Appendix A. The minimal product-quotient surfaces of general type with $p_g = 0$ and $K^2 < 8$

In this section we describe all the minimal product-quotient surfaces we have listed in tables 1 and 2, with the exception of the one whose singular model X has at worse canonical singularities (these are already described in [BCG08] and [BCGP08]).

In the sequel we will follow the scheme below:

- G: here we define the group G (usually as permutation group);
- t_i: here we specify the respective signatures of the pair of spherical generators of the group G;
- S_1: here we list the first set of spherical generators;
- S_2: here we list the second set of spherical generators;
- H_1: the first homology group of the surface;
- π_1: the fundamental group of the surface;

A.1. $K^2 = 5$, basket $\{\frac{1}{3}(1,1) + \frac{1}{3}(1,2)\}$.

A.1.1. *Group* $\mathfrak{S}_4 \times \mathbb{Z}_2$:

- G: $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$;
- t_i: $(3, 2^4)$ and $(6, 4, 2)$;
- S_1: (134),(34)(56),(13)(24)(56),(23)(56), (13)(24)(56);
- S_2: (234)(56), (4321)(56), (14);
- H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_4$;
- π_1: the fundamental group of this surface fits in two exact sequences

$$1 \to \mathbb{Z}^2 \to \pi_1 \to D_{2,8,3} \to 1$$

$$1 \to \mathbb{Z}^2 \to \pi_1 \to Q(16) \to 1$$

where $Q(16)$ is the generalized quaternion group of order 16.

The normal subgroups of index 16 of π_1 on the left have minimal index among the normal subgroups of π_1 with free abelianization. Let us recall that $D_{2,8,3}$ is the group $< x, y | x^2, y^8, xyx^{-1}y^{-3} >$ and $Q(16)$ is the group $< x, y | x^8, x^4y^{-2}, yxy^{-1}x >$.

A.1.2. *Group* $\mathfrak{S}_4$.

- G: $\mathfrak{S}_4$;
- t_i: $(3, 2^4)$ and $(4^2, 3)$;
- S_1: $(124), (23), (24), (14), (13)$;
- S_2: $(1243), (1234), (123)$;
- H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_8$;
- π_1: the fundamental group of this surface fits in an exact sequence

$$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$$

and the normal subgroup of index 8 of π_1 on the left has minimal index among the normal subgroups of π_1 with free abelianization.

A.1.3. *Group* $\mathfrak{S}_4 \times \mathbb{Z}_2$*:*

G: $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$;
t_i: $(3, 2^3)$ and $(6, 4^2)$;
S_1: $(143), (12), (24)(56), (12)(34)(56)$;
S_2: $(134)(56), (1342)(56), (1234)$;
H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_8$;
π_1: the fundamental group fits in an exact sequence

$$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$$

and the normal subgroup of index 8 of π_1 on the left has minimal index among the normal subgroups of π_1 with free abelianization.

A.1.4. *Group* $\mathfrak{S}_5$*.*

G: $\mathfrak{S}_5$;
t_i: $(6, 5, 2)$ and $(4^2, 3)$;
S_1: $(13)(245), (14253), (34)$;
S_2: $(4321), (1534), (235)$;
H_1: $\mathbb{Z}_8$;
π_1: $D_{8,5,-1} =< x, y | x^8, y^5, xyx^{-1}y >$.

A.1.5. *Group* $\mathfrak{A}_5$*.*

G: $\mathfrak{A}_5$;
t_i: $(3, 2^3)$ and $(5^2, 3)$;
S_1: $(152), (14)(23), (23)(45), (14)(25)$;
S_2: $(15423), (13425), (254)$;
H_1: $\mathbb{Z}_2 \times \mathbb{Z}_{10}$;
π_1: $\mathbb{Z}_5 \times Q_8$, where Q_8 is the quaternion group $< x, y | x^4, x^2y^{-2}, xyx^{-1}y >$.

A.1.6. *Group* $\mathbb{Z}_2^4 \rtimes \mathfrak{S}_3$*:* this is the semidirect product obtained by letting (12) and (123) act on $\mathbb{Z}_2^4$ respectively as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

G: $< x_1, x_2, x_3, x_4, y_2, y_3 | x_i^2, y_i^i, [x_i, x_j], (y_2y_3)^2, y_2x_{2i-1}y_2x_{2i}, y_3^{-1}x_{2i-1}y_3x_{2i},$ $y_3^{-1}x_{2i}y_3x_{2i-1}x_{2i} >$;
t_i: $(3, 2^3)$ and $(4^2, 3)$;
S_1: $y_3x_1, y_2y_3^2x_3, x_1x_3x_4, y_2y_3x_4$;
S_2: $y_2x_1x_4, y_2y_3x_1, y_3^2x_1x_3$;
H_1: $\mathbb{Z}_2 \times \mathbb{Z}_8$;
π_1: the fundamental group of this surface fits in an exact sequences

$$1 \to H \to \pi_1 \to D_{8,4,3} \to 1$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = D_{8,4,3}$ but the computer could not solve the problem. Recall that $D_{8,4,3}$ is the group $< x, y | x^8, y^4, xyx^{-1}y^{-3} >$.

A.1.7. *Group* $\mathfrak{A}_5$.

G: $\mathfrak{A}_5$;
t_i: $(3, 2^3)$ and $(5^2, 3)$;
S_1: $(152), (14)(23), (23)(45), (14)(25)$;
S_2: $(14235), (15243), (123)$;
H_1: $\mathbb{Z}_2 \times \mathbb{Z}_{10}$;
π_1: $\mathbb{Z}_2 \times \mathbb{Z}_{10}$.

A.2. $K^2 = 4$, **basket** $\{2 \times \frac{1}{5}(1, 2)\}$.

A.2.1. *Group* $\mathfrak{A}_5$.

G: $\mathfrak{A}_5$;
t_i: $(5, 2^3)$ and $(5, 3^2)$;
S_1: $(13245), (12)(34), (15)(23), (14)(35)$;
S_2: $(13542), (123), (345)$;
H_1: $\mathbb{Z}_2 \times \mathbb{Z}_6$;
π_1: $\mathbb{Z}_2 \times \mathbb{Z}_6$.

A.2.2. *Group* $\mathbb{Z}_2^4 \rtimes D_5$: this is the semidirect product obtained by letting a symmetry and a rotation of D_5 act on $\mathbb{Z}_2^4$ respectively as $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

G:

$$< x_1, x_2, x_3, x_4, y_2, y_5 | \quad x_i^2, y_i^i, [x_i, x_j], (y_2y_5)^2, \\ y_2x_1y_2x_1x_2, y_2x_2y_2x_2, y_2x_3y_2x_1x_2x_4, y_2x_4y_2x_1x_3, \\ y_5^{-1}x_1y_5x_1x_2, y_5^{-1}x_2y_5x_2x_3, y_5^{-1}x_3y_5x_3x_4, y_5^{-1}x_4y_5x_1 >$$

t_i: $(5, 4^2)$ and $(5, 4, 2)$;
S_1: $y_5^2x_1, y_2y_5^2x_2x_4, y_2x_4$;
S_2: $y_5x_2x_3, y_2y_5x_1x_2x_3x_4, y_2x_1x_3x_4$;
H_1: $\mathbb{Z}_8$;
π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_8 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.2.3. *Group* $\mathbb{Z}_2^4 \rtimes D_5$*:*

G: as above;

t_i: $(5, 4^2)$ and $(5, 4, 2)$;

S_1: $y_5^3 x_1 x_4, y_2 x_3, y_2 y_5^2 x_2 x_4$;

S_2: $y_5^4 x_1 x_2 x_3, y_2 x_2 x_4, y_2 y_5$;

H_1: $\mathbb{Z}_8$;

π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_8 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.2.4. *Group* $\mathbb{Z}_2^4 \rtimes D_5$*:*

G: as above;

t_i: $(5, 4^2)$ and $(5, 4, 2)$;

S_1: $y_5^3 x_1 x_4, y_2 x_3, y_2 y_5^2 x_2 x_4$;

S_2: $y_5 x_2 x_3, y_2 y_5 x_1 x_2 x_3 x_4, y_2 x_1 x_3 x_4$;

H_1: $\mathbb{Z}_8$;

π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_8 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.2.5. *Group* $\mathfrak{A}_6$.

G: $\mathfrak{A}_6$;

t_i: $(5, 4, 2)$ and $(5, 3^2)$;

S_1: $(14623), (13)(2564), (12)(56)$;

S_2: $(14562), (134)(265), (243)$;

H_1: $\mathbb{Z}_6$;

π_1: $\mathbb{Z}_6$.

A.3. $K^2 = 3$, basket $\{\frac{1}{5}(1,1) + \frac{1}{5}(1,4)\}$.

A.3.1. *Group* $\mathfrak{A}_5$.

G: $\mathfrak{A}_5$;

t_i: $(5, 2^3)$ and $(5, 3^2)$;

S_1: $(14235), (23)(45), (13)(45), (14)(35)$;

S_2: $(13542), (123), (345)$;

H_1: $\mathbb{Z}_2 \times \mathbb{Z}_6$;

π_1: $\mathbb{Z}_2 \times \mathbb{Z}_6$.

A.3.2. *Group* $\mathbb{Z}_2^4 \rtimes D_5$*:* this is the semidirect product obtained by letting a symmetry and a rotation of D_5 act on $\mathbb{Z}_2^4$ respectively as $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

G:

$$< x_1, x_2, x_3, x_4, y_2, y_5 | \quad x_i^2, y_i^i, [x_i, x_j], (y_2y_5)^2, \\ y_2x_1y_2x_1x_2, y_2x_2y_2x_2, y_2x_3y_2x_1x_2x_4, y_2x_4y_2x_1x_3, \\ y_5^{-1}x_1y_5x_1x_2, y_5^{-1}x_2y_5x_2x_3, y_5^{-1}x_3y_5x_3x_4, y_5^{-1}x_4y_5x_1 >;$$

t_i: $(5, 4^2)$ and $(5, 4, 2)$;
S_1: $y_5^2x_1, y_2y_5^2x_2x_4, y_2x_4$;
S_2: $y_5^3x_1x_3, y_2y_5^3x_4, y_2x_1x_3x_4$;
H_1: $\mathbb{Z}_8$;
π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_8 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.3.3. *Group* $\mathbb{Z}_2^4 \rtimes D_5$*:*

G: as above;
t_i: $(5, 4^2)$ and $(5, 4, 2)$;
S_1: $y_5^3x_1x_4, y_2x_3, y_2y_5^2x_2x_4$;
S_2: $y_5^3x_2x_4, y_2y_5^2x_1x_4, y_2y_5^4x_1x_2x_4$;
H_1: $\mathbb{Z}_8$;
π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_8 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.3.4. *Group* $\mathbb{Z}_2^4 \rtimes D_5$*:*

G: as above;
t_i: $(5, 4^2)$ and $(5, 4, 2)$;
S_1: $y_5^3x_1x_4, y_2x_3, y_2y_5^2x_2x_4$;
S_2: $y_5^3x_1x_3, y_2y_5^3x_4, y_2x_1x_3x_4$;
H_1: $\mathbb{Z}_8$;
π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_8 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.3.5. *Group* $\mathfrak{A}_6$.

G: $\mathfrak{A}_6$;
t_i: $(5, 4, 2)$ and $(5, 3^2)$;
S_1: $(14623), (13)(2564), (12)(56)$;
S_2: $(15342), (164), (135)(246)$;
H_1: $\mathbb{Z}_6$;
π_1: $\mathbb{Z}_6$.

A.4. $K^2 = 3$**, basket** $\{2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)\}$**.**

A.4.1. *Group* $\mathfrak{S}_4 \times \mathbb{Z}_2$*:*

G: $\langle(12), (13), (14), (56)\rangle < \mathfrak{S}_6$;
t_i: $(4, 3, 2^2)$ and $(6, 4, 2)$;
S_1: $(1234), (234), (13)(24)(56), (34)(56)$;
S_2: $(234)(56), (4321)(56), (14)$;
H_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$;
π_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$.

A.5. $K^2 = 2$**, basket** $\{2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)\}$**.**

A.5.1. *Group* $\mathfrak{A}_4 \times \mathbb{Z}_2$*:*

G: $\langle(123), (12)(34), (56)\rangle < \mathfrak{S}_6$;
t_i: $(6^2, 2)$ and $(3^2, 2^2)$;
S_1: $(132)(56), (142)(56), (13)(24)$;
S_2: $(234), (123), (13)(24)(56), (14)(23)(56)$;
H_1: $\mathbb{Z}_2^2$;
π_1: Q_8.

A.5.2. *Group* $\mathfrak{S}_4$*:*

G: $\mathfrak{S}_4$;
t_i: $(4^2, 3)$ and $(3^2, 2^2)$;
S_1: $(123), (134), (12), (24)$;
S_2: $(1234), (1243), (124)$;
H_1: $\mathbb{Z}_8$;
π_1: $\mathbb{Z}_8$.

A.5.3. *Group* $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$*:* this is the semidirect product obtained by letting a generator of $\mathbb{Z}_3$ act on $\mathbb{Z}_5^2$ as $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

G: $\langle x_1, x_2, y | x_i^5, [x_1, x_2], y^3, y^{-1}x_1^{-1}yx_1x_2^2, y^{-1}x_2^{-1}yx_1x_2^3 \rangle$;
t_i: both $(5, 3^2)$;
S_1: $x_1^3x_2^2, y^2x_1^3x_2^4, y$;
S_2: $x_1^3, yx_1, y^2x_1^4x_2^2$;
H_1: $\mathbb{Z}_5$;
π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_5 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_5$ but the computer could not solve the problem.

A.5.4. *Group* $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$*:*

G: as above
t_i: both $(5, 3^2)$;
S_1: $x_1^3x_2^2, y^2x_1^3x_2^4, y$;
S_2: $x_1^4x_2^3, yx_1x_2, y^2x_1^4x_2^3$;
H_1: $\mathbb{Z}_5$;
π_1: the fundamental group fits in an exact sequences

$$1 \to H \to \pi_1 \to \mathbb{Z}_5 \to 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_5$ but the computer could not solve the problem.

A.5.5. *Group* $\mathfrak{A}_5$.

G: $\mathfrak{A}_5$;
t_i: $(5, 3^2)$ and $(3, 2^3)$;
S_1: $(13542), (123), (345)$;
S_2: $(152), (14)(23), (23)(45), (14)(25)$;
H_1: $\mathbb{Z}_2^2$;
π_1: $\mathbb{Z}_2^2$.

A.6. $K^2 = 2$**, basket** $\{2 \times \frac{1}{2}(1,1) + \frac{1}{4}(1,1) + \frac{1}{4}(1,3)\}$**.**

A.6.1. *Group* $PSL(2,7)$*:*

G: $\langle (34)(56), (123)(457) \rangle < \mathfrak{S}_7$;
t_i: $(7, 4, 2)$ and $(4, 3^2)$;
S_1: (1436275),(14)(2357),(36)(45);
S_2: (1236)(47), (245)(376), (164)(257);
H_1: $\mathbb{Z}_3$;

π_1: $\mathbb{Z}_3$.

A.6.2. *Group* $PSL(2,7)$*:*

G: $\langle(34)(56),(123)(457)\rangle < \mathfrak{S}_7$;
t_i: $(7,4,2)$ and $(4,3^2)$;
S_1: (1436275),(14)(2357),(36)(45);
S_2: (34)(1675), (164)(257), (134)(265);
H_1: $\mathbb{Z}_3$;
π_1: $\mathbb{Z}_3$.

A.6.3. *Group* $\mathfrak{A}_6$.

G: $\mathfrak{A}_6$;
t_i: $(5,4,2)$ and $(4,3^2)$;
S_1: $(14623),(13)(2564),(12)(56)$;
S_2: $(16)(2435),(246),(162)(345)$;
H_1: $\mathbb{Z}_3$;
π_1: $\mathbb{Z}_3$.

A.6.4. *Group* $\mathfrak{A}_6$.

G: $\mathfrak{A}_6$;
t_i: $(5,4,2)$ and $(4,3^2)$;
S_1: $(14623),(13)(2564),(12)(56)$;
S_2: $(1365)(24),(124)(356),(125)$;
H_1: $\mathbb{Z}_3$;
π_1: $\mathbb{Z}_3$.

A.6.5. *Group* $\mathfrak{S}_5$.

G: $\mathfrak{S}_5$;
t_i: $(5,4,2)$ and $(6,4,3)$;
S_1: $(15432),(1235),(45)$;
S_2: $(15)(234),(2453),(153)$;
H_1: $\mathbb{Z}_3$;
π_1: $\mathbb{Z}_3$.

A.6.6. *Group* $\mathfrak{S}_5$.

G: $\mathfrak{S}_5$;
t_i: $(5,4,2)$ and $(6,4,3)$;
S_1: $(15432),(1235),(45)$;
S_2: $(14)(235),(1254),(432)$;
H_1: $\mathbb{Z}_3$;
π_1: $\mathbb{Z}_3$.

A.7. $K^2 = 1$**, basket** $\{4 \times \frac{1}{2}(1,1) + \frac{1}{3}(1,1) + \frac{1}{3}(1,2)\}$**.**

A.7.1. *Group* $\mathfrak{S}_5$.

G: $\mathfrak{S}_5$;
t_i: $(3, 2^3)$ and $(4^2, 3)$;
S_1: $(123), (34), (23), (13)(24;$
S_2: $(1234), (1243), (124)$;
H_1: $\mathbb{Z}_4$;
π_1: $\mathbb{Z}_4$.

A.7.2. *Group* $PSL(2, 7)$*:*

G: $\langle(34)(56), (123)(457)\rangle < \mathfrak{S}_7$;
t_i: $(7, 3, 2)$ and $(4^2, 3)$;
S_1: $(1476532), (164)(235), (26)(47)$;
S_2: $(1765)(23), (17)(3645), (236)(475)$;
H_1: $\mathbb{Z}_2$;
π_1: $\mathbb{Z}_2$.

A.7.3. *Group* $\mathfrak{S}_4 \times \mathbb{Z}_2$*:*

G: $\langle(12), (13), (14), (56)\rangle < \mathfrak{S}_6$;
t_i: $(3, 2^3)$ and $(6, 4, 2)$;
S_1: $(134), (13)(24)(56), (23), (24)(56)$;
S_2: $(143)(56), (1234)(56), (23)$;
H_1: $\mathbb{Z}_2$;
π_1: $\mathbb{Z}_2$.