Abstracts

Surfaces with $p_g = 0$: computer aided constructions

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The surfaces of general type with $p_g = 0$ have been recently object of intensive research since they arise naturally from many different directions, as (*e.g.*) the Chow groups (because of the Bloch conjecture) and the analysis of the "exceptional" behaviours of the pluricanonical maps of the surfaces of general type.
We are interested in the following situation (#):

- C_1, C_2 be compact complex curves of respective genera g_1, $g_2 \geq 2$;
- G be a finite group acting faithfully on each C_i;
- $X := (C_1 \times C_2)/G$ be the quotient by the diagonal action;
- $S \to X$ be the minimal resolution of the singularities of X.

We have constructed many new surfaces of general type S with $p_g(S) = 0$ by performing a systematic search of surfaces as in (#). It is remarkable that by a result of Kimura for all these surfaces the Bloch conjecture holds.

Theorem 1.([2]) There are exactly 17 families of smooth surfaces of general type $X = C_1 \times C_2/G$, with G finite and $p_g(X) = 0$. They form 17 connected components of the moduli space of the surfaces of general type.

Theorem 2.([3]) There are exactly 27 families of surfaces as in (#) such that S is of g.t. with $p_g(S) = 0$, X is singular and has canonical singularities.

Theorem 3.([4]) There are exactly 32 families of surfaces as in (#) such that S is minimal of g.t. with $p_g(S) = 0$, and X has at least a non-canonical singularity.

We use the following algebraic recipe.
Let C be a curve and $p_1, \ldots, p_r$ be the branching points of a G-cover $\xi\colon C \to C' := C/G$ of respective branching indices $m_1, \ldots, m_r$. We assume $C' \cong \mathbb{P}^1$, although the argument works with minor modifications when $g(C') > 0$, which is relevant for constructing irregular surfaces (cf. [5], [6], [7]).
ξ has a monodromy representation $\psi\colon \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}) \to G$, which factors through the map $\varphi\colon \Pi(0; m_1, \ldots, m_r) := \langle c_1, \ldots, c_r | c_i^{m_i}, c_1 \cdots c_r \rangle \to G$ defined by $\varphi(c_i) := a_i := \psi(\gamma_i)$ where the γ_i are geometric loops around the p_i chosen so that $\prod \gamma_i = 1$. The a_i generate G, $\prod a_i = 1$ and each a_i has order m_i: for short $[a_1, \ldots, a_r]$ is a *sequence of spherical generators* of G of signature $(m_1, \ldots, m_r)$.
By the Riemann Existence Theorem ξ is determined by the p_i, the γ_i and the a_i. Therefore the surfaces as in (#) are given by two finite subsets of $\mathbb{P}^1$, loops around these points as above, and two sequences of spherical generators $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ (of respective signatures, say, $(m_1, \ldots, m_r)$ and $(n_1, \ldots, n_s)$).

Lemma. There are numbers D^2, M, R and B, explicit functions (only) of the singularities of X such that

i) if $\chi(\mathcal{O}_S) = 1$, then $K_S^2 = 8 - B$;
ii) $r \leq R$ and $\forall i$, $m_i \leq M$;

iii) $|G| = \frac{K_S^2 - D^2}{2\left(-2+\sum_1^r\left(1-\frac{1}{m_i}\right)\right)\left(-2+\sum_1^s\left(1-\frac{1}{n_s}\right)\right)}$.

It follows an algorithm to compute all surfaces S as in (#) with $p_g(S) = q(S) = 0$ and a given fixed value of K_S^2:

1) find all possible *baskets* of singularities with $B = 8 - K_S^2$;
2) for each basket list all signatures respecting the inequalities in ii);
3) for each pair of signatures, search all groups of the order predicted by iii) for sequences of spherical generators of the prescribed signatures;
4) check the resulting surfaces: most of them will be too singular, and not even of general type!

Some remarks:

- In few dozens of cases, the computer can't perform step 3) since the predicted $|G|$ is too big, and no database contains all the necessary groups: we proved theoretically that these cases do not occur.
- The algorithm is heavy and in this form there is little chance that a computer can complete it for small values of K_S^2. We proved and inserted in the algorithm much stronger conditions on Sing X and on the signatures, to obtain the full list of surfaces with $K_S^2 \geq 1$.
- If X has a non-canonical singularity, K_S may be not nef, and therefore K_S^2 may be nonpositive: we may have missed some nonminimal surfaces.

To understand if these surfaces are topologically pairwise distinct, we compute their fundamental groups. We proved the following theorem.

Theorem 4.([3]) For every surface S as in (#), $\pi_1(S)$ contains a normal subgroup of finite index isomorphic to $\Pi_g \times \Pi_{g'}$, where Π_g, $\Pi_{g'}$ are the fundamental groups of a smooth curve of genus g resp. g' (here $g, g' \geq 0$).

The proof is purely algebraic and indirect. Theorem 4 suggests a *geometrical* description of $\pi_1(S)$ (as in table 1) which can be used to study the deformations of these surfaces. This has been already done in a case: see [1].
We listed in table 1, for each family from Theorem 2 or 3, K_S^2, the singularities (where q/n^k means "k points of type 1/n(1,q)") the signatures T_i (with an analogous *exponential* notation), G, the number N of different families we obtain with these data, $H_1(S, \mathbb{Z})$ and $\pi_1(S)$. More details are in [3] and [4].

References

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K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
6	$1/2^2$	$2^3,4$	$2^4,4$	$\mathbb{Z}_2\times D_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4^2$	$1\to\mathbb{Z}^2\times\Pi_2\to\pi_1\to\mathbb{Z}_2^2\to 1$
6	$1/2^2$	$2^4,4$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	$1\to\Pi_2\to\pi_1\to\mathbb{Z}_2\times\mathbb{Z}_4\to 1$
6	$1/2^2$	$2,5^2$	$2,3^3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_3\times\mathbb{Z}_{15}$	$\mathbb{Z}^2\rtimes\mathbb{Z}_{15}$
6	$1/2^2$	$2,4,10$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathfrak{S}_3\times D_{4,5,-1}$
6	$1/2^2$	$2,7^2$	$3^2,4$	PSL(2,7)	2	$\mathbb{Z}_{21}$	$\mathbb{Z}_7\times\mathfrak{A}_4$
6	$1/2^2$	$2,5^2$	$3^2,4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_{15}$	$\mathbb{Z}_5\times\mathfrak{A}_4$
5	$1/3,2/3$	$2,4,6$	$2^4,3$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$1\to\mathbb{Z}^2\to\pi_1\to D_{2,8,3}\to 1$
5	$1/3,2/3$	$2^4,3$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_8$	$1\to\mathbb{Z}^2\to\pi_1\to\mathbb{Z}_8\to 1$
5	$1/3,2/3$	$4^2,6$	$2^3,3$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2\times\mathbb{Z}_8$	$1\to\mathbb{Z}^2\to\pi_1\to\mathbb{Z}_8\to 1$
5	$1/3,2/3$	$2,5,6$	$3,4^2$	$\mathfrak{S}_5$	1	$\mathbb{Z}_8$	$D_{8,5,-1}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_{10}$	$\mathbb{Z}_5\times Q_8$
5	$1/3,2/3$	$2^3,3$	$3,4^2$	$\mathbb{Z}_2^4\rtimes\mathfrak{S}_3$	1	$\mathbb{Z}_2\times\mathbb{Z}_8$	$D_{8,4,3}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_{10}$	$\mathbb{Z}_2\times\mathbb{Z}_{10}$
4	$1/2^4$	2^5	2^5	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	$1\to\mathbb{Z}^4\to\pi_1\to\mathbb{Z}_2^2\to 1$
4	$1/2^4$	$2^2,4^2$	$2^2,4^2$	$\mathbb{Z}_2\times\mathbb{Z}_4$	1	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	$1\to\mathbb{Z}^4\to\pi_1\to\mathbb{Z}_2^2\to 1$
4	$1/2^4$	2^5	$2^3,4$	$\mathbb{Z}_2\times D_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$1\to\mathbb{Z}^2\to\pi_1\to\mathbb{Z}_2\times\mathbb{Z}_4\to 1$
4	$1/2^4$	$3,6^2$	$2^2,3^2$	$\mathbb{Z}_3\times\mathfrak{S}_3$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2\rtimes\mathbb{Z}_3$
4	$1/2^4$	$3,6^2$	$2,4,5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2\rtimes\mathbb{Z}_3$
4	$1/2^4$	2^5	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}^2\rtimes\mathbb{Z}_2$
4	$1/2^4$	$2^2,4^2$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$\mathbb{Z}^2\rtimes\mathbb{Z}_4$
4	$1/2^4$	2^5	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$\mathbb{Z}^2\rtimes\mathbb{Z}_4$
4	$1/2^4$	$2^3,4$	$2^3,4$	$\mathbb{Z}_2^4\rtimes\mathbb{Z}_2$	1	$\mathbb{Z}_4^2$	$G(32,2)$
4	$1/2^4$	$2,5^2$	$2^2,3^2$	$\mathfrak{A}_5$	1	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$
4	$1/2^4$	$2^2,3^2$	$2^2,3^2$	$\mathbb{Z}_3^2\rtimes Z_2$	1	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$
4	$2/5^2$	$2^3,5$	$3^2,5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_6$	$\mathbb{Z}_2\times\mathbb{Z}_6$
4	$2/5^2$	$2,4,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
4	$2/5^2$	$2,4,5$	$3^2,5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$
3	$1/5,4/5$	$2^3,5$	$3^2,5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_6$	$\mathbb{Z}_2\times\mathbb{Z}_6$
3	$1/5,4/5$	$2,4,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
3	$1/3,1/2^2,2/3$	$2^2,3,4$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$
3	$1/5,4/5$	$2,4,5$	$3^2,5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$
2	$1/3^2,2/3^2$	$2,6^2$	$2^2,3^2$	$\mathbb{Z}_2^3\rtimes\mathbb{Z}_3$	1	$\mathbb{Z}_2^2$	Q_8
2	$1/2^6$	4^3	4^3	$\mathbb{Z}_4^2$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
2	$1/2^6$	$2^3,4$	$2^3,4$	$\mathbb{Z}_2\times D_4$	1	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$
2	$1/3^2,2/3^2$	$2^2,3^2$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_8$	$\mathbb{Z}_8$
2	$1/3^2,2/3^2$	$3^2,5$	$3^2,5$	$\mathbb{Z}_5^2\rtimes\mathbb{Z}_3$	2	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2^3,4$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/3^2,2/3^2$	$3^2,5$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2,3,7$	4^3	PSL(2,7)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2,6^2$	$2^3,3$	$\mathfrak{S}_3\times\mathfrak{S}_3$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/2^6$	$2,6^2$	$2,4,5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4,1/2^2,3/4$	$2,4,7$	$3^2,4$	PSL(2,7)	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4,1/2^2,3/4$	$2,4,5$	$3^2,4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4,1/2^2,3/4$	$2,4,6$	$2,4,5$	$\mathfrak{S}_5$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
1	$1/3,1/2^4,2/3$	$2^3,3$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$
1	$1/3,1/2^4,2/3$	$2,3,7$	$3,4^2$	PSL(2,7)	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$1/3,1/2^4,2/3$	$2,4,6$	$2^3,3$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$

TABLE 1. The minimal surfaces of general type S as in (#) with $p_g(S)=0$ and X singular