MIXED QUASI-ÉTALE QUOTIENTS WITH ARBITRARY SINGULARITIES

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Abstract. A mixed quasi-étale quotient is the quotient of the product of a curve of genus at least 2 with itself by the action of a group which exchanges the two factors and acts freely out of a finite subset. A mixed quasi-étale surface is the minimal resolution of its singularities.

We produce an algorithm computing all mixed quasi-étale surfaces with given geometric genus, irregularity, and self-intersection of the canonical class. We prove that all irregular mixed quasi-étale surfaces of general type are minimal.

As application, we classify all irregular mixed quasi étale surfaces of general type with genus equal to the irregularity, and all the regular ones with $K^2 > 0$, thus constructing new examples of surfaces of general type with $\chi = 1$. We mention the first example of a minimal surface of general type with $p_g = q = 1$ and Albanese fibre of genus bigger than K^2.

Introduction

In the last decade, after the seminal paper [Cat00], there has been growing interest on the surfaces birational to the quotient of the product of two curves of genus at least 2 by the action of a subgroup of its automorphism group.

These have shown to be a very productive source of examples, expecially in the very interesting and still mysterious case of the surfaces of general type with $\chi(S) = 1$ (equivalently $p_g(S) = q(S)$). Here and in the following we use the standard notation of the theory of the complex surfaces, as in [Bea78, BHPV04]. For the motivations and for the state of the art (few years ago) of the research on the surfaces of general type with $p_g = q = 0$ we suggest to the reader the survey [BCP11], while some informations on the more general case $\chi(S) = 1$ can be found in [BCP06, Section 2]. We just mention here that the case $p_g = q \geq 3$ has been classified ([Bea82, CCML98, Pir02, HP02]), whereas the case $p_g = q \leq 2$ is still rather unknown.

Recently several new surfaces of general type with $p_g = q$ have been constructed as quotient of a product of two curves by the action of a finite group; see [BC04, BCG08, BCGP12, BP12] for $p_g = 0$, [CP09, Pol08, Pol09,

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MP10] for $p_g = 1$, [Pen11, Zuc03] for $p_g = 2$. In all these articles the authors assume either that the action be free, or *unmixed*, which means that the action is the diagonal action induced by actions on the factors.

In [Fra11] the first author considered a more general case, assuming the action to be free out of a finite set of points: it is not difficult to show that this includes both the cases above. We call this case *quasi-étale* since the induced map into the quotient is quasi-étale in the sense of [Cat07]. Since the above mentioned papers give a satisfactory description of the *unmixed* case, [Fra11] concentrated on the *mixed* case, which is the complement case. After some preliminary results, [Fra11] restricted to the case of the surfaces of general type with $p_g = 0$, and imposed a strong condition on the singularities of the quotient surface, obtaining several interesting new examples.

In this paper we study the general *mixed quasi-étale* case, dropping all assumptions in [Fra11].

The situation is the following. Let C be a Riemann surface of genus $g(C) \geq 2$, and let G be a finite group that acts on $C \times C$. We say that $X = (C \times C)/G$ is a *quasi-étale quotient* if the action of G is free out of a finite set of points. Let $S \to X$ be the minimal resolution of the singularities of X, we call S a *quasi-étale surface*. The action is *mixed* if $G \subset \mathrm{Aut}(C \times C) \cong \mathrm{Aut}(C)^2 \rtimes \mathbb{Z}_2$ is not contained in $\mathrm{Aut}(C)^2$; if the action is mixed we say that X is a *mixed q.e. quotient*, S is a *mixed q.e. surface* and we denote by $G^0 \triangleleft G$ the subgroup $G \cap \mathrm{Aut}(C)^2$.

The main result of this paper is an algorithm which, for each fixed integers p_g, q and K^2, produces all mixed q.e. surfaces with those invariants. We implemented the algorithm in the program MAGMA [MAG]; the script may be freely downloaded from

`http://www.science.unitn.it/~pignatel/papers/Mixed.magma`

As application, running the program for all possible positive values of K^2 and $p_g = q$, we obtained the following theorems A, B and C. Note that the programs works also for arbitrary values of K^2, p_g and q, so more surfaces may be produced with it.

Theorem A. *The mixed q.e. surfaces S with $p_g = q = 0$ and $K^2 > 0$ form the 17 irreducible families collected in Table 1. In all cases S is minimal and of general type.*

In Table 1, every row correspond to an irreducible family. Two columns need some explanation: the column $\mathcal{B}(X)$ represents the basket of singularities of X (see Definition 2.15), the column Sign. gives the signature of the generating vector of G^0 (see Definition 1.5) in a compacted way, *e.g.* $2^3, 4$ stands for $(q; 2, 2, 2, 4)$. Throughout the paper we denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathfrak{S}_n$ the symmetric group on n letters, by A_n the alternating group on n letters, by Q_8 the group of quaternions, by D_n the dihedral group of order $2n$, by $D_{p,q,r}$ the group $\langle x, y \mid x^p = y^q = 1, xyx^{-1} = y^r \rangle$, by BD_n the group $\langle x, y \mid y^{2n} = x^2y^n = 1, xyx^{-1} = y^{-1} \rangle$ and by G(a,b) the b^{th} group of order a in the MAGMA database of finite group.

There may exist more mixed q.e. surfaces of general type with $p_g = q = 0$: they would have $K^2 \leq 0$ and therefore they would be not minimal. In the

K_S^2	$\mathcal{B}(X)$	Sign.	G^0	G	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
1	$2\,C_{2,1}, 2\,D_{2,1}$	$2^3, 4$	$D_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$
2	$6\,C_{2,1}$	2^5	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
2	$6\,C_{2,1}$	4^3	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	G(64,82)	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
2	$C_{2,1}, 2\,D_{2,1}$	$2^3, 4$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$
2	$C_{2,1}, 2\,D_{2,1}$	$2^2, 3^2$	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$2\,C_{4,1}, 3\,C_{2,1}$	$2^3, 4$	G(64,73)	G(128,1535)	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
2	$2\,C_{3,1}, 2\,C_{3,2}$	$3^2, 4$	G(384,4)	G(768,1083540)	$\mathbb{Z}_4$	$\mathbb{Z}_4$
2	$2\,C_{3,1}, 2\,C_{3,2}$	$3^2, 4$	G(384,4)	G(768,1083541)	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
3	$C_{8,3}, C_{8,5}$	$2^3, 8$	G(32, 39)	G(64, 42)	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
4	$4\,C_{2,1}$	2^5	$D_4 \times \mathbb{Z}_2$	$D_{2,8,5} \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_8$
4	$4\,C_{2,1}$	2^5	$\mathbb{Z}_2^4$	$(\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	∞
4	$4\,C_{2,1}$	4^3	G(64, 23)	G(128, 836)	$\mathbb{Z}_2^3$	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$
8	$\emptyset$	2^5	$D_4 \times \mathbb{Z}_2^2$	$(D_{2,8,5} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$	∞
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	$\mathbb{Z}_4^3$	∞
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	∞
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	∞
8	$\emptyset$	4^3	G(128, 36)	G(256, 3679)	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	∞

TABLE 1. Mixed q.e. surfaces of general type with $K^2 > 0$ and $p_g = q = 0$

irregular ($q > 0$) case, the situation is, from this point of view, much more clear since we could prove the following Theorem (4.5).

Theorem. *Let S be an irregular mixed q.e. surface of general type, then S is minimal.*

The result does not extend to the unmixed case, counterexamples can be found in [MP10]. Then we could give a complete classification of the mixed q.e. irregular surfaces of general type with $p_g = q$.

Theorem B. *The mixed q.e. surfaces of general type S with $p_g = q = 1$ form the 19 irreducible families collected in Table 2.*

In Table 2 we use the same notation of the previous Table 1; we also report the genus g_{alb} of a general fibre of the Albanese map, and we do not report $\pi_1(S)$, always infinite. Note that there is a surface with $K_S^2 = 6$ and $g_{alg} = 7$; to our knowledge it is the first example of a minimal surface of general type with $p_g = q = 1$ and $g_{alb} > K_S^2$; we recall that this is not possible for $K_S^2 \leq 3$ by their classification [Cat10, CC91, CC93, CP06]. We noticed also the first example with $K^2 = 6$ and $g_{alb} = 5$. Also the other examples with $4 \leq K^2 \leq 6$ may be, to our knowledge, new, although other surfaces with those invariants have been already constructed (see [Pig09, Pol09, MP10, Rit10a, Rit07, Rit10b]).

Theorem C. *There exists a unique irreducible family of mixed q.e. surfaces of general type with $p_g = q \geq 2$, and it has $p_g = 2$ and $K^2 = 8$, see Table 3.*

The mixed q.e. surfaces with $K_S^2 = 8\chi(S)$ are those for which the action is free; indeed all the examples in Tables 1, 2 and 3 appeared in the papers

K_S^2	g_{alb}	$\mathcal{B}(X)$	Sign.	G^0	G	$H_1(S,\mathbb{Z})$
2	2	$C_{2,1}, 2D_{2,1}$	2^2	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}^2$
2	2	$C_{2,1}, 2D_{2,1}$	2	D_8	$D_{2,8,3}$	$\mathbb{Z}^2$
2	2	$C_{2,1}, 2D_{2,1}$	2	Q_8	BD_4	$\mathbb{Z}^2$
4	3	$4C_{2,1}$	2^2	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_2 \times \mathbb{Z}^2$
4	3	$4C_{2,1}$	2^2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$
4	2	$4C_{2,1}$	2	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	G(32,29)	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$
4	3	$4C_{2,1}$	2	$D_{4,4,3}$	$D_{4,8,3}$	$\mathbb{Z}_2 \times \mathbb{Z}^2$
4	3	$4C_{2,1}$	2	$D_{4,4,3}$	$D_{4,8,7}$	$\mathbb{Z}_2 \times \mathbb{Z}^2$
4	2	$4C_{2,1}$	2	$D_{4,4,3}$	G(32,32)	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$
4	2	$4C_{2,1}$	2	$D_{4,4,3}$	G(32,35)	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$
4	3	$4C_{2,1}$	2	$D_{2,8,5}$	G(32,15)	$\mathbb{Z}_2 \times \mathbb{Z}^2$
5	3	$C_{3,1}, C_{3,2}$	3	BD_3	BD_6	$\mathbb{Z}_2 \times \mathbb{Z}^2$
5	3	$C_{3,1}, C_{3,2}$	3	D_6	$D_{2,12,5}$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$
6	3	$2C_{2,1}$	2	$A_4 \times \mathbb{Z}_2$	G(48,30)	$\mathbb{Z}_2 \times \mathbb{Z}^2$
6	7	$2C_{2,1}$	2	$A_4 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}^2$
6	5	$C_{5,3}$	5	D_5	G(20,3)	$\mathbb{Z}_2 \times \mathbb{Z}^2$
8	5	$\emptyset$	2^2	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$D_{2,8,5}$	$\mathbb{Z}_4 \times \mathbb{Z}^2$
8	5	$\emptyset$	2^2	D_4	$D_{2,8,3}$	$\mathbb{Z}_4 \times \mathbb{Z}^2$
8	5	$\emptyset$	2^2	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$

TABLE 2. Mixed q.e. surfaces of general type with $p_g = q = 1$

K_S^2	$\mathcal{B}(X)$	Sign.	G^0	G	$H_1(S,\mathbb{Z})$
8	$\emptyset$	-	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}^4$

TABLE 3. Mixed q.e. surfaces of general type with $p_g = q = 2$

cited at the beginning of this introduction. In particular, the list in [Pen11] is the complete list of all the q.e. surfaces with $p_g = q = 2$.

An expanded version of these tables can be downloaded from:

`http://www.science.unitn.it/~pignatel/papers/TablesMixed.pdf`

The paper is organized as follows.

In section 1 we give the algebraic recipe which, using Riemann Existence Theorem, constructs mixed q.e. surfaces.

In section 2 we give a complete description of the analytic type of the possible singularities of X. We show moreover how to compute the number of singular points of X and the analytic type of each singularity directly by the *ingredients* of the algebraic recipe above, and we give formulas for K_S^2, $p_g(S)$ and $q(S)$. We think is worth mentioning here an unexpected consequence of those formula (Corollary 2.20): we proved that the number of branch points of the double cover $(C \times C)/G^0 \to (C \times C)/G$ is even and bounded from above by $2(p_g(S) + 1)$.

Section 3 is devoted to the Albanese map of a mixed q.e. surface with $q = 1$. The main result is a formula to compute the genus of its general fibre.

In section 4 we show that all irregular mixed q.e. surfaces are minimal. In the regular case, we prove it under a strong assumption on the singularities of X (Proposition 4.9).

Finally, in section 5, we present our algorithm to construct all mixed quasi-étale surfaces with given values of K^2, p_g and q, and prove Theorems A, B and C.

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1. The algebraic recipe

Throughout this paper we will denote by C a Riemann surface of genus $g \geq 2$ and by G a finite subgroup of $\mathrm{Aut}(C \times C)$ whose action is free out of a finite subset and *mixed*, which means that there are elements in G which exchange the two natural isotrivial fibrations of $C \times C$. We will denote by G^0 the index 2 subgroup of the elements that do not exchange the factors.

We will say that the quotient surface $X = (C \times C)/G$ is a *mixed q.e. quotient*. We will denote by $\rho\colon S \to X$ the minimal resolution of the singularities of X, and we say that S is a *mixed q.e. surface*.

Remark 1.1. By [Fra11, Remark 2.3] every mixed q.e. quotient is induced by a unique *minimal* action, which means that G^0 acts faithfully on both factors: therefore in this paper we will only consider minimal actions. If X is a *mixed q.e. surface*, then the quotient map factors as follows:

$$C \times C \xrightarrow{\sigma} Y := (C \times C)/G^0 \xrightarrow{\pi} X.$$

[Cat00, Proposition 3.16] gives the following description of minimal mixed actions:

Theorem 1.2. *Let $G \subseteq \mathrm{Aut}(C \times C)$ be a minimal mixed action. Fix $\tau' \in G \setminus G^0$; it determines an element $\tau := \tau'^2 \in G^0$ and an element $\varphi \in \mathrm{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$. Then, up to a coordinate change, G acts as follows:*

$$\begin{aligned} g(x,y) &= (gx, \varphi(g)y) \\ \tau' g(x,y) &= (\varphi(g)y, \tau g\, x) \end{aligned} \qquad \text{for } g \in G^0 \tag{1.1}$$

Conversely, for every $G^0 \subseteq \mathrm{Aut}(C)$ and G extension of degree 2 of G^0, fixed $\tau' \in G \setminus G^0$ and defined τ and φ as above, (1.1) defines a minimal mixed action on $C \times C$.

We recall the following results:

Theorem 1.3 ([Fra11, Theorem 2.6])**.** *Let X be a quotient surface of mixed type provided by a minimal mixed action of G on $C \times C$. The quotient map $C \times C \to X$ is quasi-étale if and only if the exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{1.2}$$

does not split.

Moreover, if the quotient map is quasi-étale, then $\mathrm{Sing}(X) = \pi(\mathrm{Sing}(Y))$.

Lemma 1.4 ([Fra11, Lemma 2.9])**.** *Let $S \to X = (C \times C)/G$ be a mixed q.e. surface. Then $q(S)$ equals the genus of $C' := C/G^0$.*

The study of varieties birational to a quotient of a product of curves is strictly connected with the study of Galois coverings of Riemann surfaces. Now we collect some results that allow us to shift from the geometrical setup to the algebraic one and viceversa.

Definition 1.5. Let H be a finite group and let

$$g \geq 0 \text{ and } m_1, \dots, m_r > 1$$

be integers. A *generating vector* for H of signature $(g; m_1, \dots, m_r)$ is a $(2g+r)$-tuple of elements of H:

$$V := (d_1, e_1, \dots, d_g, e_g; h_1, \dots, h_r)$$

such that V generates H, $\prod_{i=1}^{g}[d_i, e_i] \cdot h_1 \cdot h_2 \cdots h_r = 1$ and there exists a permutation $\sigma \in \mathfrak{S}_r$ such that $\operatorname{ord}(h_i) = m_{\sigma(i)}$ for $i = 1, \dots, r$. In this case, we also say that H is $(g; m_1, \dots, m_r)$*-generated.*

By Riemann's existence theorem (see [BCP11]), any curve C of genus g together with an action of a finite group H on it, such that C/H is a curve C' of genus g', is determined (modulo automorphisms) by the following data:

1. the branch point set $\{p_1, \dots, p_r\} \subset C'$;
2. loops $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_r \in \pi_1(C' \setminus \{p_1, \dots, p_r\})$, where $\{\alpha_i, \beta_i\}_i$ generates $\pi_1(C')$, each γ_i is a simple geometric loop around p_i and $\prod_{i=1}^{g'}[\alpha_i, \beta_i] \cdot \gamma_1 \cdot \ldots \cdot \gamma_r = 1$;
3. a *generating vector* for H of signature $(g; m_1, \dots, m_r)$ with the property that the *Hurwitz's formula* holds:

$$2g - 2 = |H| \left(2g' - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \right). \qquad (1.3)$$

Remark 1.6. Analogously, a mixed q.e. quotient $X = (C \times C)/G$ determines a finite group G^0, a degree 2 extension $1 \to G^0 \to G \to \mathbb{Z}_2 \to 1$, the curve $C' = C/G^0$, a set of points $\{p_1, \dots, p_r\} \subset C'$, and, for every choice of $\alpha_i, \beta_j, \gamma_k \in \pi_1(C' \setminus \{p_1, \dots, p_r\})$ as in (2), a generating vector V for G^0,

Conversely, the following algebraic data:

- a finite group G^0;
- the curve C';
- points $p_1, \dots, p_r \in C'$, and $\alpha_i, \beta_j, \gamma_k \in \pi_1(C' \setminus \{p_1, \dots, p_r\})$ as in (2);
- integers $m_1, \dots, m_r > 1$;
- a generating vector V for G^0 of signature $(g(C'); m_1, \dots, m_r)$;
- a degree 2 extension $1 \to G^0 \to G \to \mathbb{Z}_2 \to 1$ which does not split;

give a uniquely determined mixed q.e. quotient. Indeed by Riemann's existence theorem the first 5 data give the Galois cover $c \colon C \to C/G^0 \cong C'$ branched over $\{p_1, \dots, p_r\}$. The last datum determines, by Theorem 1.2, a minimal mixed action on $C \times C$ and by Theorem 1.3 the action is free out of a finite set of points.

If $V := (d_1, e_1, \dots, d_g, e_g; h_1, \dots, h_r)$ we will denote by K_i the cyclic subgroup of G^0 generated by h_i.

2. The singularities of a mixed q.e. quotient

This section is devoted to the study of the singularities of a mixed q.e. quotient $X = (C\times C)/G$. We will need to consider the intermediate quotient $Y = (C \times C)/G^0$, and the two isotrivial fibrations $\alpha_i \colon Y \to C' = C/G^0$ induced by the projections of $C \times C$ on the two factors.

The double cover $\pi \colon Y \to X$ determines an involution $\iota \colon Y \to Y$ such that $X = Y/\iota$. We need to consider the singularities of Y and the action of ι on them; indeed ι splits the singularities of X in two classes: the singularities not in the branch locus of π (analytically isomorphic to each of its preimages in Y), and the images of the fixed points of ι which are, by the last statement of Theorem 1.3, singular points of X.

Y is a product-quotient surface, whose singularities are now well understood (see [BP12, MP10, Pol10]). They are *cyclic quotient singularities*, isomorphic to the quotient $\mathbb{C}^2/\langle\sigma\rangle$, where σ is the diagonal linear automorphism with eigenvalues $\exp(\frac{2\pi i}{n})$ and $\exp(\frac{2\pi i a}{n})$ with $n > a > 0$ and $\gcd(a, n) = 1$. We will say that this is a *singularity of type* $C_{n,a}$. Two singularities of respective types $C_{n,a}$ and $C_{n',a'}$ are locally analytically isomorphic if and only if $n = n'$ and either $a = a'$ or $aa' \equiv 1 \mod n$. We read from [BP12] how to determine the singular points of Y and their respective n and a.

Proposition 2.1 ([BP12, Propositions 1.16 and 1.18])**.** *Let X be a mixed q.e. quotient given by data as in Remark 1.6, let $Y = (C \times C)/G^0$ be the intermediate product-quotient surface, and consider the induced map $Q = (\alpha_1, \alpha_2) \colon Y \to C' \times C'$. The singular points of Y are the points $y = \sigma(u, v)$ such that*

$$\mathrm{Stab}_{G^0}(u) \cap \varphi^{-1}(\mathrm{Stab}_{G^0}(v)) \neq \{1\}\,,$$

where φ is the automorphism of G^0 in Theorem 1.2. In particular, if $y \in \mathrm{Sing}(Y)$ then $Q(y) = (p_i, p_j)$. Let i, $j \in \{1, \ldots, r\}$. Then

i) *there is a G^0-equivariant bijection $(Q \circ \sigma)^{-1}(p_i, p_j) \to G^0/K_i \times G^0/K_j$, where the action on the target is $g(aK_i, bK_j) = (gaK_i, \varphi(g)bK_j)$;*
ii) *there is a K_i-equivariant bijection between the orbits of the above G^0-action on $G^0/K_i \times G^0/K_j$ with the orbits of the K_i-action on $\{\overline{1}\} \times G^0/K_j$.*
iii) *An element $[g] \in \{\overline{1}\} \times G^0/K_j$ corresponds to a point of type $C_{n,a}$, where $n = |K_i \cap \varphi^{-1}(gK_jg^{-1})|$, and a is given as follows: let δ_i be the minimal positive integer such that there exists $1 \leq \gamma_j \leq \mathrm{ord}(h_j)$ with $h_i^{\delta_i} = g\varphi^{-1}(h_j^{\gamma_j})g^{-1}$. Then $a = \dfrac{n\gamma_j}{\mathrm{ord}(h_j)}$.*

By Proposition 2.1 we can compute the singularities of Y from the algebraic data of Remark 1.6. In order to compute the basket of singularities of X, we first need to know which of them are ramification points for π.

Lemma 2.2 ([Fra11, Proposition 3.8])**.** *Let $y \in Y$ be a fixed point for ι. Then $Q(y) = (p_i, p_i)$ for some i. In other words, Q maps all fixed points of ι on the diagonal of $C' \times C'$.*

Proposition 2.3. *An element $[g] \in \{\overline{1}\} \times G^0/K_i$ corresponds to a fixed point for ι if and only if there exists an element $h \in G^0$ such that:*

$$\begin{cases} \varphi(h)\tau h \in K_i \\ \varphi(h)g \in K_i \end{cases}$$

Proof. The point (K_i, gK_i) corresponding to $[g]$ is a ramification point for π if and only if there exists an element $\tau' h \in G \setminus G^0$ such that $(K_i, gK_i) = \tau' h(K_i, gK_i) = (\varphi(h)gK_i, \tau h K_i)$, that is

$$\begin{cases} \varphi(h)gK_i = K_i \\ gK_i = \tau h K_i \end{cases} \iff \begin{cases} \varphi(h)gK_i = K_i \\ \varphi(h)\tau h K_i = (\tau' h)^2 K_i = K_i \end{cases}$$

□

We study now the action of ι on a neighbourhood of a singular point of Y. We denote by $\lambda \colon T \to Y$ the minimal resolution of the singularities of Y. The exceptional divisor E of the minimal resolution of a cyclic quotient singularities of type $C_{n,a}$ is a *Hirzebruch-Jung string*, that is $E = \sum_{i=1}^{l} E_i$, where the E_i are smooth rational curves with $E_i.E_{i+1} = 1$, $E_i.E_j = 0$ for $|i-j| \geq 2$, and $E_i^2 = -b_i$ where the b_i are the coefficients of the continous fraction of $\frac{n}{a}$:

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} =: [b_1, \dots, b_l] \,.$$

Remark 2.4. $a \cdot a' \equiv 1 \mod n$ if and only if the continued fraction of $\frac{n}{a'}$ is $[b_l, \dots, b_1]$.

We need the following

Lemma 2.5. *The involution ι on Y lifts to a morphism $\mu \colon T \to T$.*

Proof. Consider $\mu := \lambda^{-1} \circ \iota \circ \lambda \colon T \dashrightarrow T$. Let $\Gamma \subset T \times T$ be the graph of μ; let $f_1, f_2 \colon \Gamma \to T$ be the projections on the factors.

If μ is not defined in a point $p \in T$, then Γ contains a (-1)-curve C contracted to p by f_1. $D := f_2(C) \subset T$ is a curve contracted to $\iota(\lambda(p))$ by λ, so a component of a H-J string: in particular $D^2 \leq -2$. On the other hand, since f_2 is a birational morphism, $D^2 \geq C^2 = -1$, a contradiction. □

To study the action of μ on the H-J strings, we will need the following

Proposition 2.6 (see [Ser96, Theorem 2.1])**.** *Let $y \in Y$ be a singular point of type $C_{n,a}$, and consider the two fibres $F_1 := \alpha_1^*(\alpha_1(y))$ and $F_2 := \alpha_2^*(\alpha_2(y))$ taken with the reduced structure. Let $\tilde{F}_i := \lambda_*^{-1}(F_i)$ be the strict transforms of F_i ($i = 1, 2$) and let E be the exceptional divisor of y.*

Then $\tilde{F}_1$ intersects one of the extremal curves of E, say E_1, while $\tilde{F}_2$ intersects the other extremal curve, say E_l.

Proposition 2.6 motivates the following

Definition 2.7. Let $\alpha \colon Y \to C'$ be one of the two natural fibrations. Let $y \in \operatorname{Sing}(Y)$ be a point of type $C_{n,a}$. Let $E := \sum_{i=1}^{l} E_i$ be the exceptional divisor over y, where the E_i are rational curves ordered so that $E_i^2 = -b_i$, $E_i.E_{i+1} = 1$. Let $\tilde{F}$ be the strict transform in T of the fibre $F = \alpha^*(\alpha(y))$

taken with the reduced structure.
We say that y is *of type $C_{n,a}$ with respect to* α if $\tilde{F}$ intersects E_1.

Remark 2.8. If y is of type $C_{n,a}$ with respect to α_1 then y is of type $C_{n,a'}$ with respect to α_2, with $a \cdot a' \cong 1 \mod n$.

Lemma 2.9. *If y is a point of type $C_{n,a}$ with respect to α_1, then $\iota(y)$ is a point of type $C_{n,a'}$ with respect to α_1, with $a \cdot a' \cong 1 \mod n$.*

Proof. Since ι is an isomorphisms, y and $z := \iota(y)$ have the same analytic type, so z is either of type $C_{n,a}$ or of type $C_{n,a'}$ with respect to α_1.

Let Y_i, resp. Z_i be the fibre of α_i containing y, resp. z, all of them taken with the reduced structure and let $\tilde{Y}_i := \lambda_*^{-1}(Y_i)$ and $\tilde{Z}_i := \lambda_*^{-1}(Z_i)$ $(i = 1, 2)$ be their strict transforms in T. Note that ι is the map induced on Y by the action on $C \times C$ of any $\tau' \in G \setminus G^0$. Since τ' exchanges the two factors, then $\iota(Y_1) = Z_2$ and therefore $\mu(\tilde{Y}_1) = \tilde{Z}_2$.

Let $E = \sum_{i=1}^{l} E_i$ resp. $E' = \sum_{i=1}^{l} E'_i$ be the exceptional divisor of y resp. z, with the E_i resp. E'_i ordered as in Definition 2.7 for $\alpha = \alpha_1$. By assumption $\tilde{Y}_1$ intersects E_1, $\tilde{Z}_1$ intersects E'_1, $\tilde{Z}_2$ intersects E'_l.

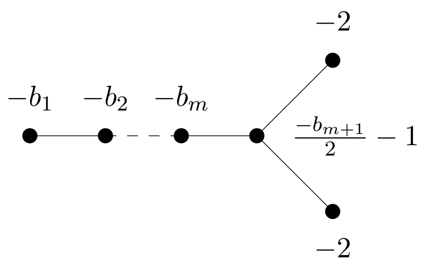
Since $\mu(\tilde{Y}_1) = \tilde{Z}_2$, then $\mu(E_1) = E'_l$. It follows that z is of type $C_{n,a}$ with respect to α_2 and of type $C_{n,a'}$ with respect to α_1. □

We give now a full description of the singular points of X arising from fixed points of ι.

Proposition 2.10. *Let $X = (C \times C)/G$ be a mixed q.e. quotient and let $y \in Y$ be a fixed point of ι. Then y is a singularity of type $C_{n,a}$ with $a^2 \equiv 1$ mod n; so $a = a'$ and the continued fraction $\dfrac{n}{a} = [b_1, \ldots, b_l]$ is palindrome: $b_i = b_{l+1-i}$ $\forall i$.*

Moreover

(i) *n is even;*
(ii) *l is odd: $l = 2m + 1$ and b_{m+1} is even;*
(iii) *the exceptional divisor of the minimal resolution of the singular point $\pi(y)$ is a tree of $m + 3$ smooth rational curves with decorated dual graph:*



Proof. By Lemma 2.9 y is of type $C_{n,a}$ with respect to both α_j, so $a = a'$ and $b_i = b_{l+1-i}$. More precisely, the proof of Lemma 2.9 shows that, if $E = \sum_1^l E_i$ is the H-J string of y, $\mu(E_i) = E_{l+1-i}$.

(i) If $y = \sigma(u, v)$, $|\mathrm{Stab}_{G^0}(u, v)| = n$ and $|\mathrm{Stab}_G(u, v)| = 2n$. If n is odd, then by Sylow's theorem there exists an element g of order 2 in $\mathrm{Stab}_G(u, v) \setminus \mathrm{Stab}_{G^0}(u, v)$, splitting the exact sequence (1.2), a contradiction.

(ii) Let $D = \sum_{i=1}^{l} D_i := \lambda^{-1}(y)$, and assume that $l = 2m$ is even. The involution μ exchanges D_i with D_{l+1-i}, hence $p = D_m \cap D_{m+1}$ is the unique point of D fixed by μ. $d\mu_p$ exchanges the directions of the tangent spaces of D_m and D_{m+1} and therefore it is not a multiple of the identity. Since it is an involution, then up to a linear coordinate change $d\mu_p = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which implies that the fixed locus of μ contains a curve through p in the direction of the eigenspace with eigenvalue 1, a contradiction. We delay the proof that b_{m+1} is even.

(iii) By point (ii), $l = 2m+1$ and all fixed points of μ in D belong to $D_{m+1} = \mu(D_{m+1})$. The restriction of μ to D_{m+1} is an involution and therefore by Hurwitz's formula it fixes exactly two points p_1 and p_2, that are distinct from the points of intersection of D_{m+1} with D_m or D_{m+2}.

Let V be a small μ-invariant open set of T containing D and not intersecting any other exceptional divisor, so that $\lambda(V)$ is an open set of Y containing only one singular point: y. Let $\epsilon\colon V' \to V$ be the blow-up in p_1 and p_2, we denote by D'_i the strict transform of D_i and by A_1 and A_2 the two (-1)-exceptional curves. The involution μ lifts to an involution μ' on V' whose fixed locus is the smooth curve $A_1 \cup A_2$.

Then V'/μ' is smooth, and therefore a resolution of the singular point $\pi(x)$ whose exceptional divisor is D/μ. The computation of the dual graph of D/μ is a standard computation that we leave to the reader. We notice that there is no curve with self-intersection -1, so the resolution is the minimal resolution. Moreover there is a curve of self-intersection $-(1 + b_{m+1}/2)$, showing that b_{m+1} is even. □

It follows that the analytic type of a singularity on X only depends on its preimage on Y. Indeed, these quotient singularities can be described as follows:

Proposition 2.11. *Let $X = (C \times C)/G$ be a mixed q.e. quotient and let $y \in \mathrm{Sing}(Y)$ be a point of type $C_{n,a}$ with $\frac{n}{a} = [b_1, \ldots, b_m, 2b, b_m, \ldots, b_1]$. Let*

$$\frac{p}{q} := [b_1, \ldots, b_m], \qquad \text{and} \qquad \xi := bp - q\,.$$

If y is a ramification point for π, then $x := \pi(y)$ is a quotient singularity isomorphic to $\mathbb{C}^2/H$ with:

- *if $\xi = 0$ (i.e. $p = 0$), then*

$$H = \left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{n+1} \end{pmatrix} \right\rangle, \qquad \text{with } \epsilon = e^{\frac{2\pi i}{2n}},$$

- *if $\xi \neq 0$ and odd, then*

$$H = \left\langle \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \text{ with } \eta = e^{\frac{2\pi i}{2\xi}}, \omega = e^{\frac{2\pi i}{2p}},$$

- *if $\xi \neq 0$ and even, then*

$$H = \left\langle \begin{pmatrix} 0 & \zeta \\ -\zeta & 0 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \right\rangle, \text{ with } \zeta = e^{\frac{2\pi i}{4\xi}} \text{ and } \omega = e^{\frac{2\pi i}{2p}}.$$

Proof. The statement follows immediately from the classification of finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ without quasi-reflections, see [Bri68, Satz 2.11] or [Mat67, Theorem 4.6.20]. □

Definition 2.12. We say that a singular point x as in Proposition 2.11 is a singular point of type $D_{n,a}$.

Remark 2.13.

(1) A singular point of type $D_{n,a}$ is a Rational Double Point if and only if $a = n-1$, in which case we have a Rational Double Point of type $D_{\frac{n}{2}+2}$ (if $n = 2$, this is most commonly known as A_3).
(2) A singular point of type $D_{n,a}$ is a cyclic quotient singularity if and only if $a = 1$. More precisely singularities of type $D_{n,1}$ are isomorphic to singularities of type $C_{2n,n+1}$. We will distinguish between them, to keep track of the branching locus of π. So we will consider the singularity as of Dynkin diagram $D_{n,1}$ if it belongs to the branch locus of π, else of type $C_{2n,n+1}$.
(3) We noted that a point of type $C_{n,a}$ is also a point of type $C_{n,a'}$ with $a' = a^{-1}$ in $\mathbb{Z}_n$. We consider these different representations as equal and usually we do not distinguish between them.

In the following the term *multiset* will be used in the sense of MAGMA [MAG]. So a multiset is a set whose elements have a *multiplicity*, a positive integer.

Definition 2.14. Let Y be an unmixed surface. Then we define the *basket of singularities of* Y to be the multiset

$$\mathcal{B}(Y) := \left\{\lambda \times C_{n,a} \,:\, Y \text{ has exactly } \lambda \text{ singularities of type } C_{n,a}\right\}.$$

Let $X = (C \times C)/G$ be a mixed q.e. quotient. We define the following two multisets:

$$\mathcal{B}_C := \Big\{\eta \times C_{n,a} \,:\, X \text{ has exactly } \eta \text{ singularities of type } C_{n,a} \\ \text{not in the branch locus of } \pi \Big\}.$$

$$\mathcal{B}_D := \Big\{\zeta \times D_{m,b} \,:\, X \text{ has exactly } \zeta \text{ singularities of type } D_{m,b} \\ \text{in the branch locus of } \pi \Big\}.$$

Definition 2.15. The *basket of singularities of* X is the multiset

$$\mathcal{B}(X) = \mathcal{B}_C \cup \mathcal{B}_D\,.$$

The following is an useful constraint on the basket of singularities.

Proposition 2.16. *Let $X = (C \times C)/G$ be a mixed q.e. quotient. Let $\mathcal{B}(X) = \mathcal{B}_C \cup \mathcal{B}_D$ be the basket of singularities of X with $\mathcal{B}_C := \{\eta_i \times C_{n_i,a_i}\}_i$ and $\mathcal{B}_D := \{\zeta_j \times D_{m_j,b_j}\}_j$. Then*

$$\sum_i \eta_i \frac{a_i + a_i'}{n_i} + \sum_j \zeta_j \frac{b_j}{m_j} \in \mathbb{Z}\,.$$

Proof. If $x \in X$ is a singular point of type $C_{n,a}$, then by Lemma 2.9 $\pi^{-1}(x)$ is given by two singular points, one of type $C_{n,a}$ with respect to α_1 and the other of type $C_{n,a'}$ with respect to α_1. If $x \in X$ is a singular point of type $D_{m,b}$, then $\pi^{-1}(x)$ is given by a unique singular point of type $C_{m,b}$ with respect to α_1. The result follows now directly from [Pol10, Proposition 2.8]. □

Definition 2.17. Let x be a singular point of type $C_{n,a}$ with $\frac{n}{a} := [b_1, \dots, b_l]$. We define the following nonnegative rational numbers

i) $k_x = k(C_{n,a}) := -2 + \dfrac{2+a+a'}{n} + \sum_{i=1}^{l}(b_i - 2)$;

ii) $e_x = e(C_{n,a}) := l + 1 - \dfrac{1}{n} \geq 0$;

Let x be a singular point of type $D_{n,a}$ with $\frac{n}{a} := [b_1, \dots, b_m, 2b, b_m, \dots, b_1]$. We define the analogous nonnegative rational numbers

i) $k_x = k(D_{n,a}) := \frac{k(C_{n,a})}{2} = -2 + \frac{a+1}{n} + \sum_{i=1}^{m}(b_i - 2) + b$;

ii) $e_x = e(D_{n,a}) := \frac{e(C_{n,a})}{2} + 3 = m + 4 - \dfrac{1}{2n}$;

In both cases we set $B_x := 2e_x + k_x$. Note that $B(D_{n,a}) = \frac{B(C_{n,a})}{2} + 6$. Let $\mathcal{B}$ be a basket of singularities. We use the following notation:

$$k(\mathcal{B}) = \sum_{x \in \mathcal{B}} k_x, \quad e(\mathcal{B}) = \sum_{x \in \mathcal{B}} e_x, \quad B(\mathcal{B}) = \sum_{x \in \mathcal{B}} B_x\,.$$

These correction terms determine the invariants of S as follows:

Proposition 2.18. *Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface, and let $\mathcal{B}$ be the basket of singularities of X. Then*

$$K_S^2 = \frac{8(g-1)^2}{|G|} - k(\mathcal{B})\,; \tag{2.1}$$

$$e(S) = \frac{4(g-1)^2}{|G|} + e(\mathcal{B})\,. \tag{2.2}$$

Proof. Since the quotient map $C \times C \to X$ is quasi-étale, we get

$$K_X^2 = \frac{K_Y^2}{2} = \frac{8(g-1)^2}{|G|}\,.$$

Let $\mathcal{B} = \mathcal{B}_C \cup \mathcal{B}_D = \{\eta_i \times C_{n_i,a_i}\}_i \cup \{\zeta_j \times D_{n_j,a_j}\}_j$, then the basket of singularities of Y is $\mathcal{B}(Y) = \{2\eta_i \times C_{n_i,a_i}\}_i \cup \{\zeta_j \times C_{n_j,a_j}\}_j$, hence by definition $k(\mathcal{B}(Y)) = 2k(\mathcal{B})$. By [BCGP12, Proposition 2.6], we get

$$K_T^2 = \frac{8(g-1)^2}{|G^0|} - k(\mathcal{B}(Y))\,.$$

Let $\epsilon\colon T' \to T$ be the blow-up of T in the $2d$ ($d = |\mathcal{B}_D|$) points fixed by μ:

$$K_{T'}^2 = K_T^2 - 2d = K_Y^2 - k(\mathcal{B}(Y)) - 2d = 2(K_X^2 - k(\mathcal{B}) - d)\,. \tag{2.3}$$

By the proof of Proposition 2.10 we have a double cover $\tilde{\pi}\colon T' \to S$ branched over $F := F_1 + \ldots + F_{2d}$, where the F_i are smooth rational curves with

$F_i^2 = -2$ and and $F_i.F_j = 0$ if $i \neq j$. Then numerically ([CD89, pages 13-14]) $K_{T'}$ equals $\tilde{\pi}^*(K_S + F/2)$ and, since $K_S.F = 0$, it follows:

$$K_{T'}^2 = 2\left(K_S + \frac{F}{2}\right)^2 = 2\left(K_S^2 + \frac{-4d}{4}\right) = 2(K_S^2 - d)\,. \tag{2.4}$$

From equations (2.3) and (2.4), we get:

$$K_S^2 = K_X^2 - k(\mathcal{B}) = \frac{8(g-1)^2}{|G|} - k(\mathcal{B})\,.$$

Let $X^0 := X \setminus \mathrm{Sing}(X)$ be the smooth locus of X; arguing as in [BCGP12] we get:

$$e(S) = e(X^0) + \sum_{x \in \mathcal{B}_C} (l_x + 1) + \sum_{x \in \mathcal{B}_D} (m_x + 4)$$

and

$$e(X^0) = \frac{e(C \times C)}{|G|} - \sum_{x \in \mathcal{B}_C} \frac{1}{n_x} - \sum_{x \in \mathcal{B}_D} \frac{1}{2n_x}$$

It follows that

$$e(S) = \frac{4(g-1)^2}{|G|} + e(\mathcal{B})$$

□

Using Noether's formula and Proposition 2.18 we get:

Corollary 2.19. *Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface, and let $\mathcal{B}$ be the basket of singularities of X. Then*

$$K_S^2 = 8\chi(S) - \frac{1}{3}B(\mathcal{B})\,.$$

We conclude this section by showing a very strong restriction on the cardinality of $\mathcal{B}_D$.

Corollary 2.20. *Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface. The cardinality d of $\mathcal{B}_D$ is even and*

$$\frac{d}{2} \leq p_g(S) + 1\,.$$

Corollary 2.20 follows from the next proposition since the singular points of X of type $D_{n,a}$ are the branch points of π.

Proposition 2.21. *Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface and let $\lambda\colon T \to Y$ be the minimal resolution of the singularities of Y. Let d be the number of fixed points for ι, then*

$$p_g(S) \leq p_g(T) = 2p_g(S) + 1 - \frac{d}{2}\,.$$

Proof. Let $\epsilon\colon T' \to T$ be the blow-up of T in the $2d$ points fixed by μ; we have a double cover $\tilde{\pi}\colon T' \to S$ branched along $2d$ smooth pairwise disjoint rational curve. Pulling back the forms on S to forms on T' we note that

$p_g(S) \leq p_g(T') = p_g(T)$. Moreover $e(T') = 2e(S) - 4d$, $e(T) = e(T') - 2d = 2e(S) - 6d$ and $K_T^2 = 2K_S^2$, by (2.3) and (2.4). By Noether's formula:

$$\chi(\mathcal{O}_T) = \frac{1}{12}(K_T^2 + e(T)) = \frac{1}{12}(2K_S^2 + 2e(S) - 6d) = 2\chi(\mathcal{O}_S) - \frac{d}{2}$$

Since $T \to Y$ is a product-quotient surface, $q(T) = 2g(C/G^0) = 2q(S)$ and

$$p_g(T) = 2 + 2p_g(S) - 2q(S) - \frac{d}{2} + q(T) - 1 = 2p_g(S) + 1 - \frac{d}{2}.$$

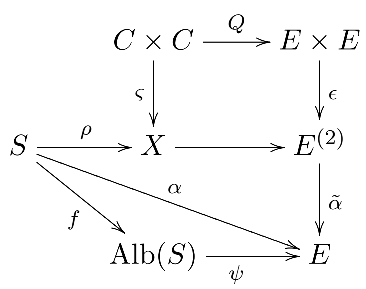
□

3. The Albanese fibre of a mixed q.e. surface with irregularity 1

The Albanese map of a surface of general type S with irregularity 1 is a fibration onto the elliptic curve $\mathrm{Alb}(S)$. The genus g_{alb} of the general Albanese fibre is a deformation invariant, very important from the point of view of the geography of the surfaces of general type. In this section we show how to compute it for mixed q.e. surfaces.

Let $S \stackrel{\rho}{\to} X = (C \times C)/G$ be a mixed q.e. surface with $q(S) = 1$. By Lemma 1.4, $C' = C/G^0$ is an elliptic curve; in this section we will set $E := C'$, to remind that it is elliptic. We have the following commutative diagram:

(3.1)



where $\tilde{\alpha}$ is the Abel-Jacobi map. By the properties of the Albanese torus (see [BHPV04, Proposition I.13.9]), the Stein factorization of α is given by the Albanese map $f\colon S \to \mathrm{Alb}(S)$ and a (unique) homomorphism $\psi\colon \mathrm{Alb}(S) \to E$.

The Galois cover $c\colon C \to E$ has branching set $B := \{p_1, \dots, p_r\}$; up to translation we may assume that the neutral element 0 of E is not in B, and that $-p_i \notin B$ for each $i \in \{1, \dots, r\}$.

Let $E' := \epsilon^*(\tilde{\alpha}^*(0)) = \{(x, -x) \mid x \in E\} \cong E$, consider $F^* := Q^*(E')$ and let $F := \alpha^*(0)$. Note that $\rho(F) = \varsigma(F^*)$. Our assumption $-p_i \notin B$ ensures that F^* and F are smooth, and the arithmetic genus of F can be easily computed by Hurwitz Formula (more precisely it equals the genus of C, see the proof of Proposition 3.2). F is disjoint union of $\deg \psi$ fibres of the Albanese map, so to compute g_{alb} we need to compute $\deg \psi$ first.

We will need the points $q_i := (p_i, -p_i)$ and $q_i' := (-p_i, p_i)$ of E'; we set $B' := \{q_i, q_i'\}_i$. We note that $0' = (0, 0) \in E' \setminus B'$.
By Remark 1.6, chosen suitable loops $\alpha, \beta, \gamma_1, \dots, \gamma_r \in \pi_1(E \setminus B, 0)$, the cover $c\colon C \to E$ is determined by a generating vector $(a, b; h_1, \dots, h_r)$ of G^0, representing the monodromy map $\mu\colon \pi_1(E \setminus B, 0) \to G^0$ of c.

Since $Q = c \times c$, the monodromy map of the $G^0 \times G^0$-cover Q is given by two copies of μ. Q induces by restriction the $G^0 \times G^0$-cover $F^* \to E'$, whose branching locus is B'. To describe its monodromy map we choose generators $\delta, \theta, \gamma_1', \dots, \gamma_r', \gamma_1'', \dots, \gamma_r'' \in \pi_1(E' \setminus B', 0')$ as follows:

- $\delta = (\alpha, -\alpha)$
- $\theta = (\beta, -\beta)$
- $\gamma_i' = (\gamma_i, -\gamma_i)$ are geometric loops around q_i
- $\gamma_i'' = (-\gamma_i, \gamma_i)$ are geometric loops around q_i'

Please note that we need some care in the choice of the loops α, β, γ_i to ensure that $\delta, \theta, \gamma_i'$ and γ_i'' does not meet B'.

Moreover, the class of them in $\pi_1(E' \setminus B', 0')$ depends on the choice of the loops α, β, γ_i and not only on their class in $\pi_1(E \setminus B, 0)$. Anyway, the classes of $\delta, \theta, \gamma_i'$ and γ_i'' generate $\pi_1(E' \setminus B', 0')$ and the monodromy map of $Q_{|F^*} \colon F^* \to E'$ is the unique homomorphism $\pi_1(E' \setminus B', 0') \xrightarrow{\mu'} G^0 \times G^0$ such that

$$\begin{array}{rcl} \delta & \stackrel{\mu'}{\longmapsto} & (a, a^{-1}) \\ \theta & \stackrel{\mu'}{\longmapsto} & (b, b^{-1}) \\ \gamma_i & \stackrel{\mu'}{\longmapsto} & (h_i, 1) \\ \gamma_i' & \stackrel{\mu'}{\longmapsto} & (1, h_i) \end{array} \tag{3.2}$$

We define

$$M := \left| \bigcup_{g \in G^0} g \mathrm{Im} \mu' \right| .$$

Lemma 3.1. *Let S be a mixed q.e. surface with $q(S) = 1$. Then* $\deg \psi = \dfrac{|G^0|^2}{M}$.

Proof. Let $u \in E'$. The action of $G^0 \times G^0$ on $Q^{-1}(u)$ induces a bijection among $Q^{-1}(u)$ and $G^0 \times G^0$; two points of $Q^{-1}(u)$ belong to the same connected component of F^* if and only if the corresponding elements in $G^0 \times G^0$ differ by an element in $\mathrm{Im}(\mu')$.
Moreover, two points h, $h' \in F^*$, map to the same point of X if and only if there exists $g \in G$ such that $g(h') = h$. So exactly M points of $Q^{-1}(u)$ are mapped in each connected component of $\varsigma(F^*)$. We conclude since $\deg \psi$ equals the number of connected components of F. □

Proposition 3.2. *Let S be a mixed q.e. surface with $q(S) = 1$, then*

$$g_{alb} = 1 + \frac{g(C) - 1}{|G^0|^2} M \, .$$

Proof. Let us look at diagram (3.1). Since G^0 is $(1; m_1, \dots, m_r)$-generated, then $e(C) = -|G^0| \sum_{i=1}^{r} \left(\dfrac{m_i - 1}{m_i} \right)$. The $(G^0 \times G^0)$-cover Q is branched exactly along the union of r "horizontal" copies of E and r "vertical" copies of E; moreover for each i there are one horizontal copy and one vertical copy with branching index m_i. Since $\epsilon^*(\Gamma)$ is an elliptic curve that intersects all these copies of E transversally in one point, by the Hurwitz formula applied to $F^* \to \epsilon^*(\Gamma)$ we obtain

$$e(F^*) = -|G^0|^2 \sum_{i=1}^{r} 2\left(\frac{m_i - 1}{m_i}\right).$$

On the other hand, the G-cover ς is q.e. and we get

$$e(F) = \frac{e(F^*)}{|G|} = -|G^0| \sum_{i=1}^{r} \left(\frac{m_i - 1}{m_i}\right) = e(C).$$

By Lemma 3.1, F is the disjoint union of $\deg \psi = \frac{|G^0|^2}{M}$ curves of genus g_{alb}, and therefore

$$2 - 2g(C) = e(F) = \frac{|G^0|^2}{M}(2 - 2g_{alb}).$$

□

4. The minimal model

In this section we want to determine the minimal model of the surfaces we construct. We start by recalling some useful results:

Lemma 4.1 ([Bom73, Proposition 1])**.** *On a smooth surface S of general type every irreducible curve C satisfies $K_S.C \geq -1$.*

Lemma 4.2 ([BP12, Remark 4.3])**.** *On a smooth surface S of general type every irreducible curve C with $K_S.C \leq 0$ is smooth and rational.*

Proposition 4.3. *Let S be a smooth surface of general type. If E, C are distinct smooth rational curves with $E^2 = -1$, $C^2 \geq -4$, then $C.E \leq 1$.*

Proof. Assume by contradiction $C.E \geq 2$. Let $b\colon S \to S'$ be the blow-down given by the contraction of E and set $C' := b(C)$. By the assumption $C.E \geq 2$, C' is singular, so by Lemma 4.2:

$$\begin{aligned} 0 < K_{S'}.C' &= (K_S - E).(C + (C.E)E) \\ &= (K_S - E).C \\ &= -C^2 - 2 - E.C \leq 0, \end{aligned}$$

a contradiction. □

Corollary 4.4. *Let S be a smooth surface of general type. Assume that E is a (-1)-curve in S, then E intersects at most one (-2) curve.*

Proof. Suppose E intersects two (-2) curves. By Proposition 4.3 it intersects them transversally in a point. Then contracting E we get two (-1) curves intersecting in a point, which is not possible on a surface of general type. □

The following is the main result of this section, showing that in the irregular case, the surfaces obtained are automatically minimal.

Theorem 4.5. *Let S be an irregular mixed q.e. surface of general type, then S is minimal.*

Proof. Aiming for a contradiction, let E be a (-1)-curve on S.

Consider the intermediate quotient $Y = (C \times C)/G^0$, the minimal resolution of its singularities $\lambda\colon T \to Y$ and the involution μ on T (Lemma 2.5). Let $\epsilon\colon T' \to T$ be the blow up of the fixed points of μ. By Proposition 2.10 and its proof, there is a map $\tilde{\pi}\colon T' \to S$ which is a double cover ramified along the exceptional divisors of ϵ, so branched along a disjoint union of $(-2)-$curves.
Since E can intersect at most one (-2)-curve then $\tilde{\pi}^*(E)$ is union of two rational curves; let R be one of them. By construction R is not exceptional for the resolution $T' \to Y$, and therefore one of the fibrations $\alpha_i\colon Y \to C'$ is a surjective map from a rational curve to C', contradicting $g(C') = q(S) > 0$. □

In the case $q = 0$ we borrow an argument of [BP12]. Let $\Gamma \subset X = (C \times C)/G$ be a rational curve. Let $\Gamma' := (\pi \circ \sigma)^*(\Gamma) = \sum_1^k n_i \Gamma_i$ be the decomposition in irreducible components of its pull back to $C \times C$. We observe that, since $\pi \circ \sigma$ is quasi-étale, $\forall i\ n_i = 1$ and that G acts transitively on the set $\{\Gamma_i \mid i = 1, \dots, k\}$. Hence there is a subgroup $H \triangleleft G$ of index k acting on Γ_1 such that $\pi(\sigma(\Gamma_1)) = \Gamma_1/H = \Gamma$.
Normalizing Γ_1 and Γ, we get the following commutative diagram:

$$\begin{array}{ccccc} \tilde{\Gamma}_1 & \xrightarrow{\alpha} & \Gamma_1 & \xrightarrow{\beta} & C \times C \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\nu} & \Gamma & \hookrightarrow & X \end{array}$$

Since each automorphism lifts to the normalization, H acts on $\tilde{\Gamma}_1$ and f is the quotient map $\tilde{\Gamma}_1 \to \tilde{\Gamma}_1/H \cong \mathbb{P}^1$. Moreover $\beta(\alpha(\Gamma_1))$ is a curve in $C \times C$, and therefore surjects on C, hence $g(\Gamma_1) \geq g(C) \geq 2$ and so f is branched in at least 3 points.

Lemma 4.6. *Let p be a branch point of f, then $\nu(p)$ is a singular point of X.*

Proof. Let $p' \in f^{-1}(p) \subset \tilde{\Gamma}_1$ be a ramification point of f, then $\mathrm{Stab}_H(p') := H_1 \neq \{1\}$ and so $\mathrm{Stab}_G(\beta(\alpha(p'))) \supseteq H_1$. Hence $\nu(f(p')) = \nu(p) \in \mathrm{Sing}(X)$. □

Corollary 4.7. *Any rational curve in X passes at least 3 times through singular points.*

We will need the following consequence of Proposition 4.3.

Corollary 4.8. *Let S be a smooth surface of general type. Assume that E is a (-1)-curve in S, then E cannot intersect three distinct smooth rational curves with self-intersection -2 or -3.*

Proof. By Proposition 4.3 E intersect each of the three curves transversally in a point.

Contracting E we get three smooth rational curves with self-intersection -1 or -2 with a common point. If one of them has self-intersection -1, by Corollary 4.4 also a second one has self-intersection -1, and we find two intersecting $(-1)-$curves, which is impossible on a surface of general type. So all have self-intersection -2.

We go to the minimal model of S by contracting all possible $(-1)-$curves. If one of the contracted curves would intersect one of our three (-2)-curves, we get the same contradiction as above. So the image of our configuration gives three smooth rational curves with self-intersection -2 on a minimal surface of general type with a common point. This is impossible (see *e.g.*, [Bom73, Proposition 2]). □

Proposition 4.3 and Corollary 4.8 imply that, if the basket of singularities of X is simple enough, then S is minimal. More precisely

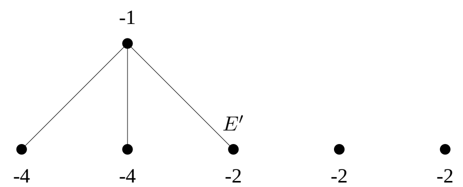
Proposition 4.9. *Let $S \to X$ be a mixed q.e. surface of general type. Assume one of the following*

i) *either all exceptional curves for $S \to X$ have self-intersection -2 or -3*
ii) *or $\mathcal{B}(X) = \{2 \times C_{4,1}, 3 \times C_{2,1}\}$*

Then S is minimal.

Proof. i) In this case, if S were not minimal, by Corollary 4.7 and Proposition 4.3 then there would be a (-1)-curve E which intersects three different smooth rational curves with self-intersection -2 or -3, contradicting Corollary 4.8.

ii) In this case the exceptional divisor is given by five rational curves which do not intersect each other, two of self-intersection -4 and three of self-intersection -2. If S were not minimal, by Corollary 4.7, Proposition 4.3 and Corollary 4.4 the dual graph of the resulting configuration of rational curves is:



After the contraction of the (-1)-curve we can also contract E', finding a surface of general type with two (-2) curves which are tangent in a point. Contracting all possible further (-1)-curves we find a contradiction as in the end of the proof of Corollary 4.8 . □

Proposition 4.9 is obviously not sharp: we can prove the same result for many different baskets of singularities exactly by the same argument. We decided to state it in this weak form for sake of simplicity, since *a posteriori* (inspecting the output of the program we describe in the next section) cases i) and ii) are the only cases that occur for mixed q.e. surfaces of general type with $p_g = 0$ and $K^2 > 0$.

5. The classification

We wrote a MAGMA script which computes all mixed q.e. surfaces with fixed (input of the script) p_g, q and K^2. To write the algorithm we needed to overcome some theoretical problems, namely to find explicit bounds for the basket of singularities $\mathcal{B}$ and for the signatures $(q; m_1, \ldots, m_r)$.

For the basket of singularities, since by Corollary 2.19 $B(\mathcal{B}) = 24(1-q+p_g)-3K^2$, it is enough to prove that there are finitely many possible baskets with fixed invariant $B(\mathcal{B})$, and show how to produce the whole list:

Lemma 5.1. *Let $B_0 \in \mathbb{Q}$. Then there are finitely many baskets $\mathcal{B}$ such that*

$$B(\mathcal{B}) = B_0\,.$$

More precisely $|\mathcal{B}| \leq B_0/3$. Moreover, if $n/a = [b_1, \dots, b_l]$ then

(1) $B(C_{n,a}) \geq \sum b_i$;
(2) $B(D_{n,a}) \geq 6 + \frac{1}{2}\sum b_i$.

Proof. We note that $B(C_{n,a}) = \frac{a+a'}{n} + \sum b_i \geq 3$, while $B(D_{n,a}) = \frac{B(C_{n,a})}{2} + 6 \geq 15/2 > 3$: this prove $|\mathcal{B}| \leq B_0/3$, bounding from above the number of singular points .
(1) and (2) are trivial consequences of the formulas; they show that there are only finitely many possible $[b_1, \dots, b_l]$, so finitely many pairs (n, a). □

The second problem is to bound the possible signatures, once we have fixed K^2, p_g, q and the basket $\mathcal{B}$. We have to find upper bounds for r and for the m_i.

Definition 5.2. Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface. Let $(q; m_1, \dots, m_r)$ be the signature of the induced generating vector for G^0. Let $\mathcal{B}$ be the basket of singularities of X. Then we define the following numbers:

$$\Theta := 2q(S) - 2 + \sum_{i=1}^{r}\left(\frac{m_i - 1}{m_i}\right),$$

$$\beta := \frac{12\chi(\mathcal{O}_S) + k(\mathcal{B}) - e(\mathcal{B})}{3\Theta},$$

Definition 5.3 (see [Rei87])**.** The minimal positive integer I_x such that I_xK_X is Cartier in a neighborhood of $x \in X$ is called the *index of the singularity x*. The *index* of a normal variety X is the minimal positive integer I such that IK_X is Cartier. In particular, $I = \mathrm{lcm}_{x\in \mathrm{Sing}(X)} I_x$ depends only on the basket of singularities.

The index of a singularity of type $C_{n,a}$ is

$$I_x = \frac{n}{\gcd(n, a+1)}\,.$$

We can now give the bounds we need.

Proposition 5.4. *Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface. Let $(q; m_1, \dots, m_r)$ be the signature of the induced generating vector for G^0. Let $\mathcal{B} = \mathcal{B}_C \cup \mathcal{B}_D$ be the basket of singularities of X. Then*

a) $\Theta > 0$ *and* $\beta = g(C) - 1$;
b) $r \leq 2\Theta + 4(1-q)$;
c) $m_i \leq 4\beta + 6$;
d) *each* m_i *divides* $2\beta I$ *where* I *is the index of* Y;
e) $m_i \leq \dfrac{2I\beta\Theta + 1}{M}$, *with* $M := \max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\}$;
f) *except at most* $|\mathcal{B}_C| + |\mathcal{B}_D|/2$ *indices* i, $m_i \leq \dfrac{\beta\Theta + 1}{M}$ *and divides* β.

Proof. a) Since $q(S) = g(C/G^0)$, by Hurwitz's formula:

$$2(g(C)-1) = |G^0| \cdot \Theta\,,$$

hence $\Theta = \dfrac{2(g(C)-1)}{|G^0|} > 0$, since $g(C) \geq 2$. Let $k := k(\mathcal{B})$ and $B := B(\mathcal{B})$. By Corollary 2.19 and Proposition 2.18 we get

$$\beta = \frac{24\chi + 3k - B}{6\Theta} = \frac{K_S^2 + k}{2\Theta} = \frac{8(g(C)-1)^2}{4\Theta|G^0|} = g(C) - 1\,,$$

b) By definition $\Theta \geq 2q - 2 + \frac{r}{2}$, hence $r \leq 2\Theta - 4(q-1)$.

c) Since $m_i = \mathrm{ord}(h_i)$ and h_i is an automorphism of a curve of genus $g \geq 2$, by Wiman's Theorem (see [Wim95]) $m_i \leq 4g + 2 = 4\beta + 6$.

d) Since $|\mathcal{B}(Y)| = 2|\mathcal{B}_C| + |\mathcal{B}_D|$, the claim follows by [BP12, Proposition 1.14, d].

e) We first show

$$\Theta + \frac{1}{m_i} \geq \max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\}.$$

Since $\Theta = 2q - 2 + r - \sum_{j=1}^{r} \dfrac{1}{m_j}$, we get

$$\Theta + \frac{1}{m_i} = 2q - 2 + r - \sum_{j \neq i} \frac{1}{m_j} \geq 2q - 2 + r - \frac{r-1}{2} = \frac{r-3+4q}{2}\,.$$

Since $\Theta > 0$, $\frac{r-3+4q}{2} \geq \frac{1}{6}$ unless $r = 3$ and $q = 0$. In this case, $\Theta > 0$ implies that at most one m_i can be equal to 2. Hence also in this case $\Theta + \frac{1}{m_i} \geq 0 - 2 + 3 - \sum_{j \neq i} \frac{1}{m_j} \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

By d), we get

$$\left(\max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\}\right) m_i \leq 1 + \Theta \cdot m_i \leq 1 + 2I\beta\Theta.$$

f) By [BP12, Proposition 1.14, e], except for at most $|\mathcal{B}(Y)|/2 = |\mathcal{B}_C| + |\mathcal{B}_D|/2$ indices, m_i divides β. From $m_i \leq \beta$ follows

$$\left(\max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\}\right) m_i \leq 1 + \Theta m_i \leq 1 + \Theta\beta\,.$$

□

We used the inequalities proved in this section to produce an algorithm to compute all mixed q.e. surfaces with fixed p_g, q and K^2, following the same strategy of the algorithm of [BP12] which computed the product-quotient surfaces with $p_g = q = 0$ (input was just K^2). The algorithm uses also the following simple remarks:

Remark 5.5. By Hurwitz formula $|G| = 2|G^0| = \dfrac{4(g(C)-1)}{\Theta} = \dfrac{4\beta}{\Theta}$.

Remark 5.6. Let $\rho\colon S \to X = (C \times C)/G$ be a mixed q.e. surface. Let $(q; m_1, \ldots, m_r)$ be the signature of the induced generating vector for G^0. If X has a singular point of type $C_{n,a}$ or $D_{n,a}$, then there exists m_i such that n divides m_i.

Indeed, the singular point is the class of a point $(x, y) \in C \times C$ such that $\mathrm{Stab}_{G^0}(x, y) = \langle \eta \rangle$ with $o(\eta) = n$. x is a ramification point of $c\colon C \to C/G^0$, and its ramification index, that equals $|\mathrm{Stab}_{G^0}(x)|$, is one of the m_i. So by $\eta \in \mathrm{Stab}_{G^0}(x)$ follows that n divides m_i.

We explain here very briefly the strategy of the algorithm.

Fixed the values of K_S^2, $p_g(S)$ and $q(S)$, by Corollary 2.19 we know $B(\mathcal{B})$, and Lemma 5.1 gives easily a procedure to produce the finite list of baskets with that invariant B. Then, for each basket, we produce the finite list of all signatures $(q; m_1, \dots, m_r)$ respecting all conditions in Proposition 5.4, including the requirement that β be an integer.

Now, for each basket and for each associated signature, the orders of G and G^0 are computed by Remark 5.5. Then the script checks all the finitely many groups G^0 of that order, and their unsplit degree 2 extensions G.

Then we have a list of quintuples (basket, signature, G^0, generating vector, extension), each quintuple gives a family of mixed q.e. surfaces (just determined by (G^0, generating vector, extension) as explained in Remark 1.6), and all mixed q.e. surfaces with the prescribed invariants are here. Anyway, in this list there are also surfaces with different invariants: these are those whose singularities does not correspond to the basket. Then the scripts computes these singularities in each case, using the results of section 2, (in particular Propositions 2.1 and 2.3), and it discards the surfaces with wrong basket.

Moreover, different generating vectors give isomorphic surfaces if they differs by some *Hurwitz moves*, which are described, in the cases we need, in [Pen11, Section 5]. The scripts computes this action on the remaining generating vectors, and returns only a representative for each orbit. Finally, the script computes, using a result by Armstrong ([Arm65], [Arm68]), the fundamental groups (see [Fra11]) of the resulting surfaces.

Our code skips some signatures giving rise to groups of large order, either not covered by the MAGMA SmallGroup database, or causing extreme computational complexity. The program returns the list of the skipped cases, which have to be studied separately.

A commented version of the full program can be downloaded from:

`http://www.science.unitn.it/~pignatel/papers/Mixed.magma`

Using it, we proved Theorems A, B and C as follows.

Sketch of the proof of Theorems A, B and C. By Corollary 2.19 every mixed q.e. surface has $K^2 \leq 8\chi$; so the possible invariants of a minimal surface of general type with $\chi = 1$ are $K_S^2 = 1, 2, 3, 4, 5, 6, 7, 8$ and, by Beauville's inequality [Bea82] $p_g \geq 2q - 4$, $p_g = q = \{0, 1, 2, 3, 4\}$. We runned our program for all these values; it returned the surfaces in the tables 1, 2 and 3.

As mentioned, the surfaces returned by the program may be not all mixed q.e. surfaces with the required invariants, since the program is forced to skip some signatures, giving rise to groups of large order. The program returns the list of these "skipped" case.

For the cases $p_g = q \neq 0$, this list is empty, so the tables 2 and 3 are complete. We report the list of the "skipped" signatures for $p_g = q = 0$

K_S^2	Sing$X$	Sign.	$\|G^0\|$	K_S^2	Sing$X$	Sign.	$\|G^0\|$
1	$2 \times C_{8,1}, C_{4,1}$	2,3,8	6336	4	$C_{4,3}, C_{4,1}$	2,3,8	2880
1	$3 \times C_{4,1}, C_{4,3}$	2,3,8	2304	4	$4 \times C_{2,1}$	2,3,8	2304
1	$C_{8,1}, C_{4,1}, C_{8,5}$	2,3,8	4032	4	$C_{3,1}, C_{3,2}, C_{2,1}$	2,3,8	2496
1	$4 \times C_{4,1}, C_{2,1}$	2,3,8	2880	4	$2 \times C_{4,1}, C_{2,1}$	2,4,5	2400
1	$2 \times C_{8,3}, C_{4,1}, C_{2,1}$	2,3,8	2304	4	$2 \times C_{4,1}, C_{2,1}$	2,3,8	3456
1	$2 \times C_{2,1}, C_{8,3}, C_{8,1}$	2,3,8	3744	5	$C_{5,2}, C_{2,1}$	2,4,5	2160
2	$2 \times C_{8,3}, C_{4,1}$	2,3,8	2880	5	$3 \times C_{2,1}$	2,3,8	2880
2	$C_{8,3}, C_{8,1}, C_{2,1}$	2,3,8	4320	5	$C_{3,1}, C_{3,2}$	2,3,8	3072
2	$4 \times C_{4,1}$	2,4,5	2400	5	$2 \times C_{4,1}$	2,4,5	2800
2	$4 \times C_{4,1}$	2,3,8	3456	5	$2 \times C_{4,1}$	2,3,8	4032
2	$C_{8,3}, C_{8,5}, C_{2,1}$	2,3,8	2016	6	$2 \times C_{2,1}$	2,4,5	2400
2	$2 \times C_{4,1}, 3 \times C_{2,1}$	2,3,8	2304	6	$2 \times C_{2,1}$	2,3,8	3456
2	$2 \times C_{4,1}, C_{3,1}, C_{3,2}$	2,3,8	2496	6	$2 \times C_{5,3}$	2,4,5	2560
3	$2 \times C_{4,1}, 2 \times C_{2,1}$	2,3,8	2880	7	$C_{2,1}$	2,3,9	2268
3	$C_{8,3}, C_{8,1}$	2,3,8	4896	7	$C_{2,1}$	2,4,5	2800
3	$2 \times C_{4,1}, C_{5,3}$	2,4,5	2160	7	$C_{2,1}$	2,3,8	4032
3	$C_{8,3}, C_{8,5}$	2,3,8	2592	8	$\emptyset$	2,3,9	2592
3	$C_{4,3}, C_{4,1}, C_{2,1}$	2,3,8	2304	8	$\emptyset$	2,4,5	3200
				8	$\emptyset$	2,3,8	4608

TABLE 4. The skipped cases for $p_g = q = 0$ and $K^2 > 0$

in Table 4. We proved that no one of these cases occur by arguments very similar to the analogous proofs in the papers [BCGP12, BP12, Fra11] and therefore we do not include them here. The interested reader will find the details in

`http://www.science.unitn.it/~pignatel/papers/skipped.pdf`

Now let us consider the surfaces in 1, 2 and 3. A surface with $K^2 > 0$ is either of general type or rational, therefore regular and simply connected: a quick inspection of the tables shows that this last case do not occur, so all constructed surfaces are of general type. By Theorem 4.5 and Proposition 4.9 all the constructed surfaces are minimal. Moreover, again by Proposition 4.5, since every minimal surface of general type has positive K^2, we have found all irregular mixed q.e. surfaces with $p_g = q$. □

References

[Arm65] M.A. Armstrong. On the fundamental group of an orbit space. Proc. Cambridge Phil. Soc., 61:639–646, 1965.

[Arm68] M.A. Armstrong. The fundamental group of the orbit space of a discontinuous group. Proc. Cambridge Phil. Soc., 64:299–301, 1968.

[BC04] I. Bauer and F. Catanese. Some new surfaces with $p_g = q = 0$. In Turin Univ. Torino, editor, The Fano Conference, pages 123–142, 2004.

[BCG08] I. Bauer, F. Catanese, and F. Grunewald. The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves. Pure Appl. Math. Q., 4(2):547–586, 2008.

[BCGP12] I. Bauer, F. Catanese, F. Grunewald, and R. Pignatelli. Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups. American Journal of Mathematics, 134(4):993–1049, 2012.

[BCP06] I. Bauer, F. Catanese, and R. Pignatelli. Complex Surfaces of General Type: Some Recent Progress. In Global aspects of complex geometry, pages 1–58. Springer, Berlin, 2006.

[BCP11] I. Bauer, F. Catanese, and R. Pignatelli. Surfaces of general type with geometric genus zero: A survey. In Complex and Differential Geometry, volume 8, pages 1–48. Springer Proceedings in Mathematics, 2011.

[Bea78] A. Beauville. Surfaces algébriques complexes. Asterisque **54**, Soc.Math. France, 1978.

[Bea82] A. Beauville. L' inégalité $p_g \geq 2q - 4$ pour les surfaces de type général. Bull. Soc. Math. France, 110(3):343–346, 1982. Appendix to [Deb82].

[BHPV04] A.P. Barth, K. Hulek, C.A.M. Peters, and A. Van de Ven. Compact Complex Surfaces, volume 4. Springer, Berlin, 2004. Ergebnisse der Mathematik und ihrer Grenzgebiete.

[Bom73] E.Bombieri. Canonical models of surfaces of general type. Inst. Hautes Études Sci. Publ. Math., 42:171–219, 1973.

[BP12] I. Bauer and R. Pignatelli. The classification of minimal product-quotient surfaces with $p_g = 0$. Mathematics of Computation, 81(280):2389–2418, 2012.

[Bri68] E. Brieskorn. Rationale singularitäten komplexer Flächen. Inventiones mathematicae, 4:336–358, 1968.

[Cat00] F. Catanese. Fibred surfaces, varieties isogenous to a product and related moduli spaces. American Journal of Mathematics, 122(1):1–44, 2000.

[Cat07] F. Catanese. Q.E.D. for algebraic varieties. J. Differential Geom., 77(1):43–75, 2007.

[Cat10] F. Catanese. On a class of surfaces of general type. In Algebraic surfaces, volume 76 of C.I.M.E. Summer Sch., pages 267–284. Springer, Heidelberg, 2010.

[CC91] F. Catanese and C. Ciliberto. Surfaces with $p_g = q = 1$. In Problems in the theory of surfaces and their classification (Cortona, 1988), Sympos. Math., XXXII, pages 49–79. Academic Press, 1991.

[CC93] F. Catanese and C. Ciliberto. Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$. J. Algebraic Geom., 2(3):389–411, 1993.

[CCML98] F. Catanese, C. Ciliberto, and M Mendes Lopes. On the classification of irregular surfaces of general type with nonbirational bicanonical map. Trans. Amer. Math. Soc., 350(1):275–308, 1998.

[CD89] F. Cossec and I. Dolgachev. Enriques Surfaces I. Birkhäuser, 1989.

[CP06] F. Catanese and R. Pignatelli. Fibrations of low genus. I. Ann. Sci. École Norm. Sup. (4), 39(6):1011–1049, 2006.

[CP09] G. Carnovale and F. Polizzi. The classification of surfaces with $p_g = q = 1$ isogenous to a product of curves. Advances in Geometry, 9(2):233–256, 2009.

[Deb82] O. Debarre. Inégalités numériques pour les surfaces de type général. Bull. Soc. Math. France, 110(3):319–342, 1982. With an appendix by A. Beauville.

[Fra11] D. Frapporti. Mixed surfaces, new surfaces of general type with $p_g = 0$ and their fundamental group. arXiv:1105.1259, 2011.

[HP02] C. D. Hacon and R. Pardini. Surfaces with $p_g = q = 3$. Trans. Amer. Math. Soc., 354(7):2631–2638 (electronic), 2002.

[MAG] MAGMA Database of Small Groups. `http://magma.maths.usyd.edu.au/magma/htmlhelp/text404.htm`.

[Mat67] K. Matsuki. Introduction to the Mori program, volume 56 of Graduate Texts in Mathematics. Springer Verlag, 1967.

[MP10] E. Mistretta and F. Polizzi. Standard isotrivial fibrations with $p_g = q = 1$, II. J. Pure and Applied Algebra, 214(4):344–369, 2010.

[Pen11] M. Penegini. The classification of isotrivially fibred surfaces with $p_g = q = 2$. Collect. Math., 62(3):239–274, 2011. With an appendix by Sönke Rollenske.

[Pig09] R. Pignatelli. Some (big) irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 4$. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 20(3):207–226, 2009.

[Pir02] G. P. Pirola. Surfaces with $p_g = q = 3$. Manuscripta Math., 108(2):163–170, 2002.

[Pol08] F. Polizzi. On surfaces of general type with $p_g = q = 1$ isogenous to a product of curves. Comm. Algebra, 36(6):2023–2053, 2008.

[Pol09] F. Polizzi. Standard isotrivial fibrations with $p_g = q = 1$. J. Algebra, 321(6):1600–1631, 2009.

[Pol10] F. Polizzi. Numerical properties of isotrivial fibrations. Geom. Dedicata, 147:323–355, 2010.

[Rei87] M. Reid. Young person's guide to canonical singularities. In Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), volume 46 of Proc. Sympos. Pure Math., pages 345–414. Amer. Math. Soc., 1987.

[Rit07] C. Rito. On surfaces with $p_g = q = 1$ and non-ruled bicanonial involution. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 6(1):81–102, 2007.

[Rit10a] C. Rito. Involutions on surfaces with $p_g = q = 1$. Collect. Math., 61(1):81–106, 2010.

[Rit10b] C. Rito. On equations of double planes with $p_g = q = 1$. Math. Comp., 79(270):1091–1108, 2010.

[Ser96] F. Serrano. Isotrivial fibred surfaces. Ann. Mat. Pura Appl., 171(4):63–81, 1996.

[Wim95] A. Wiman. Über die hyperelliptischen Kurven und diejenigen vom Geschlechte $p = 3$, welche eindeutige Transformationen in sich zulassen. Bihang Kongl. Svenska Vetenkamps-Akademiens Handlingar, 21:1–23, 1895.

[Zuc03] F. Zucconi. Surfaces with $p_g = q = 2$ and an irrational pencil. Canad. J. Math., 55(3):649–672, 2003.

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