Abstracts

Fibrations of low genus and surfaces with $q = p_g = 1$

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(joint work with Fabrizio Catanese)

Let X be a projective surface, and let $f : X \to B$ be a *fibration*, *i.e.* a surjective morphism with connected fibres (of genus g) onto a smooth curve B of genus b. We may assume without loss of generality the fibration to be relatively minimal, contracting every rational curve with self-intersection (-1) contained in a fibre.

In [6] we announced classifications theorems for *low genus fibrations*, *i.e.* for $g = 2$ and 3; more precisely in the case $g = 3$ we consider only the case when the general fibre is not hyperelliptic and we moreover need to assume that every fibre is $2-$connected (excluding, *e.g.*, fibrations with double fibres).

1. Genus 2 fibrations

The canonical map of a genus 2 curve is a double cover of $\mathbb{P}^1$. Given a genus 2 fibration f one can *glue* the canonical map of the fibres to a rational map from X to a $\mathbb{P}^1$-bundle over B, more precisely $\mathbb{P}(V_1)$ where $V_1 = f_*\omega_{X|B}$ with $\omega_{X|B} := \omega_X \otimes f^*\omega_B^{-1}$.

This map allows to construct X as double cover of a ruled surface (as done by many authors, see, *e.g.* [7], [8]). One first constructs the ruled surface $\mathbb{P}(V_1)$, and then has to find a divisor on it, the branch curve of the double cover. This double cover is called *relative canonical map*. The difficulty of this construction is that one has to find a curve in a suitable linear system *with prescribed singularities*.

The problem comes from the *special* fibres, fibres that can be decomposed as $E_1 + E_2$ with $E_1E_2 = 1$. Let us consider by sake of simplicity only the easier example: two elliptic curves intersecting transversally in a point. The relative canonical map of this reducible curve blows up the intersection point (sending the exceptional divisor isomorphically to the corresponding fibre of the ruled surface) and contracts the two elliptic curves to two points.

The branch curve on $\mathbb{P}(V_1)$ is a relative sextic, *i.e.* a curve intersecting a general fibre in six points. On the special fibres corresponding to the above example, three of these points converge to the image of each elliptic curve. Therefore the branch curve contains the fibre and has two quadruple points on it.

Our approach is slightly different: we use the bicanonical map of the fibres. It induces a morphism of degree 2 from X onto a relative conic (a singular birational model of $\mathbb{P}(V_1)$) contained in the $\mathbb{P}^2$-bundle $\mathbb{P}(V_2)$ over B, where $V_2 = f_*\omega_{X|B}^{\otimes 2}$. The advantage is that the branch curve has no essential singularities.

Definition 1. *We define the* **associated 5-tuple** (B, V_1, τ, ξ, w) *of a genus* 2 *fibration* f *where:*

- B *is a curve;*

- *V_1 is a rank 2 vector bundle;*
- *τ is an effective divisor on B;*
- *$\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$;*
- *$w \in \mathbb{P}(H^0(B, \mathcal{F}))$, for a suitable vector bundle $\mathcal{F}$ on B (depending on ξ).*

We refer for a precise definition of this 5−tuple to [6]. We just recall that B is the base curve, $V_1 := f_* \omega_{X|B}$, τ is the set of the points corresponding to the *special* fibres, ξ is the extension corresponding to the exact sequence

$$0 \to S^2(V_1) \stackrel{\sigma_2}{\to} V_2 \to \mathcal{O}_\tau \to 0,$$

induced by the natural multiplication map σ_2: from this 4 data we can reconstruct the conic bundle. The last datum, w, depending from a vector bundle $\mathcal{F}$ obtained by ξ with a procedure we do not repeat here (see [6]), gives the branch curve.

Definition 2. *We will say that a a 5−tuple (B, V_1, τ, ξ, w) is* **admissible** *if*

- *B is a smooth curve;*
- *V_1 is a vector bundle on B of rank 2;*
- *$\tau \in Div^+(B)$;*
- *$\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ yields a vector bundle V_2;*
- *$w \in \mathbb{P}(H^0(B, \mathcal{F}))$, where $\mathcal{F}$ is obtained by ξ following the above mentioned procedure;*

and if moreover they satisfy some open conditions[1] ensuring that the associated double cover has only Rational Double Points as singularities.

Theorem 1. *The associated 5−tuple of a genus 2 fibration is admissible. Viceversa, every admissible 5−tuple is associated to a genus 2 fibration $f: X \to B$, with invariants $\chi(\mathcal{O}_X) = \deg(V_1) + (b-1), K^2 = 2\deg V_1 + \deg \tau + 8(b-1)$. Two genus 2 fibration having the same associated 5−tuple are isomorphic.*

2. Surfaces with $q = p_g = 1$

Let S be a minimal surface of general type with $q = p_g = 1$; in this case $2 \le K_S^2 \le 9$, and the Albanese map is a morphism $f: S \to B$ where B is a smooth elliptic curve. In fact, for $K_S^2 = 2$ it was proved in [2] that the Albanese is a genus 2 fibration, S is a double cover of $B^{(2)}$, the second symmetric power of B, and the moduli space is generically smooth, unirational of dimension 7.

Definition 3. *We will denote by $\mathcal{M}$ the family, in the moduli space of the minimal surfaces of general type, corresponding to the surfaces S with $p_g(S) = q(S) = 1$, $K_S^2 = 3$.*

$\mathcal{M}$ is studied in [3], [4]. In [3] it is proved that for this class of surfaces the Albanese is a genus g fibration with $g = 2$ or 3. The second case is completely classified in [4], where it is shown that it gives a generically smooth, unirational connected component of $\mathcal{M}$ of dimension 5.

[1] We do not specify here the open conditions by lack of space.

We are left with the case $g = 2$, where we can use theorem 1. In [3] was shown the existence of this case, and conjectured that this family of surfaces should form an unirational component of the moduli space (so $\mathcal{M}$ would have two unirational connected components). By use of theorem 1 we have disproved this conjecture giving a complete description of $\mathcal{M}$ as follows:

Theorem 2. *$\mathcal{M}$ has 4 connected components, all unirational of dimension 5.*

We use the classification of vector bundles over elliptic curves given in [1]. In [3] is proved that the vector bundle V_1 is undecomposable, and then one can assume (up to translations) without loss of generality $V_1 = E_{[0]}(2,1)$ (in Atiyah's notation). By a result of Clemens ([5]) every irreducible component of this moduli space has dimension at least 5. We have one parameter for B, one for τ ($\deg \tau = 1$ so τ is a point of B) and one can easily compute that ξ varies in a 2-parameters space. For general choice of the above data, the resulting vector bundle $\mathcal{F}$ has $h^0(\mathcal{F}) = 2$, and therefore its projective space gives one further parameter for w: this gives the *main stream* component, unirational of dimension 5.

To understand if there are other components, one need a case-by-case analysis, since for special choice of (τ, ξ), $h^0(\mathcal{F})$ grows. Most cases give *strata* of dimension smaller than 5. There is one exception, if ξ is such that V_2 decomposes as sum of three line bundles (finitely many choices for ξ).

In this case, for general choice of τ, $h^0(\mathcal{F}) = 4$ (so 2 parameters for the pair (B, τ) and 3 for w) but the resulting linear system of branch curves has a fixed part not reduced: this case does not fulfil our open conditions. If we assume τ special ($\tau = [0]$ or τ 2−torsion), $h^0(\mathcal{F}) = 5$. This very special case gives two more unirational families with 5 parameters (1 for B and 4 for w).

We can show that all the three families exist and do not intersect pairwise.

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