RULED SURFACES AND GENERIC COVERINGS

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1. Introduction

In this paper we study normal singularities of complex analytic surfaces. In particular, since a normal surface has only isolated singularities (see, e.g., [Nar]), we restrict our attention to germs (S, s) where S is a connected normal surface, $s \in S$ and $S \setminus s$ is non-singular.

We begin by setting some standard notation.

A *normal generic covering* (S, π) (ngc in the sequel) is a finite holomorphic map $\pi : S \longrightarrow \mathbb{C}^2$ from a connected normal surface S to the complex plane $\mathbb{C}^2$, which is an analytic covering branched over a curve $B \subset \mathbb{C}^2$, such that the fiber over a smooth point of B is supported on $\deg \pi - 1$ distinct points. A ngc is called *smooth* if S is non-singular.

Two ngcs (S_1, π_1), (S_2, π_2) are called (analytically) *equivalent* if there exists an isomorphism $\phi : S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$. In the sequel, we will consider equivalent ngcs to be the same covering.

The main interest in ngcs comes from the well known fact that, by Weierstrass preparation theorem, given an analytic surface $S \subset \mathbb{C}^n$, a generic projection $S \xrightarrow{\pi} \mathbb{C}^2$ is (at least locally, in order to insure $\deg \pi < \infty$) a ngc branched over a curve (see [GuRo]).

Classically, one would like to reconstruct every ngc starting from *downstairs* data (*i.e.* in $\mathbb{C}^2$, like the branch curve B).

Over a non-singular point of B, π is locally (in S) equivalent to the map of the complex plane to itself which takes (x, y) to (x^a, y) with $a = 1, 2$. The main point is then to study germs of ngcs where the branch curve is a singular germ of a plane curve. We can then restrict to the case in which the branch locus B has only one singular point, which we may assume to be the origin O.

For a fixed curve B there are three natural problems related to ngcs: the *existence* problem (there exists a ngc branched over B?), the *smoothness* problem (there exists a smooth ngc branched over B?), and the *uniqueness* problem (under which hypothesis is the covering unique? This is related to a conjecture of Chisini, see [MaPi2]).

Notice that for $\deg \pi = 2$ the problems are trivial, since there always exists a (unique) ngc of degree 2 branched over any curve B: namely, if B has equation $f(x, y) = 0$, it is sufficient to consider the projection on the x, y-plane of the surface in $\mathbb{C}^3$ defined by the equation $z^2 = f(x, y)$.

This paper is addressed to the smoothness problem in case the branch curve has (up to analytic equivalence) the equation $\{x^n = y^m\}$. Let us point out that, according to the Puisieux classification (see [BrKn]), this class of singularities is a natural first step for a complete classification.

A standard way to study ngcs is the following: given a ngc (S, π) with branch curve B, one defines the *monodromy* homomorphism $\rho : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_{\deg \pi}$ as the action of this fundamental group on the fiber of π over a fixed regular value. The "generic" condition means that for each geometric loop (i.e. a simple loop in $\mathbb{C}^2 \setminus B$ around a smooth point of the curve B) its monodromy is a transposition.

It is well known that one can reconstruct the covering from the pair (B, ρ) (cfr. [GrRe]). However, despite the explicit construction, understanding the singularity of the covering in this way is very difficult (except in specific cases), and in particular the smoothness problem is far to be solved.

Recall that a point P in a normal surface S is a smooth point if and only if the local fundamental group $\pi_1(S \setminus \{P\})$ is trivial (see [Mum]).

For a combinatorial approach to the problem see [MaPi1] and [Man] in which we represent the monodromy ρ of a ngc of degree d branched on the curve $\{x^n = y^m\}$ by a connected graph with d vertices and n labeled edges called *monodromy graph*, and we give an answer in some cases (irreducible branch locus, $n|m$, $d - n \geq 0$). For example, n and m cannot be both odd and if $n|m$, the ngc is smooth if and only if the monodromy graph is a tree.

In this paper we start from a different construction of ngcs branched on $\{x^n = y^m\}$: roughly speaking, we take a ruled surface $\tilde{X}$ on a smooth curve C, contract a section C_0 and quotient by a suitable action of a finite cyclic group G.

More specifically, we take a curve C of genus $\frac{\mathrm{lcm(n,m)}}{2} - d + 1$, where d will be the degree of the covering; $\tilde{X} \cong \mathbf{P}(\mathcal{E}) = \mathbf{Proj}(Sym(\mathcal{E}))$ where $\mathcal{E} = \mathcal{O}_C(-L) \oplus \mathcal{O}_C$ with L a "generic" divisor on C of degree d; and the action is induced by an automorphism σ of C preserving L of order $\frac{\mathrm{lcm}(n,m)}{\mathrm{gcd(n,m)}}$.

We denote the surface obtained in this way as $X_{C,L,G}$, and we denote by P the point image of the contracted curve C_0: the construction yields a natural germ of ngc $\pi_{C,L,G} : (X_{C,L,G}, P) \to (\mathbb{C}^2, O)$ (all the details of these constructions are in the section 2) branched on $\{x^n = y^m\}$.

In section 2 we prove that all ngcs branched over $\{x^n = y^m\}$ are equivalent to these.

In section 3 we compute the local fundamental groups in case $n = m$.

In section 4 we exploit the singularities of $\tilde{X}/G$.

In section 5 we give a numerical characterization of all the smooth ngcs branched over $\{x^n = y^m\}$.

Theorem 1.1 (Smoothness criterion). *$X_{C,L,G}$ is smooth, if and only if the following occur:*

a) $d|\frac{\mathrm{lcm(n,m)}}{\mathrm{gcd(n,m)}}$*;*
b) *some numerical condition about the orbits of the action of σ on C.*

The precise statement will be given in theorem 5.5 (we cannot give it here since it uses a few definitions that we will give in section 4).

We give also a similar (and easier) rationality criteria (*i.e.* we can decide whether the germ $(X_{C,L,G}, P)$ is a germ of a rational singularity or not).

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2. Ruled surfaces and singularities

Let $p : \tilde{X} \to C$ be a ruled surface (on a smooth curve C): then by [Har], prop. V.2.8, $\tilde{X} \cong \mathbf{P}(\mathcal{E}) = \mathbf{Proj}(Sym(\mathcal{E}))$ for some rank 2 vector bundle $\mathcal{E}$ on C with $H^0(\mathcal{E}) \neq 0$ and $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for each line bundle $\mathcal{L}$ on C of negative degree. Recall that there is a line bundle on $\mathbf{P}(\mathcal{E})$, called $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, yielding $\mathcal{O}_{\mathbb{P}^1}(1)$ on each fibre of p, and such that $p_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$ (cf. [Har], prop. II.7.11).

We are interested in the case $\mathcal{E} = \mathcal{O}_C(-L) \oplus \mathcal{O}_C$ where L is a line bundle on C of degree $d := \deg \mathcal{O}_C(L) > 0$. In particular $h^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = 1$ and the only divisor in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$, usually denoted by C_0, has self-intersection $-d$ by [Har], proposition V.2.9.

Contracting this curve we get a normal surface X with at most a singular point: the image P of C_0.

In this section we will see how this simple example do in fact 'generate' (up to quotient by a finite cyclic group) every generic covering branched on $\{x^n = y^m\}$.

First, let us consider the case $n = m$: by theorem 5.2 in [MaPi1] the singularity has a resolution given by a single irreducible curve with self-intersection $-d$: this suggests us to try to construct the covering starting from a ruled surface.

Definition 2.1. *Let C be a smooth curve, L a linear pencil (not necessarily complete) on C. Let d be the degree of L. We say that the pencil is generic if it has no fixed points and every divisor in L is supported in at least $d-1$ distinct points.*

Definition 2.2. *In the following C will be always a smooth curve, L a generic pencil on C, d the degree of L.*

We denote

- *by $\tilde{X}_{C,L}$ the ruled surface $\mathbf{P}(\mathcal{O}_C(-L) \oplus \mathcal{O}_C)$;*
- *by $(C_0)_{C,L}$ the only curve in $|\mathcal{O}_{\tilde{X}_{C,L}}(1)|$;*
- *by $p_{C,L} : \tilde{X}_{C,L} \to C$ the natural projection;*
- *by $X_{C,L}$ the surface obtained from $\tilde{X}_{C,L}$ by contracting $(C_0)_{C,L}$;*
- *by $P_{C,L}$ the image of $(C_0)_{C,L}$ in $X_{C,L}$.*

We will sometimes drop the subindices C, L (i.e. we will write $\tilde{X}$, C_0, p, X and P resp.) when no confusion arises.

Proposition 2.3. *Let C be a curve of genus g, L a generic pencil on C of degree d.*

Then there is a generic covering $\pi_{C,L} : X_{C,L} \to \mathbb{P}^2$ of degree d such that:

i) *$\pi_{C,L}^{-1}((1:0:0)) = P_{C,L}$;*
ii) *the branch locus of $\pi_{C,L}$ is the union of $2(g+d-1)$ distinct lines through $(1:0:0)$;*
iii) *there are isomorphisms between C and the preimage of each line $l \subset \mathbb{P}^2$ not passing through $(1:0:0)$ such that the pull-back of $|\mathcal{O}_l(1)|$ to C gives exactly L.*

Proof. We fix two sections f_1 and f_2 in $H^0(\mathcal{O}_C(L))$ generating the pencil. By the projection formula

$$p_*(\mathcal{O}_{\tilde{X}_{C,L}}(1) \otimes p^*\mathcal{O}_C(L)) \cong \mathcal{O}_C \oplus \mathcal{O}_C(L);$$

in particular we have an isomorphism

$$\begin{aligned} H^0(\mathcal{O}_{\tilde{X}_{C,L}}(1) \otimes p^*\mathcal{O}_C(L)) &\cong H^0(\mathcal{O}_C \oplus \mathcal{O}_C(L)) \\ &= H^0(\mathcal{O}_C) \oplus H^0(\mathcal{O}_C(L)). \end{aligned} \tag{1}$$

Then, the three pairs $(1,0)$, $(0,f_1)$ and $(0,f_2)$ in the vector space on the right side of (1) induce three sections in $H^0(\mathcal{O}_{\tilde{X}_{C,L}}(1) \otimes p^*\mathcal{O}_C(L))$ and therefore a rational map $\tilde{\pi}_{C,L} : X_{C,L} \dashrightarrow \mathbb{P}^2$. The last two chosen pairs generate the image of the natural map

$$H^0(\mathcal{O}_{\tilde{X}_{C,L}}(1)) \otimes Span(f_1, f_2) \to H^0(\mathcal{O}_{\tilde{X}_{C,L}}(1) \otimes p^*\mathcal{O}_C(L))$$

and therefore, since by assumption L has no base points, the intersection of the corresponding divisors is exactly C_0. On the contrary the pair $(1,0)$ gives a divisor that does not contain C_0, and therefore (having intersection 0 with it) does not even intersect it: we conclude that $\tilde{\pi}_{C,L}$ is a morphism and $\tilde{\pi}_{C,L}^{-1}((1:0:0)) = C_0$.

Therefore $\tilde{\pi}_{C,L}$ induces a morphism $\pi_{C,L} : X_{C,L} \to \mathbb{P}^2$, and $\pi_{C,L}^{-1}((1 : 0 : 0)) = P_{C,L}$.

Let $l \subset \mathbb{P}^2$ be a line not passing through $(1:0:0)$. Then $\tilde{\pi}^{-1}(l)$ is a divisor D in $|\mathcal{O}_{\tilde{X}_{C,L}}(1) \otimes p^*\mathcal{O}_C(L)|$ that does not intersect C_0.

In particular D does not contain any fibre; having intersection 1 with each fibre, it is a section of p, hence $p_{|D}$ gives an isomorphism from D to C.

The space $H^0(\mathcal{O}_l(1))$ is generated (modulo the equation of l) by the restriction of the linear forms x_1 and x_2; by definition

$$(\tilde{\pi}_{C,L} \circ p_{|D}^{-1})^*(x_i) = f_i, \ \forall i = 1, 2.$$

In other words, if we compose $\tilde{\pi}_{C,L} \circ p_{|D}^{-1}$ with the projection of center $(1:0:0)$ we get the map $C \to \mathbb{P}^1$ defined by L (regardless of the choice of the line l). In particular $\tilde{\pi}_{C,L}$ is generic (by the genericity assumption on L), of degree d and branched on a union of lines through $(1:0:0)$. The number of these lines follows from the Riemann-Hurwitz formula. □

Remark 2.4. *The map $\pi_{C,L}$ is not uniquely determined: the construction in the above proof depends on the choice of the two sections f_1 and f_2. Changing this choice corresponds to composing the map with an automorphism of $\mathbb{P}^2$ (fixing $(1:0:0)$), so $\pi_{C,L}$ is uniquely determined up to automorphisms.*

Theorem 2.5. *Let (S, p) be a germ of normal surface singularity, $\pi : (S, p) \to (\mathbb{C}^2, 0)$ a generic covering of degree $d \geq 3$ branched on n lines through the origin.*

Then there is a curve C of genus $\frac{n}{2} - d + 1$ and a generic pencil L of degree d on C such that the map π is equivalent to the germ of $\pi_{C,L}$ at $P_{C,L}$.

Proof. π is the germ at p of a generic covering onto $\mathbb{C}^2$ that we will (with a 'light' abuse of notation: the reader will forgive us) write as $\pi : S \to \mathbb{C}^2$, branched on n lines through the origin.

We define $C^\circ := \pi^{-1}(\{y = 1\})$; $\pi^\circ := \pi_{|C^\circ} : C^\circ \to \{y = 1\}$.

Let $\gamma_1, \ldots, \gamma_n$ be a set of minimal standard generators on $\{y = 1\}$ of the fundamental group of the complement of the branch curve (see [MaPi1], section 1, proposition 1.1).

Let $\overline{\Gamma}$ be their product: since $\overline{\Gamma}$ is in the center of the group and $d \geq 3$, the associated monodromy is the identity. It follows that we have a natural compactification of π° to a branched covering $\overline{\pi} : C \to \mathbb{P}^1$ with C compact and the fibre at ∞, $C \setminus C^\circ$, given by d distinct points: the genus of C follows from the Riemann-Hurwitz formula.

Note that $\overline{\pi}$ is a generic covering. The pull back of $\mathcal{O}_{\mathbb{P}^1}(1)$ to C is then a generic pencil L: we choose two elements f_1 and f_2 of $H^0(\mathcal{O}_C(L))$

such that the pull-back of the line $\{ax + by = 0\}$ is the divisor of $af_1 + bf_2$.

The map $\pi_{C,L}$ (in the coordinates induced by the choice of f_1 and f_2), restricted to the preimage of the affine space $\{x_0 \neq 0\}$, is a generic projection to $\mathbb{C}^2$ with the same branch locus and the same monodromy as π, so they are equivalent. □

Theorem 2.5 gives a description of all coverings branched on $\{x^n = y^n\}$ in terms of ruled surfaces. If, more generally, we have a generic covering $\pi : (S,p) \to (\mathbb{C}^2, 0)$ branched on $\{x^n = y^m\}$, we define $m' = m/gcd(m,n)$, $n' = n/gcd(m,n)$, and consider the map $f_{m',n'} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ defined by $f_{m',n'}(x,y) = (x^{m'}, y^{n'})$.

Taking the fibre product $(\overline{S}, \overline{p}) = (S,p) \times_{(\mathbb{C}^2,0)} (\mathbb{C}^2, 0)$ we get a Cartesian diagram

$$\begin{array}{ccc} (\overline{S}, \overline{p}) & \to & (S,p) \\ \overline{\pi} \downarrow & & \pi \downarrow \\ (\mathbb{C}^2, 0) & \stackrel{f_{m',n'}}{\to} & (\mathbb{C}^2, 0) \end{array}.$$

By theorem 2.2 of [MaPi1] $\overline{S}$ is normal and $\overline{\pi}$ is a generic covering branched on $\{x^k = y^k\}$, with $k = lcm(m,n)$. This allows us to represent all generic coverings branched on $\{x^n = y^m\}$ as 'quotients' of the $\pi_{C,L}$'s.

Definition 2.6. *Let C be a smooth curve of genus g and let L be a generic pencil on C of degree d; even if not necessary, we will assume that there are suitable coordinates in $\mathbb{P}^1$ such that the induced map $\varphi_L : C \to \mathbb{P}^1$ branches on $\{(\omega, 1) | \omega^{2(g+d-1)} = 1\}$; it simplifies the arguments and we need only this case.*

Let m', n' be relatively prime natural numbers and let G be a cyclic group of automorphisms of C of order $m'n'$.

We assume L to be G-invariant. We fix a generator σ of G and we assume, moreover, that, we can find coordinates y_1, y_2 on $\mathbb{P}^1$ such that, chosen $f_i := \varphi_L^*(y_i)$,

$$\sigma^*(f_1) = e^{\frac{2\pi i}{m'}} f_1,$$
$$\sigma^*(f_2) = e^{\frac{2\pi i}{n'}} f_2;$$

note that it follows that $m'n' | 2(g + d - 1)$.

The induced action of G on $\mathcal{O}_C \oplus \mathcal{O}_C(L)$ gives an action on $\tilde{X}_{C,L}$ fixing C_0, therefore we get an action of G on $X_{C,L}$.

We define $\tilde{X}_{C,L,G} = \tilde{X}_{C,L}/G$; $X_{C,L,G} = X_{C,L}/G$, whence $(C_0)_{C,L,G}$, and $P_{C,L,G}$ are the images of $(C_0)_{C,L}$ and $P_{C,L}$ in $\tilde{X}_{C,L,G}$, and $X_{C,L,G}$ respectively.

Definition 2.7. *Considering the action of G on $\mathbb{P}^2$ defined by $\sigma((x_0 : x_1 : x_2)) = (x_0 : e^{\frac{2\pi i}{m'}} x_1 : e^{\frac{2\pi i}{n'}} x_2)$, we see that $\pi_{C,L}$ is $G-$equivariant; passing to the quotient we get a morphism $\pi_{C,L,G} : X_{C,L,G} \to \mathbb{P}^2/G$.*

By the definition of the action, the germ at $(1:0:0)$ of the projection $\mathbb{P}^2 \to \mathbb{P}^2/G$ is the map $f_{m',n'}$; it follows that $\mathbb{P}^2/G$ is smooth at the class of $(1:0:0)$ and that the germ at $P_{C,L,G}$ of $\pi_{C,L,G}$ is a generic covering branched on $\{x^{\frac{2(g+d-1)}{m'}} = y^{\frac{2(g+d-1)}{n'}}\}$.

We can then state the main result of this section.

Theorem 2.8. *Let $\pi : (S,p) \to (\mathbb{C}^2, 0)$ be a generic covering of degree d branched on $\{x^n = y^m\}$; consider $k := lcm(n,m)$, $m' := k/n$, $n' := k/m$.*

Then there are a curve C of genus $\frac{k}{2} - d + 1$, a generic pencil L on C of degree d and a cyclic subgroup G of $Aut(C)$ of order $m'n'$ fulfilling all assumptions in definition 2.6 such that (S,p) is isomorphic to the germ of $X_{C,L,G}$ at $P_{C,L,G}$, and π is isomorphic to the germ of $\pi_{C,L,G}$.

Proof. With the nowadays standard abuse of notation, π is the germ of a generic covering $\pi : S \to \mathbb{C}^2$ branched on the (whole) curve $\{x^n = y^m\}$.

We consider the map $f_{m',n'} : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $f_{m',n'}(x,y) = (x^{m'}, y^{n'})$, and we use it to define the fibre product $\overline{S} := S \times_{\mathbb{C}^2} \mathbb{C}^2$: by theorem 2.2 of [MaPi1] $\overline{S}$ is a normal surface and the projection on the second factor is a generic covering of degree d branched on $\{x^k = y^k\}$.

Theorem 2.5 applies to $\overline{\pi}$: we conclude that $\overline{\pi}$ is the germ of $\pi_{C,L}$ for some curve C of genus $\frac{k}{2} - d + 1$, and some pencil L on C.

The map $f_{m',n'}$ is the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$ where G is the group generated by the map

$$(x,y) \mapsto (e^{\frac{2\pi i}{m'}} x, e^{\frac{2\pi i}{n'}} y);$$

the action lifts to the fiber product $\overline{S}$ (acting trivially on the second factor), and the composition of maps $S \to \overline{S} \to \overline{S}/G$ is clearly an isomorphism.

We want to describe the action of G on $\overline{S}$: at the special point it is the germ of an action on $X_{C,L}$ like the ones considered in definition 2.6.

To do that, we 'compactify' $\overline{\pi}$ to a map (forgive the further abuse) $\overline{\pi} : X_{C,L} \to \mathbb{P}^2$. The action of G naturally extends to this compactification, and the induced action of the above mentioned generator of G on $\mathbb{P}^2$ is

$$(x_0, x_1, x_2) \mapsto (x_0 : e^{\frac{2\pi i}{m'}} x_1, e^{\frac{2\pi i}{n'}} x_2).$$

The line at ∞, $\{x_0 = 0\}$ is $G-$invariant, therefore the same holds for its preimage C_∞, which is isomorphic to C by proposition 2.3. This shows that the line bundle $\mathcal{O}_{X_{C,L}}(1) \otimes p^*\mathcal{O}_C(L)$ is $G-$ invariant. Moreover, restricting the action to C_∞, we see that L (cut by the pull-back of the lines of $\mathbb{P}^2$) is $G-$invariant.

The reader can easily check the remaining properties: we just note that the map φ_L is the map of C_∞ to the line at ∞ and the coordinates y_1, y_2 on it of definition 2.6 are restrictions of x_1, x_2. □

This gives a concrete geometrical way to write every germ of a generic covering branched on $\{x^n = y^m\}$; let us first introduce the following

Definition 2.9. *Let C be a curve, L a linear system (not necessarily complete) on it. We will denote by $Aut(C, L)$ the subgroup of $Aut(C)$ of those automorphisms which preserve L.*

Remark 2.10. *We have the following recipe to construct all generic coverings of fixed degree d branched on $\{x^n = y^m\}$: we define*

$$k := lcm(m, n), \quad m' := k/n, \quad n' := k/m.$$

We take the following ingredients:

- *a curve C of genus $\frac{k}{2} - d + 1$;*
- *a generic pencil L of degree d on C;*
- *$\sigma \in Aut(C, L)$ of order $m'n'$ acting on L as in definition 2.6.*

Then we cook the ingredients as described in definition 2.6: the covering is the germ of $\pi_{C,L,G}$ of definition 2.7, where G is the subgroup of $Aut(C)$ generated by σ.

Example. Assume $d \geq \frac{k}{3} + 1$. Then the degree of L is at least $2g + 1$, and therefore $\varphi_{|L|}$ embeds C in $\mathbb{P}^{2d-\frac{k}{2}-1}$. In this case $X_{C,L}$ is simply the cone over $\varphi_{|L|}(C)$ in $\mathbb{P}^{2d-\frac{k}{2}}$.

The action of σ on $|\mathcal{O}_{\tilde{X}}(1) \otimes p^*\mathcal{O}_C(L)|$ has finite order, and therefore it is diagonalizable: we can choose a basis of eigenvectors that starts with our x_0, x_1, x_2.

Therefore, if $d \geq \frac{k}{3} + 1$ our recipe reduces to considering an automorphism σ of $\mathbb{P}^{2d-\frac{k}{2}}$ acting as $\sigma^* x_j = e^{\frac{2a_j\pi i}{m'n'}} x_j$ with $a_0 = m'n'$, $a_1 = n'$, $a_2 = m'$, and take the cone over a σ-invariant smooth curve C in $\mathbb{P}^{2d-\frac{k}{2}-1}$; the germ at $(1 : 0 : \cdots : 0)$ of the quotient is the corresponding germ of singularity.

3. The local fundamental groups

In this section we want to compute a presentation for the local fundamental group of $X_{C,L}$ at $P_{C,L}$.
Recall that if S is a complex surface and $P \in S$, a neighborhood of $S \setminus P$ retracts onto the link $\mathrm{Lnk}(S, P)$ of S at P, obtained for instance by immerging a neighborhood of P in some $\mathbb{C}^n$ and intersecting with a small ball centered at P. Note that $\mathrm{Lnk}(S, P)$ is a real oriented 3-manifold.

We want to show that in our case, $\mathrm{Lnk}(X_{C,L}, P_{C,L})$ admits a S^1-action and thus it is a Seifert manifold. Seifert manifolds are a classical subject and their fundamental group is known once one knows the genus of the quotient by the action, the auto-intersection of a section and how S^1 acts on exceptional orbits (see [Sei], [SeTh] or [Orl]).
In order to do this, we define a $\mathbb{C}^*$-action on $\tilde{X}_{C,L}$ which fixes $(C_0)_{C,L}$

and $\tilde{C}_\infty$ the preimage under $\tilde{\pi}_{C,L}$ of the line at infinity. This then iduces a good C^*-action on $X_{C,L} \setminus C_\infty$ for which the only point with a non-trivial isotropy group is $P_{C,L}$.
The action is the following: writing $\tilde{X}_{C,L} = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{O}_C(L))$, we let $\mathbb{C}^*$ act by multiplication on the first factor and trivially on the second factor.
Then, choosing two sections f_1, f_2 in $H^0(\mathcal{O}_C(L))$ which generate the pencil as in the proof of 2.3, we see that the line in a fiber of $\mathcal{O}_C \oplus \mathcal{O}_C(L)$ generated by the vector $(x_0, x_1 f_1 + x_2 f_2)$ is fixed under the action if and only if $x_0 = 0$ or $f_1 = f_2 = 0$ as we wanted.

This shows that $\mathrm{Lnk}(X_{C,L}, P_{C,L})$ is a Seifert manifold without exceptional orbits, i.e. a Seifert bundle.
We can then state the following

Proposition 3.1. *The local fundamental group of $X_{C,L}$ at $P_{C,L}$ can be presented as*

$$< \tau, \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g | [\tau, \alpha_i], [\tau, \beta_i], \tau^{-d}[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] >,$$

where τ is the class of a small loop (in $\tilde{X}$) around C_0 in a fiber of $\tilde{\pi}$, $d = \deg L$ and the other geneators map to the standard generators of the fundamental group of C.

For a proof see e.g. [Orl]. □

Corollary 3.2. *A smooth generic covernig branched on $\{x^n = y^m\}$ has degree that divides $m'n' = \frac{lcm(m,n)}{gcd(m,n)}$.*

Proof.
By theorem 2.8, we have to show that if the germ $(X_{C,L,G}, P_{C,L,G})$ is smooth, then $d = \deg L$ divides $o(G)$.

The local fundamental group of the germ $(X_{C,L,G}, P_{C,L,G})$ is the fundamental group of a smooth real manifold $S_{C,L}$ of dimension 3, boundary of a tubular neighborhood of C_0 that we may assume $G-$invariant: the smoothness of $X_{C,L,G}$ is, by Mumford criterion [Mum], the simple connectedness of the quotient S/G.

We give a $G-$invariant simplicial decomposition of $S_{C,L}$ such that τ is supported on its $1-$skeleton: in particular it induces a simplicial decomposition of S/G.

By proposition 3.1 the first homology group of $S_{C,L}$ is $\mathbb{Z}/_{d\mathbb{Z}} \times \mathbb{Z}^{2g}$ with torsion subgroup generated by the class of τ: let $\pi : S \to S/G$ be the natural projection: by simply connectedness $\pi(\tau)$ is the boundary of a $2-$simplex Δ.

The $2-$simplex $\pi^{-1}(\Delta)$ has boundary that equals the orbit of τ; that is $o(G)$ copies of τ (around different points, but they are of course pairwise homotopic to each other): in particular $o(G)\tau$ is trivial in $H_1(S)$ and therefore d divides $o(G)$.

□

4. The singularities of $\tilde{X}_{C,L,G}$

In order to understand the germ (X,P) we need to describe the singularities of $\tilde{X}$ in a neighbourhood of C_0. In fact, the ruled surface $\tilde{X}_{C,L}$ is by definition always smooth but the quotient surface $\tilde{X}_{C,L,G}$ has, in general, quotient singularities.

Remark 4.1. *Let (C,L,σ) be as in remark 2.10; let G be the cyclic group generated by σ and consider the induced action of G on $\varphi_L(C)\cong\mathbb{P}^1$: by assumption, in the coordinates y_1,y_2 of definition 2.6, the generator σ of G acts as $\sigma(y_1,y_2)=(\omega^{n'}y_1,\omega^{m'}y_2)$ where $\omega=e^{\frac{2\pi i}{m'n'}}$.*

Since by assumption $gcd(m',n')=1$, it follows that the only points of C having a non trivial stabilizer are contained in the divisor $(f_1)+(f_2)$.

Definition 4.2. *We denote by ν_i the number of orbits of the action of σ on (f_i).*

The branch points of the map $C\to C/G$ are then contained in the $\nu_1+\nu_2$ points image of $(f_1)+(f_2)$.

Lemma 4.3. *Let (C,L,σ) be as in remark 2.10, and consider the numbers ν_i introduced in definition 4.2. Then*

$$g\left(C/G\right)=\frac{gcd(m,n)-(\nu_1+\nu_2)}{2}+1;$$

in particular C/G is rational if and only if $\nu_1+\nu_2=gcd(m,n)+2$.

Proof. The map $C\to C/G$ has degree $m'n'$ and ramifies at most in the points of the divisors (f_1) and (f_2): by the Riemann-Hurwitz formula

$$\chi\left(C\right)=m'n'\chi\left(C/G\right)-\delta_1-\delta_2,$$

where δ_i is the sum of the ramification indices in the points image of (f_i).

The formula follows from $g(C)=\frac{lcm(m,n)}{2}-d+1$ and $\delta_i=m'n'\nu_i-d$. □

Example. Consider the classical case of the generic covering obtained projecting the smooth surface $\{z^3-3zx+2y=0\}\subset\mathbb{C}^3$ onto the $(x,y)-$plane. To obtain it with our recipe $((n,m,d)=(3,2,3))$ we take an elliptic curve (the cubic $\{x_3^3-3x_3x_1^2+2x_2^3=0\}\subset\mathbb{P}^2$), a generic rational pencil L of degree d (cut by Span(x_1,x_2)) and a $\mathbb{Z}/_{6\mathbb{Z}}$ action on it $((x_1:x_2:x_3)\mapsto(-x_1:e^{\frac{2\pi i}{3}}x_2:x_3))$.

Non trivial stabilizer can happen only on the two sets of $d=3$ 'special' points $(\{x_i=0\}$ for $i=1,2)$; in fact the action of σ on one of this $(\{x_1=0\})$ is a $3-$cycle (and therefore $\nu_1=1$), whence on the other one $(\{x_2=0\})$ is a transposition (fixing $(1:0:0)$, therefore $\nu_2=2$). By the lemma we have $g(C/G)=0$ as expected.

More generally, in [MaPi2] we introduced the germ of generic coverings $(S_{h,k,a,b},\pi_{h,k,a,b})$, for $h,k,a,b\in\mathbb{N}$, $gcd(h,k)=1$: we have just discussed the case $h = k = a = b = 1$. All the germs of surfaces $S_{h,k,1,1}$ are smooth with $(n, m, d) = (h + k, hk, h + k))$: a similar discussion (using the equations given in [MaPi2]) shows that $\nu_1 = 1$ and $\nu_2 = 2$, the two orbits corrisponding to ν_2 having respective cardinality h and k.

The description of the singularities of $\tilde{X}$ needs the following definition

Definition 4.4. *Let σ be as above a generator of G and let Q be a point of $C \cong (C_0)_{C,L}$. We will denote by $l(Q)$ the cardinality of the orbit of Q under σ, or, equivalently, the smallest positive integer l with $\sigma^l(Q) = Q$.*

We moreover introduce
$l_m := gcd(l, m')$ $\quad l_n := gcd(l, n')$
$r_m = m'/l_m$ $\quad r_n = n'/l_n$.

An orbit contained in (f_1) (resp. (f_2)) is **exceptional** *if its cardinality l is not a multiple of n' (resp. m'), or equivalently if $r_n \neq 1$ (resp. $r_m \neq 1$).*

Finally; since $gdc(r_m, r_n) = 1$, there are uniquelly determined positive integers s_m and s_n such that $s_m r_m + s_n r_n = r_n r_m - 1$.

Proposition 4.5. *$\tilde{X}_{C,L,G}$ has only isolated singularities. The singularities contained in $(C_0)_{C,L,G}$ are contained in the image of the divisor $(f_1) + (f_2) \subset (C_0)_{C,L}$.*

Assume $Q \in (f_2)$. Then there are a generator σ of G and local coordinates (z, t) centered in Q such that $(C_0)_{C,L}$ has local equation $\{t = 0\}$ and

$$(\sigma^l)^*(z, t) = (e^{\frac{2\pi i}{r_m r_n}} z, e^{\frac{2\pi i s_n}{r_m}} t).$$

In particular the image of Q in $\tilde{X}_{C,L,G}$ is a quotient singularity of type $\frac{1}{r_m}(1, s_n)$.

The same holds, exchanging m' and n', if $Q \in (f_1)$, giving a quotient singularity $\frac{1}{r_n}(1, s_m)$. In particular there is a bijection between exceptional orbits and singular points of $\tilde{X}_{C,L,G}$ contained in $(C_0)_{C,L,G}$.

Proof. Let $\overline{Q}$ be a point in $\tilde{X}_{C,L}$.

If the orbit of $\overline{Q}$ under the action of G has cardinality smaller than $m'n'$, then the same holds for its image Q on C_0, and therefore Q belongs either to (f_1) or to (f_2).

Assume then that $Q \in (f_2)$ (the case $Q \in (f_1)$ will be completely analogous). Q is then not a branch point of the map φ_L and we can choose a local parameter z over C centered in Q lifting a local parameter ζ over $\varphi_L(C)$ centered in $\varphi_L(Q)$. Since $\varphi_L(C)$ is the exceptional divisor of the blow-up of $\mathbb{C}^2$ at the origin, and we know the action of σ on $\mathbb{C}^2$, we can easily compute $\sigma^*\zeta = \omega^{m'-n'}\zeta$ where $\omega := e^{\frac{2\pi i}{m'n'}}$. It follows that

$$(\sigma^l)^* z = \omega^{l(m'-n')} z.$$

Since f_1 is invertible in Q, the bundle $\mathcal{E} = \mathcal{O}_C(-L) \oplus \mathcal{O}_C$ in a neighborhood of Q is generated by $t_0 = (f_1^{-1}, 0)$, $t_1 = (0, 1)$.

We can then write the action of σ^l on $X_{C,L}$ near Q as

$$(\sigma^l)^*(z, (t_0 : t_1)) = (\omega^{l(m'-n')} z, (t_0 : \omega^{ln'} t_1)).$$

It follows that the only singular points are in the images of $(t_0 : t_1) = (1 : 0)$ and $(0 : 1)$, as stated, and these are quotient singularities.

For an explicit description of the singularities contained in $(C_0)_{C,L,G}$, since $(C_0)_{C,L} = \{t_1 = 0\}$ we can set $t := t_1/t_0$: we get

$$(\sigma^l)^*(z, t) = (e^{\frac{2\pi i(m'-n')}{r_m r_n}} z, e^{\frac{2\pi i l_n}{r_m}} t).$$

Since $gcd(m', n') = 1$, $m' - n'$ has an inverse α in $\mathbb{Z}^*_{m'n'}$: σ^α is a generator that acts as

$$(\sigma^{l\alpha})^*(z, t) = (e^{\frac{2\pi i}{r_m r_n}} z, e^{\frac{2\pi i \alpha l_n}{r_m}} t).$$

From the definition of α it follows that r_m divides $\alpha l_n r_n + 1$, and therefore $e^{\frac{2\pi i \alpha l_n}{r_m}} = e^{\frac{2\pi i s_n}{r_m}}$. □

5. Rationality and smoothness criteria

In this section we want to give criteria to decide on the rationality and smoothness of a generic covering.

Since in section 2 we have given a recipe (with ingredients (C, L, σ)) to construct all generic coverings of degree d branched on a curve of equation $\{x^n = y^m\}$, we will give criteria to decide on the rationality and smoothness of the germ $(X_{C,L,G}, P_{C,L,G})$.

We introduce the

Definition 5.1. *Let $Y_{C,L,G}$ be a minimal resolution of the singularities of $\tilde{X}_{C,L,G}$; then the natural map $Y_{C,L,G} \to X_{C,L,G}$ gives a resolution of the germ $(X_{C,L,G}, P_{C,L,G})$. We denote moreover by $(\tilde{C}_0)_{C,L,G}$ the strict transform of $(C_0)_{C,L,G}$ in $Y_{C,L,G}$.*

We can apply to this resolution Artin's rationality criterion and prove

Theorem 5.2 (Rationality criterion). *The germ $(X_{C,L,G}, P_{C,L,G})$ is rational if and only if*

$$\nu_1 + \nu_2 = gcd(m, n) + 2.$$

Proof. The exceptional locus of the resolution of the singularity given in definition 5.1 is, by proposition 4.5, given by $(\tilde{C}_0)_{C,L,G}$, and a certain number of disjoint strings of rational curves (resolution of the quotient singularities) meeting $(\tilde{C}_0)_{C,L,G}$ transversally in a point.

By Artin's criterion ([Art]) the singularity $(X_{C,L,G}, P_{C,L,G})$ is rational if and only if $(C_0)_{C,L,G}$ is rational. But $(C_0)_{C,L,G} \cong C/G$ and we conclude by lemma 4.3. □

To proceed in the direction of a smoothness criterion, we need now a tecnical lemma on the local intersection form of a quotient singularity.

Lemma 5.3. *Let C be a smooth $\mathbb{Q}-$Cartier divisor on a surface X, and assume that C passes through a quotient singularity $\frac{1}{b}(1,a)$ as the image of the zero locus of the second coordinate. Let $Y \to X$ be a minimal resolution of the singularity, and let $\tilde{C}$ be the strict transform of C. Then $\tilde{C}^2 = C^2 - \frac{a}{b}$.*

Proof. bC is Cartier near the singularity, and therefore its pull-back on Y is $b\tilde{C} + D$ where D is an integral divisor on Y. With abuse of notation we will still write C for its (numerical) pull-back on Y. Then $(b\tilde{C} + D)D = bCD = 0$ (since C is a pull-back and D is exceptional).

It follows that $b^2(C^2 - \tilde{C}^2) = b\tilde{C}D$, and it remains to show that $\tilde{C}D = a$. By assumption D is supported on a string of rational curves and $\tilde{C}$ intersects transversally only one of these curves, one end of the string, and we have to prove that the multiplicity of this curve in D (*i.e.* in the pull-back of bC) is a.

This is a local computation; we can then assume $X = \mathbb{C}^2/\mathbb{Z}/_{b\mathbb{Z}}$ for the group action generated by $(x, y) \mapsto (e^{\frac{2\pi i}{b}}x, e^{\frac{2\pi i a}{b}}y)$, $\pi : \mathbb{C}^2 \to X$ be the quotient map. Then $C_1 := \{y = 0\}$ maps to C; more precisely the pull-back of bC is bC_1.

After blowing up the origin we get that the pull-back of bC contains b times the exceptional divisor and the induced action of the group near the intersection between the strict transform of C_1 and the exceptional divisor gives (at the quotient) a singularity $\frac{1}{b}(1, a-1)$.

Recursively after a blow-up we get a smooth quotient; the multiplicity of the last exceptional divisor is ba but the map onto Y has degree b and is totally ramified at this curve, therefore we get multiplicity a for the corresponding curve in Y. □

Applying the lemma 5.3 to our curve C_0, by proposition 4.5 immediately follows

Corollary 5.4.

$$(\tilde{C}_0)^2_{C,L,G} = -\left(\frac{d}{m'n'} + \sum_{\nu_1} \frac{s_m}{r_n} + \sum_{\nu_2} \frac{s_n}{r_m}\right).$$

where the symbol $\sum_{\nu_i}$ denote a sum on the ν_i orbits of the action of σ on (f_i), and for each of these orbits r_n, r_m, s_n, s_m are the constants associated to the length l of the corresponding orbit as in the definition 4.4.

We can now state a

Theorem 5.5 (Smoothness criterion). *The germ $(X_{C,L,G}, P_{C,L,G})$ is smooth, if and only if the following occur:*

a) *$d|m'n'$;*
b) *$\nu_1 + \nu_2 = gcd(m,n) + 2$;*
c) *there are at most two exceptional orbits;*
d) *if there are exactly two exceptional orbits, writing the two corresponding quotient singularities as $\frac{1}{r_1}(1,s_1)$ and $\frac{1}{r_2}(1,s_2)$ with $gcd(r_i,s_i)=1$, then $gcd(r_1,r_2)=1$.*

Proof.
Assume the germ is smooth: property a) follows from corollary 3.2, and property b) follows from the rationality criterion 5.2.

The resolution graph of the germ of singularity $(X_{C,L,G}, P_{C,L,G})$ is given by a vertex (the curve $\tilde{C}_0$) from which start as many strings of rational curves as the number of exceptional orbits.

Note that by construction the strings do not contain any (-1)-curves. So, if the germ is smooth, $\tilde{C}_0$ must be contractible, and what remains of the exceptional locus after its contraction must be further contractible and in particular three curves cannot pass through the same point: property c) follows.

Assume now that $\tilde{X}_{C,L,G}$ has exactly 2 singular points along C_0, quotient singularities of type $\frac{1}{r_1}(1,s_1)$ and $\frac{1}{r_2}(1,s_2)$ respectively. Let us denote by r the greatest common divisor of r_1 and r_2, and let G' be the subgroup of G of index b.

Then the quotient map $\tilde{X}_{C,L,G'} \to \tilde{X}_{C,L,G}$ yields a covering of degree r of a tubular neighbourhood of $(C_0)_{C,L,G}$ unbranched outside C_0 itself. If $X_{C,L,G}$ is smooth, Mumford's criterion [Mum] then forces $r=1$.

Conversely assume that the properties $a), b), c), d)$ hold. First we note that by property b) $(C_0)_{C,L,G}$ is a rational curve.

If there are no exceptional orbits, $\tilde{X}_{C,L,G}$ is smooth and our germ is obtained contracting $(C_0)_{C,L,G}$. We can compute its self-intersection by corollary 5.4 ($C_0 = \tilde{C}_0$ in this case): since it is an integer by property a) we conclude that it must be -1, and therefore our germ is smooth.

If there is exactly one exceptional orbit, we write as usual the corresponding singularity as $\frac{1}{r}(1,s)$ with $gcd(r,s)=1$, $0 < \frac{s}{r} < 1$. By corollary 5.4 and property a) $0 < -\tilde{C}_0^2 = \frac{d}{m'n'} + \frac{s}{r} < 2$, and therefore $\tilde{C}_0^2 = -1$.

By assumption $d|m'n'$. Setting $q := \frac{m'n'}{d}$ we have found $\frac{1}{q} + \frac{s}{r} = 1$. Since $gcd(r,s)=1$ we conclude $q = r = s+1$ and therefore we have a resolution of our germ with exceptional locus given by a (-1)-curve intersecting transversally the last curve of a string of $r-1$ rational curves with self-intersection (-2) (the exceptional locus of a minimal resolution of a quotient singularity $\frac{1}{r}(1,r-1)$). It follows that our germ is smooth.

It remains the case of two exceptional orbits. In this case the exceptional locus of a resolution of the germ is given by the curve $\tilde{C}_0$ and the strings corresponding to the resolution of the two corresponding quotient singularities that we write respectively as $\frac{1}{r_1}(1, s_1)$ and $\frac{1}{r_2}(1, s_2)$ with $0 < s_i < r_i$ and $gcd(s_i, r_i) = 1$.

By corollary 5.4

$$-\tilde{C}_0^2 = \frac{d}{m'n'} + \frac{s_1}{r_1} + \frac{s_2}{r_2} = \frac{1}{q} + \frac{s_1 r_2 + s_2 r_1}{r_1 r_2} \in \mathbb{Z}.$$

Since by property d) $gcd(r_1, r_2) = gcd(s_i, r_i) = 1$, it follows that $gcd(s_1 r_2 + s_2 r_1, r_1 r_2) = 1$, and therefore $q = r_1 r_2$.

Then $0 < -\tilde{C}_0^2 = \frac{1+s_1r_2+s_2r_1}{r_1r_2} \le \frac{1+(r_1-1)r_2+(r_2-1)r_1}{r_1r_2} < 2$. It follows that $\tilde{C_0}^2 = -1$ and s_1, s_2 are determined by the properties $0 < s_i < r_i$ and $s_1 r_2 + s_2 r_1 = r_1 r_2 - 1$.

We have then completely described the exceptional locus of the resolution $Y \to X$ as function of r_1, r_2. But what we have obtained coincides with what one obtains for the generic coverings $\pi_{r_1,r_2,1,1}$ (cf. the example at the beginning of this section): since the surfaces $S_{r_1,r_2,1,1}$ are smooth, we are done.

□

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