0, 0; t)$ (here f_2 is the one in (2)). It follows that the corresponding parameter r'' (introduced in Remark 2.6) is nonzero and then $\tau \neq \tau'$, contradicting Corollary 3.6.

□

In particular, denoting by $q \colon \mathcal{A}_2 \to \mathcal{S}_2$ the second graded piece of the surjection $\mathcal{A} \to \mathcal{S}$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(p_g + 2)x_0x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(2p_g + 2)x_1^2 \xrightarrow{(\sigma_2)_{|(x_1)}} \mathcal{A}_2 \xrightarrow{q} \mathcal{S}_2 \to 0 \tag{5}$$

where $\mathcal{S}_2 \cong \mathcal{O}_{\mathbb{P}^1}(2 + \deg\tau) = \mathcal{O}_{\mathbb{P}^1}(2p_g + \Theta)$ with $\Theta := K^2 - 4p_g + 6 \in \{0, 1, 2\}$.

$\mathcal{A}_2$ is a rank 3 vector bundle over $\mathbb{P}^1$, so there are $d_0 \leq d_1 \leq d_2$ with

$$\mathcal{A}_2 = \mathcal{O}_{\mathbb{P}^1}(d_0)y_0 \oplus \mathcal{O}_{\mathbb{P}^1}(d_1)y_1 \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)y_2$$

Lemma 3.10. $d_2 = 2p_g + 2$. *In particular, we can assume* $y_2 = \sigma_2(x_1^2)$.

Proof. The injectivity of σ_2 forces $d_2 \geq 2p_g + 2$. If $d_2 > 2p_g + 2$, then $d_2 > 2p_g + \Theta$ so $\mathcal{O}_{\mathbb{P}^1}(d_2)y_2 \subset \ker q$. But by (5) $\ker q \cong \mathcal{O}_{\mathbb{P}^1}(2p_g + 2) \oplus \mathcal{O}_{\mathbb{P}^1}(p_g + 2)$, a contradiction.

□

Proposition 3.11. *There is $\alpha \geq 0$ such that*

$$d_0 = p_g + 2 + \alpha$$
$$d_1 = 2p_g + \Theta - \alpha$$
$$d_2 = 2p_g + 2$$

Proof. We have computed d_2 in Lemma 3.10; since by (4) and Lemma 2.1 $\deg \mathcal{A}_2 = 5p_g + \Theta + 4$, we only have to prove $\alpha \geq 0$.

Else $d_1 > 2p_g+\Theta$, so $y_1 \in \ker q$. To embed $\mathcal{O}_{\mathbb{P}^1}(d_1)y_1 \oplus \mathcal{O}_{\mathbb{P}^1}(2p_g+2)y_2$ in $\ker q \cong \mathcal{O}_{\mathbb{P}^1}(2p_g + 2)x_1^2 \oplus \mathcal{O}_{\mathbb{P}^1}(p_g + 2)x_0x_1$ we need $d_1 \leq p_g + 2$, so $2p_g + \Theta < p_g + 2$, a contradiction to $\Theta \geq 0$, $p_g \geq 2$.

□

The matrix of σ_2 can now be written as

$$\sigma_2 = \begin{pmatrix} g_0 & f_0 & 0 \\ g_1 & f_1 & 0 \\ g_2 & 0 & 1 \end{pmatrix} \tag{6}$$

where we have chosen $\{x_0^2, x_0x_1, x_1^2\}$ and $\{y_0, y_1, y_2\}$ as ordered bases of the source and of the target. In other words, $\sigma_2(x_1^2) = y_2$, $\sigma_2(x_0x_1) = \sum f_iy_i$, $\sigma_2(x_0^2) = \sum g_iy_i$. The exact sequence (5) gives

$$\gcd(f_0, f_1) = 1 \tag{7}$$

which we have used to set $f_2 = 0$ by a suitable change of coordinates in the target.

Proposition 3.12. *If $\alpha = 0$, then $\mathcal{R} \cong (\operatorname{Sym} V)/\mathcal{I}$, with*
$V := \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(\chi)x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - 2\chi + 8)y \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - \chi + 7)z$, *the grading of* $\operatorname{Sym} V$ *is given by* $\deg x_i = 1$, $\deg y = 2$, $\deg z = 3$, *and $\mathcal{I}$ is the sheaf of ideals generated by* $x_0^2 + q_xx_1^2 + q_yy, z^2 + \sum_{\substack{i,j,k\geq 0\\ i+j+2k=6}} G_{ijk}x_0^ix_1^jy^k = 0$, *where q_x, q_y and G_{ijk} are homogeneous polynomial on $\mathbb{P}^1$.*

Proof. If $\alpha = 0$, then $\deg f_0 = d_0 - (p_g + 2) = 0$ and by (7) we can assume $f_0 = 1$. Then the exact sequence (5) splits and therefore by Lemma 2.1, $\mathcal{A}$ is the quotient of $\operatorname{Sym} \mathcal{W}$, where

$$\mathcal{W} \cong \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(p_g + 1)x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(2p_g + \Theta)y_1$$

by the ideal generated by

$$x_0^2 - (g_0x_0x_1 + (g_1 - f_1g_0)y_1 + g_2x_1^2). \tag{8}$$

By changing the splitting of $\mathcal{A}_1$ in (4) , *i.e.* by changing the choice of x_0, we can assume $g_0 = 0$.

The statement follows, setting $y := y_1$, from Proposition 2.3 since each map $\operatorname{Sym}^d \mathcal{W} \to \mathcal{A}_d$ splits (because of the term x_0^2 in the equation (8)). In particular the map δ lifts to $\operatorname{Sym}^6 \mathcal{W}$ giving the equation $\sum G_{ijk} x_0^i x_1^j y^k$.

□

4. The algebra of $2\mathfrak{s}$ and a lifting lemma

Definition 4.1. We consider the sheaf of algebras $\mathcal{S}' := \mathcal{A}/x_1^2$. Since $\mathcal{S} = \mathcal{A}/x_1$, the surjection $\mathcal{A} \to \mathcal{S}'$ correspond to the inclusion of the non reduced divisor $2\mathfrak{s} \subset C$.

$\mathcal{A}_6$ is a quotient of $\operatorname{Sym}^3 \mathcal{A}_2$, the cokernel of (see [CP, Lemma 4.4])

$$i_3\colon \det \mathcal{A}_1^2 \otimes \mathcal{A}_2 \to \operatorname{Sym}^3 \mathcal{A}_2$$

defined by

$$i_3((x_0 \wedge x_1)^2 \otimes v) = (\sigma_2(x_0^2)\sigma_2(x_1^2) - \sigma_2(x_0x_1)^2)v.$$

$\mathcal{S}'_6$ is the quotient of $\mathcal{A}_6$ by the multiples of x_1^2. Killing first $y_2 = x_1^2$ and then the multiples of $\sigma_2(x_0x_1)^2 = (f_0y_0 + f_1y_1)^2$, we obtain $\mathcal{S}'_6$ as cokernel of the map

$$\begin{matrix} \mathcal{O}_{\mathbb{P}^1}(2p_g+4+d_0) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^1}(2p_g+4+d_1) \end{matrix} \quad \to \quad \operatorname{Sym}^3 \mathcal{S}'_2 \cong \begin{matrix} \mathcal{O}_{\mathbb{P}^1}(3d_0)y_0^3 \\ \oplus \\ \mathcal{O}_{\mathbb{P}^1}(2d_0+d_1)y_0^2y_1 \\ \oplus \\ \mathcal{O}_{\mathbb{P}^1}(d_0+2d_1)y_0y_1^2 \\ \oplus \\ \mathcal{O}_{\mathbb{P}^1}(3d_1)y_1^3 \end{matrix}$$

given by the matrix

$$\begin{pmatrix} f_0^2 & 0 \\ 2f_0f_1 & f_0^2 \\ f_1^2 & 2f_0f_1 \\ 0 & f_1^2 \end{pmatrix}$$

and therefore by (7) $\mathcal{S}'_6$ is locally free of rank 2 and more precisely

$$\mathcal{S}'_6 \cong \mathcal{O}_{\mathbb{P}^1}(5p_g + 2\Theta + \alpha + 2) \oplus \mathcal{O}_{\mathbb{P}^1}(6p_g + 3\Theta - \alpha)$$

and the surjection $\operatorname{Sym}^3 \mathcal{S}'_2 \to \mathcal{S}'_6$ is given by the matrix

$$\begin{pmatrix} 3f_1^2 & -2f_0f_1 & f_0^2 & 0 \\ 0 & f_1^2 & -2f_0f_1 & 3f_0^2 \end{pmatrix} \tag{9}$$

It follows

Proposition 4.2. $0 \le \alpha \le \Theta$.

Proof. The surjection $\mathcal{A} \to \mathcal{S}$ factors through $\mathcal{S}'$. By Remark 3.5 $\operatorname{Hom}(\mathcal{R}_3'^2, \mathcal{S}_6') \neq 0$, so $2(3p_g + \Theta) \leq 6p_g + 3\Theta - \alpha \Leftrightarrow \alpha \leq \Theta$. □

Corollary 4.3. *If $\Theta = 0$, then $\mathcal{R} \cong (\operatorname{Sym} V)/\mathcal{I}$, with*

$$V := \mathcal{O}_{\mathbb{P}^1}(1)x_0 \oplus \mathcal{O}_{\mathbb{P}^1}(\chi)x_1 \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - 2\chi + 8)y \oplus \mathcal{O}_{\mathbb{P}^1}(K^2 - \chi + 7)z,$$

the grading of $\operatorname{Sym} V$ *is given by* $\deg x_i = 1$, $\deg y = 2$, $\deg z = 3$, *and $\mathcal{I}$ is the sheaf of ideals generated by* $x_0^2 + q_x x_1^2 + q_y y$, $z^2 + \sum_{\substack{i,j,k \geq 0 \\ i+j+2k=6}} G_{ijk} x_0^i x_1^j y^k = 0$, *where q_x, q_y and G_{ijk} are homogeneous polynomial on $\mathbb{P}^1$.*

Proof. If $\Theta = 0$, by Proposition 4.2 follows $\alpha = 0$ and we can apply Proposition 3.12. □

For the case $\Theta > 0$ we need to study some lifting properties.

One can always find a line bundle $\mathcal{M}$ and a map $m \in \operatorname{Hom}(\mathcal{R}_3'^2 \otimes \mathcal{M}^{-1}, \mathcal{R}_3'^2)$ such that $\delta \circ m$ lifts to $\operatorname{Sym}^3 \mathcal{A}_2$, giving a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_3'^2 \otimes \mathcal{M}^{-1} & \xrightarrow{\bar{\delta}} & \operatorname{Sym}^3 \mathcal{A}_2 \\ {\scriptstyle m}\downarrow & & \downarrow \\ \mathcal{R}_3'^2 & \xrightarrow{\delta} & \mathcal{A}_6 \end{array} \tag{10}$$

Once we have m, to compute $\operatorname{Hom}(\mathcal{R}_3'^2, \mathcal{A}_6)$ we need to study which $\bar{\delta} \in \operatorname{Hom}(\mathcal{R}_3'^2 \otimes \mathcal{M}^{-1}, \operatorname{Sym}^3 \mathcal{A}_2)$ belong to a diagram as (10). To simplify this computation, we need $\mathcal{M}$ of degree as small as possible. The forthcoming Lemma 4.5 shows that we can take $m = f_0^4$, so $\deg \mathcal{M} = 4\alpha$.

Definition 4.4. Consider the natural decomposition

$$\operatorname{Sym}^3 \mathcal{A}_2 = \sum_{i+j+k=3} \mathcal{L}_{ijk} y_0^i y_1^j y_2^k$$

as sum of vector bundles. Clearly $\mathcal{L}_{ijk} \cong \mathcal{O}(id_0 + jd_1 + kd_2)$. Then

$$\mathcal{V}_{i\leq 1} := \bigoplus_{i \leq 1} \mathcal{L}_{ijk} y_0^i y_1^j y_2^k \subset \operatorname{Sym}^3 \mathcal{A}_2.$$

Similarly we define $\mathcal{V}_{i\geq 2}$; clearly $\operatorname{Sym}^3 \mathcal{A}_2 = \mathcal{V}_{i\leq 1} \oplus \mathcal{V}_{i \geq 2}$.

Lemma 4.5 (Lifting Lemma)**.** *Consider a map $\mathcal{L} \to \mathcal{A}_6$ where $\mathcal{L}$ is any line bundle. Then the composition map $\mathcal{L}(-4\alpha) \xrightarrow{\cdot f_0^4} \mathcal{L} \to \mathcal{A}_6$ lifts to $\mathcal{V}_{i\leq 1} \subset \operatorname{Sym}^3 \mathcal{A}_2$.*

Proof.

Let $\mathcal{J} \subset \operatorname{Sym} \mathcal{A}_2$ be the kernel of the map $\operatorname{Sym} \mathcal{A}_2 \to \mathcal{A}_{even} \subset \mathcal{A}$. The stalk of $\mathcal{J}$ at each point is a principal ideal generated by the class of

$$\mathfrak{Q} := (f_0 y_0 + f_1 y_1)^2 - y_2 \sum g_i y_i. \tag{11}$$

Therefore the kernel $\mathcal{J}_3$ of the map $\operatorname{Sym}^3 \mathcal{A}_2 \to \mathcal{A}_6$ is generated by $\mathfrak{Q}y_0, \mathfrak{Q}y_1, \mathfrak{Q}y_2$. By (7), $f_0 \neq 0$, and then there are no nonzero relative cubics in the kernel without the terms $y_0^3, y_0^2 y_1, y_0^2 y_2$: $\mathcal{V}_{i\leq 1} \cap \mathcal{J}_3 = \{0\}$. Defining $\mathcal{T} := \operatorname{coker}\left(\mathcal{J}_3 \oplus \mathcal{V}_{i\leq 1} \to \operatorname{Sym}^3 \mathcal{A}_2\right)$ we get the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{J}_3 & \longrightarrow & \mathcal{J}_3 \oplus \mathcal{V}_{i\leq 1} & \rightleftarrows & \mathcal{V}_{i\leq 1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Sym}^3 \mathcal{A}_2 & = & \mathcal{V}_{i\geq 2} \oplus \mathcal{V}_{i\leq 1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{A}_6 & \longrightarrow & \mathcal{T} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{12}$$

from which we deduce that $\mathcal{V}_{i\leq 1}$ maps isomorphically to $\ker(\mathcal{A}_6 \to \mathcal{T})$. In other words a map to $\mathcal{A}_6$ can be lifted to $\mathcal{V}_{i\leq 1}$ if and only its image goes to 0 on $\mathcal{T}$: we conclude if we prove that $\mathcal{T}$ is annihilated by f_0^4.

Simplifying the term $\mathcal{V}_{i\leq 1}$ in the second column of (12), we see $\mathcal{T}$ as cokernel of a map between $\mathcal{J}_3$ and the $\mathcal{V}_{i\geq 2}$, both sum of 3 line bundles. Since

$$\begin{aligned}
\mathfrak{Q}y_0 &= f_0^2 y_0^3 + 2 f_0 f_1 y_0^2 y_1 - g_0 y_0^2 y_2 + \dots \\
\mathfrak{Q}y_1 &= f_0^2 y_0^2 y_1 + \dots \\
\mathfrak{Q}y_2 &= f_0^2 y_0^2 y_2 + \dots
\end{aligned}$$

(here we have explicited the terms divisible by y_0^2, which are the only terms relevant for this computation)

$$\mathcal{T} \cong \operatorname{coker} \begin{pmatrix} f_0^2 & 0 & 0 \\ 2 f_0 f_1 & f_0^2 & 0 \\ -g_0 & 0 & f_0^2 \end{pmatrix}.$$

Therefore f_0^4 annihilates $\mathcal{T}$.

□

The map $\operatorname{Sym} \mathcal{A}_2 \to \mathcal{A}$ induces an inclusion $C \subset \operatorname{Proj}(\operatorname{Sym} \mathcal{A}_2)$ and more precisely $C = \{(f_0 y_0 + f_1 y_1)^2 = y_2 \sum g_i y_i\}$. By Lemma 4.5,

$\Delta \subset C$ is cut by

$$F_\Delta := \frac{\sum_{i\le 1} F_{ijk} y_0^i y_1^j y_2^k}{f_0^4} \tag{13}$$

where F_{ijk} are polynomials on $\mathbb{P}^1$ of degree

$$\deg F_{ijk} = (id_0 + jd_1 + kd_2) - (6p_g + 2\Theta) + 4\alpha,$$

and moreover, since F_Δ has no poles when restricted to C

$$\sum_{i\le 1} F_{ijk} y_0^i y_1^j y_2^k = \left(\sum B_i y_i\right) \mathfrak{Q} \mod f_0^4 \tag{14}$$

where the B_i are rational functions on $\mathbb{P}^1$ whose denominators are invertible modulo f_0, and $\mathfrak{Q}$ is as in (11).

Lemma 4.6. *If $p_g > \alpha + 2$ then $F_{120} = 0$ and F_{030} is nonzero and divisible by f_0^2. Moreover* $\gcd(f_0, g_0) = 1$.

Proof. We compute the nonzero map $(\mathcal{R}_3')^2 \to \mathcal{S}_6'$ by composing the map $\bar{\delta}\colon (\mathcal{R}_3')^2(-4\alpha) \to \operatorname{Sym}^3 \mathcal{A}_2$ given by the numerator of (13) with the restriction $\operatorname{Sym}^3 \mathcal{A}_2 \to \operatorname{Sym}^3 \mathcal{S}_2'$, then with the matrix (9), and finally dividing the result by f_0^4.

We find that the map $(\mathcal{R}_3')^2 \to \mathcal{S}_6'$ is represented by the matrix $(\frac{F_{120}}{f_0^2}, 3\frac{F_{030}}{f_0^2} - 2\frac{f_1 F_{120}}{f_0^3})$.

It follows $f_0^2 | F_{120}$. The assumption $p_g > \alpha + 2$ is equivalent to $\deg F_{120} < 2\alpha$, so $F_{120} = 0$ and F_{030} must be nonzero and divisible by f_0^2.

Assume that there is a point p with $g_0(p) = f_0(p) = 0$. Then (see (6)) $p \in \operatorname{supp} \tau$. The node of C above p, say P_0, belongs to $\mathfrak{s} = \{y_2 = f_0 y_0 + f_1 y_1 = 0\}$ and therefore has relative coordinates $(y_0 : y_1 : y_2) = (1 : 0 : 0)$.

Near P_0, $f_1 \neq 0$ by (7), so to restrict F_Δ to $\mathfrak{s}$ in a neighbourhood of P_0 we can substitute $y_2 = 0$ and $y_1 = -y_0 f_0/f_1$: an equation for $\Delta_{|\mathfrak{s}}$ is

$$\frac{(-f_0^3 F_{030} + f_0^2 f_1 F_{120}) y_0^3}{f_0^4 f_1^3} = -\frac{F_{030}}{f_0} \frac{y_0^3}{f_1^3}.$$

Since $f_0^2 | F_{030}$, $P_0 \subset \Delta$, contradicting (1). □

Now we study the B_i in (14).

Lemma 4.7. *If $p_g > \alpha + 2$, then, modulo f_0^4,*

$$B_0 = -2 f_0 \frac{F_{030}}{f_1^3}$$

$$B_1 = \frac{F_{030}}{f_1^2}$$

$$f_0^2 B_2 = -2 f_0 \frac{g_0 F_{030}}{f_1^3}$$

Moreover F_{111} is nonzero and divisible by f_0^2.

Proof. The expressions of the B_i follow by looking at the coefficients of (14): B_1 by the term y_1^3, B_0 by $y_0y_1^2$ (since $F_{120}=0$ by Lemma 4.6), and $f_0^2B_2$ by $y_0^2y_2$.

Finally, looking at the coefficient of $y_0y_1y_2$, we find that modulo f_0^4

$$\begin{aligned} f_0F_{111} &= -f_0(B_0g_1+B_1g_0)+2B_2f_0^2f_1 \\ &= f_0\frac{F_{030}}{f_1^3}(2f_0g_1-f_1g_0-4f_1g_0) \\ &= uf_0F_{030} \end{aligned}$$

where by Lemma 4.6, $\gcd(u,f_0)=1$.

As f_0^2 divides F_{030}, it divides F_{111} too. Moreover, since $0\neq F_{030}=hf_0^2$ with h of degree $\Theta-\alpha\in\{0,1\}$, then f_0F_{030} cannot vanish modulo f_0^4 and therefore F_{111} is nonzero too. □

As a consequence we find the following

Corollary 4.8. *If $\alpha>0$ then $p_g\leq 2\alpha-\Theta+4$.*

Proof. We can assume $p_g>\alpha+2$ and apply Lemma 4.7 to find $2\alpha=\deg f_0^2\leq\deg F_{111}$ which is equivalent to $p_g\leq 2\alpha-\Theta+4$. □

Proof of Theorem 3.1. Xiao Gang's inequality guarantees that S is regular and the pencil has rational base. If $\Theta=0$, Corollary 4.3 ensures that $\mathcal{R}\cong(\operatorname{Sym}V)/\mathcal{I}$. Else, by Proposition 4.2 and Corollary 4.8 $p_g>\Theta+4\geq 2\alpha-\Theta+4$ implies $\alpha=0$ and we conclude by Proposition 3.12. □

The inequality in Corollary 4.8 is sharp. Indeed its proof shows a method for constructing more examples with $p_g=2\alpha-\Theta+4$ by fixing F_{111}/f_0^2 and consequently computing F_{030} and the B_i.

Example 4.9. We give examples for each pair (α,Θ) with $1\leq\alpha\leq\Theta\leq 2$, $p_g=2\alpha-\Theta+4$: three examples with (α,Θ,K^2,p_g) which equals respectively $(1,1,15,5)$, $(1,2,12,4)$ and $(2,2,20,6)$.

f_0 is a homogeneous polynomial on $\mathbb{P}^1$ of degree α: we choose

$$f_0:=\begin{cases} t_0 & \text{if } \alpha=1 \\ t_0(t_0-2) & \text{if } \alpha=2 \end{cases}$$

Note that $\deg g_1=2p_g+\Theta-\alpha-2=p_g+\alpha+2$ is even when $\alpha=2$, so we can choose g_1 pure power of f_0. We take

$$f_1=t_1^{\alpha+2},\ g_0=t_1^{p_g+\alpha},\ g_1=f_0^{\frac{p_g+2}{\alpha}+1},\ g_2=0$$

so that $\tau=\{f_0^{\frac{p_g+2}{\alpha}+2}=t_1^{p_g+2\alpha+2}\}$.

The reader can check that

$$f_0(t_1^{\Theta-\alpha}y_1^2-y_0y_2)(f_0y_1-2t_1^{p_g-2}y_2)-4f_0^2y_0y_1y_2$$

is, modulo f_0^4, of the form $(\sum B_i y_i)\mathfrak{Q}$ (as prescribed by (14), note $F_{111} = -5f_0^2$). So a candidate for the equation of Δ is

$$F_{\Delta_0} := \frac{(t_1^{\Theta-\alpha} y_1^2 - y_0 y_2)(f_0 y_1 - 2t_1^{p_g-2} y_2) - 4f_0 y_0 y_1 y_2}{f_0^3}.$$

The divisor Δ_0 of the restriction of F_{Δ_0} to C is effective, but could be too singular. Let $\mathfrak{D}$ be the linear system of the effective divisors on C defined by the restriction of $F_{\Delta_0} + \lambda y_2^3$, where λ is any homogeneous polynomial on $\mathbb{P}^1$ of degree $6 - 2\Theta > 0$. Since $\operatorname{Fix} \mathfrak{D} \subset \{y_2 = 0\} \cap C = 2\mathfrak{s}$, its base points are contained in $\mathfrak{s}$. On the other hand the restriction of F_{Δ_0} to $\mathfrak{s}$, computed as in the proof of Lemma 4.6, is $t_1^{\Theta-\alpha} f_0$, which has Θ distinct roots. So our pencil has Θ simple base points, all along $\mathfrak{s}$, and the general $\Delta \in \mathfrak{D}$ is smooth by Bertini.

Finally we can write a proof of Theorem 0.3. We need the following result of Xiao Tao Sun

Theorem 4.10 ([S1], [S2])**.** *If S is a minimal surface of general type whose canonical map is composed with a pencil of curves of genus g without base points then*

$$\begin{aligned} g = 3 &\Rightarrow 12K^2 \geq 63p_g - 142 \\ g = 4 &\Rightarrow 7K^2 \geq 48p_g - 134 \\ g = 5 &\Rightarrow 9K^2 \geq 80p_g - 262 \end{aligned}$$

Proof of Theorem 0.3.

If S is a minimal surface of general type with a canonical pencil, $K_S^2 \leq 4\chi(\mathcal{O}_S) - 8$ and $\chi(\mathcal{O}_S) \gg 0$, by Beauville's Theorem 0.1 the pencil is free, and by Theorem 4.10 the pencil is a pencil of genus 2 curves.

Then S is birational to the relative canonical model $X = \operatorname{Proj}(\mathcal{R})$ of the pencil and we have computed $\mathcal{R}$ in Theorem 3.1: the description as complete intersection in $\mathbb{P}$ follows trivially. All the other statements have been proved in Section 1.

□

References

[B] A. Beauville: *L'application canonique pour les surfaces de type général*. Invent. Math. **55** (1979), 121–140.

[C] F. Catanese: *Singular bidouble covers and the construction of interesting algebraic surfaces*. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97–120, Contemp. Math., **241**, Amer. Math. Soc., Providence, RI, 1999.

[CP] F. Catanese and R. Pignatelli: *Fibrations of low genus, I.* Ann. Sci. École Norm. Sup. (4) **39** (2006), 1011–1049.

[H] E. Horikawa: *On algebraic surfaces with pencils of curves of genus* 2. Complex analysis and algebraic geometry, 79–90. Iwanami Shoten, Tokyo, 1977.

[P] R. Pignatelli: *Some (big) irreducible components of the moduli space of minimal surfaces of general type with* $p_g = q = 1$ *and* $K^2 = 4$. To appear on Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.

[R] M. Reid: *Problems on pencils of small genus.* Unpublished manuscript, 1990.

[S1] X.T. Sun: *Surfaces of general type with canonical pencil.* Acta Math. Sinica **33** (1990), 769–773.

[S2] X.T. Sun: *On canonical fibrations of algebraic surfaces.* Manuscripta Math. **83** (1994), 161–169.

[X] G. Xiao: *Surfaces fibrées en courbes de genre deux.* Lecture Notes in Mathematics, **1137**. Springer-Verlag, Berlin, 1985.

[YM] J. Yang and M. Miyanishi: *Surfaces of general type whose canonical map is composed of a pencil of genus* 3 *with small invariants.* J. Math. Kyoto Univ. **38** (1998), 123–149.

[Z] F. Zucconi: *Numerical inequalities for surfaces with canonical map composed with a pencil.* Indag. Math. (N.S.) **9** (1998), 459–476.

Dipartimento di Matematica, Università di Trento. Via Sommarive 14, loc. Povo, I-38050 Trento (Italy)