

THE PICARD NUMBER OF FIBRED MORI DREAM SURFACES

FEDERICO FALLUCCA, ROBERTO PIGNATELLI, AND FRANCESCO POLIZZI

ABSTRACT. Let S be a smooth complex projective surface endowed with a fibration $f: S \rightarrow C$ onto a smooth projective curve C . We prove that, if S is a Mori dream space (or, more generally, if its pseudo-effective cone is polyhedral) then the Picard number $\rho(S)$ can be effectively computed by counting the irreducible components of the reducible fibres of f .

A first simple consequence is that, given an elliptic fibration $f: S \rightarrow \mathbb{P}^1$ with a section and such that S is a Mori dream space, the Mordell-Weil group of the general fibre of f is finite.

The main application is a simple criterion for proving that a surface fibred over a curve is not a Mori dream space.

We show that certain Horikawa surfaces, Fermat surfaces in $\mathbb{P}^3$ of every degree ≥ 4 , particular product-quotient surfaces, and the minimal simply connected numerical Godeaux surface constructed by Craighero and Gattazzo are not Mori dream spaces.

CONTENTS

1. Introduction	2
1.1. Notation and conventions.	3
2. Preliminaries on Mori dream surfaces	4
3. The main theorem	4
4. Irregular examples of non-polyhedral $\overline{\text{Eff}}(S)$: self-products of curves and diagonal double Kodaira surfaces	7
5. Double covers of ruled surfaces and Horikawa surfaces	8
6. Surfaces in $\mathbb{P}^3$ containing lines and Fermat surfaces	11
7. Product-quotient surfaces	15
7.1. Surfaces isogenous to a product, of unmixed type.	15
7.2. Product-quotient surfaces with geometric genus zero	20
8. The Craighero-Gattazzo surface	21
Acknowledgements	24
References	24

Date: August 30, 2026.

2020 Mathematics Subject Classification: 14E30, 14J29, 14D06

Keywords: Mori dream spaces, surfaces of general type, fibred surfaces.

1. INTRODUCTION

Since the pioneering paper [23], Mori dream spaces have been the subject of intense study by the mathematical community. In particular, considerable attention has been devoted to determining which algebraic varieties are Mori dream spaces and which are not.

This research has mainly concerned special classes of algebraic varieties, such as, though not exclusively, rational varieties. By contrast, very little is available in the literature concerning varieties of general type.

The main result of this paper is the following, see Theorem 3.4:

Theorem A. *Let $f : S \rightarrow C$ be a fibration of a smooth complex projective surface S onto a smooth curve C . If the pseudo-effective cone $\overline{\text{Eff}}(S)$ is polyhedral, then*

$$(1.1) \quad \rho(S) = n(f) + 2.$$

Here, as usual, $\rho(S)$ denotes the Picard number of the surface S . The number $n(f)$ is the number of reducible fibres of f , counted suitably: a fibre with exactly k distinct irreducible components counts as $k - 1$ reducible fibres (see Definition 3.1). In particular, Theorem A implies that, if $\overline{\text{Eff}}(S)$ is polyhedral, then all fibrations on S have the same number of reducible fibres (counted as explained above), which is not true in general: for example, we produce two fibrations on the Fermat quartic surface with different numbers of reducible fibres, see Remark 6.6.

As the pseudo-effective cone of a Mori dream space is polyhedral, we obtain the following criterion for proving that a fibred surface is not a Mori dream space, see Corollary 3.5:

Corollary B. *Let S be a smooth projective surface endowed with a fibration $f : S \rightarrow C$. If*

$$\rho(S) > n(f) + 2$$

then $\overline{\text{Eff}}(S)$ is non-polyhedral. In particular, S is not a Mori dream space.

Although this article mostly focuses on surfaces of general type, our result can be useful also in other contexts. As an example, we present an application to the finiteness of the Mordell-Weil group of the general fibre of an elliptic fibration, see Corollary 3.7.

By applying Corollary B, we prove that several regular surfaces of general type are not Mori dream spaces. More precisely, we apply the result to:

- certain double covers of Hirzebruch surfaces, including some Horikawa surfaces, the minimal surfaces of general type satisfying the equality in the Noether inequality $K^2 \geq 2p_g - 4$, with p_g equal to any integer ≥ 3 (Theorem 5.4 and Corollary 5.5);
- Fermat surfaces in $\mathbb{P}^3$ of every degree ≥ 4 (Theorem 6.5);

- certain product-quotient surfaces, namely minimal resolutions of the singularities, if any, of quotients of a product of two curves by the action of a finite group (Theorems 7.7, 7.11, 7.14);
- the Craighero–Gattazzo surface, a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 1$ (Theorem 8.2).

It turns out that all of them have a non-polyhedral pseudo-effective cone, so they are not Mori dream spaces.

Our work was motivated by a question posed by Keum and Lee in [25]: they asked about the existence of a minimal surface of general type $p_g = q = 0$ that is not a Mori dream space. Here we show that the Craighero–Gattazzo surface affirmatively answers Keum–Lee’s question. Another example has been recently given in [10]: we note that the example has Picard number 2, which implies that $\overline{\text{Eff}}(S)$ is polyhedral. By contrast, we show that the pseudo-effective cone of the Craighero–Gattazzo surface is non-polyhedral, so our example is of a very different nature.

We explicitly remark that the vanishing of the geometric genus in [25] is meant to be a simplifying assumption, since it makes it possible to compute the Picard number of the surface easily; indeed, we think that the examples with $p_g > 0$ provided in our work are of similar interest.

The paper is organized as follows. Section 2 contains some known preliminary results on surfaces which are Mori dream spaces. Section 3 contains the proof of Theorem A and Corollary B and the application to the elliptic surfaces. The subsequent sections prove that various families of surfaces of general type have a non-polyhedral pseudo-effective cone by studying the reducible fibres of a specific fibration defined on each of them.

1.1. Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers.

If X is a normal, $\mathbb{Q}$ -factorial projective variety and K_X is the class of a canonical Weil divisor, we will consider its geometric genus $p_g(X) = h^0(X, K_X)$ and its irregularity $q(X) = h^1(X, \mathcal{O}_X)$.

We denote by $\equiv$ the numerical equivalence relation between divisors, and we write $N^1(X) := (\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}) / \equiv$. The dimension of $N^1(X)$ as a real vector space is the *Picard number* $\rho(X)$ of X .

The *effective cone* of X is the convex cone in $N^1(X)$ generated by numerical classes of effective Weil divisors:

$$\text{Eff}(X) := \left\{ \sum a_i [D_i] \mid a_i \in \mathbb{R}_{\geq 0}, D_i \text{ is an effective divisor} \right\} \subset N^1(X).$$

The *pseudo-effective cone* $\overline{\text{Eff}}(X)$ is the closure of $\text{Eff}(X)$ in $N^1(X)$. If $\dim X = 2$, the canonical isomorphism $N^1(X) \simeq N_1(X)$ identifies $\overline{\text{Eff}}(X)$ with the *Mori cone* $\overline{\text{NE}}(X)$.

The *nef cone* $\text{Nef}(X)$ is the cone generated by the classes of nef divisors.

The *semi-ample cone* $\text{SAmp}(X)$ is the cone generated by the classes of semi-ample divisors.

We say that a cone in $N^1(X)$ is *polyhedral* if it is the positive hull of finitely many vectors. In particular, a polyhedral cone is closed.

A *fibration* is a surjective morphism whose fibres are connected.

2. PRELIMINARIES ON MORI DREAM SURFACES

We recall one of the equivalent definitions of a Mori dream space (see [23, Definition 1.10 and Proposition 2.9]):

Definition 2.1. A projective variety X is a *Mori dream space* if X is $\mathbb{Q}$ -factorial, $q(X) = 0$ and the Cox ring of X is finitely generated.

If S is a smooth projective surface with $q(S) = 0$, we have the chain of inclusions

$$\text{SAmp}(S) \subseteq \text{Nef}(S) \subseteq \overline{\text{Eff}}(S),$$

see [2, Proposition 1.2]. Moreover, $\text{Nef}(S)$ is the dual cone of $\overline{\text{Eff}}(S)$ in $N^1(S)$. In particular, if $\text{Eff}(S)$ is a polyhedral cone, then $\text{Nef}(S)$ is a polyhedral cone as well.

Theorem 2.2. [1, Corollary 2.6] *Let S be a $\mathbb{Q}$ -factorial projective surface with $q(S) = 0$. Then S is a Mori dream space if and only if $\text{Eff}(S)$ is a polyhedral cone and $\text{Nef}(S) = \text{SAmp}(S)$.*

3. THE MAIN THEOREM

We start by defining $n(f)$, the number of the reducible fibres of a fibration f , counted with a suitable weight.

Definition 3.1. Let $f: S \rightarrow C$ be a fibration of a smooth surface S to a smooth curve C . For each $p \in C$, we denote by $k_p(f)$ the number of distinct irreducible components of the fibre $f^{-1}(p)$ and we set

$$(3.1) \quad n(f) := \sum_{p \in C} (k_p(f) - 1)$$

If $f^{-1}(p)$ is irreducible (but not necessarily reduced) then $k_p(f) - 1 = 0$. Thus (3.1) is a finite sum supported over the images of the reducible fibres. Furthermore, $n(f) = 0$ if and only if all fibres of f are irreducible.

We can now prove the easy part of Theorem A, namely the inequality $\rho(S) \geq n(f) + 2$, that does not require any assumption on $\overline{\text{Eff}}(S)$.

We consider the classes in $N^1(S)$ of the irreducible curves contracted by f : the class e of a smooth fibre, and the classes $e_1, \dots, e_m$ of all distinct irreducible components of the reducible fibres of f (if any).

Lemma 3.2. *With the notation above, we have*

$$\dim_{\mathbb{R}} \text{span}(e, e_1, \dots, e_m) = n(f) + 1.$$

As a consequence, we get

$$(3.2) \quad \rho(S) \geq n(f) + 2.$$

Proof. Let us extract from the set $\{e, e_1, \dots, e_m\}$ a basis of the corresponding vector subspace. Each reducible fibre $f^{-1}(p)$ of $f: S \rightarrow C$ gives a linear relation in $N^1(S)$ involving e and the classes of components of $f^{-1}(p)$. If r is the number of reducible fibres of f , we have $r = m - n(f)$ linear relations and so we are left with

$$m + 1 - r = m + 1 - (m - n(f)) = n(f) + 1$$

generators, which are linearly independent by the classical Zariski's lemma [3, Lemma 8.2, Chap. III.8].

Finally, we observe that $\text{span}(e, e_1, \dots, e_m)$ contains no ample class, hence it is a proper subspace of $N^1(S)$ and (3.2) follows. $\square$

In order to prove the reverse inequality we need the following result by Rabinathan [36, Proposition 2.1], for which we give here a slightly different proof.

Proposition 3.3. *Let S be a smooth projective surface. Suppose that one can find three non-zero classes $e, v, h \in N^1(S)$, where e is pseudo-effective and h ample, such that*

$$e^2 = ev = hv = 0$$

and moreover

$$(3.3) \quad (e + \mathbb{R}v) \cap \overline{\text{Eff}}(S) = \{e\}.$$

Then $\overline{\text{Eff}}(S)$ is non-polyhedral.

Proof. Since $e^2 = 0$ and $h^2 > 0$, the classes e, h generate a 2-dimensional subspace $\text{span}(e, h) \subset N^1(S)$. Since Kleiman's criterion [26] implies $eh > 0$, a straightforward calculation shows that the radical of the restriction of the intersection form to $\text{span}(e, h)$ is trivial. But $v \in \text{span}(e, h)^\perp$, so e, v, h are linearly independent and the subspace $W := \text{span}(e, v, h) \subset N^1(S)$ has dimension 3. By the Hodge Index theorem the restriction of the intersection form to $h^\perp$ is negative definite: then, as $\dim W \cap h^\perp = 2$, the intersection form on W has signature $(1, 2)$.

Up to rescaling h we can assume $h^2 = 1$, and we can choose an orthogonal basis $\{h, w_1, w_2\}$ of W such that $w_1^2 = w_2^2 = -1$. With respect to this basis, setting $w = xh + y_1w_1 + y_2w_2$, the isotropic cone $\{w^2 = 0\} \subset W$ has equation $x^2 - y_1^2 - y_2^2 = 0$. Thus, the intersection of this cone with the affine plane

$$H := \{w \in W \mid (w - e)h = 0\},$$

which is orthogonal to the x -axis and does not contain the origin, is the ellipse C of equation

$$y_1^2 + y_2^2 = (eh)^2.$$

If U is the interior part of C , every rational class $w \in U$ satisfies $w^2 > 0$ and $wh > 0$: hence a sufficiently divisible integral multiple of w is big by Riemann-Roch. Thus every rational class in U is effective. Since rational classes are dense in U and the pseudo-effective cone is closed, we obtain $U \subset \overline{\text{Eff}}(S)$.

Finally, assume by contradiction that $\overline{\text{Eff}}(S)$ is polyhedral. Then $H \cap \overline{\text{Eff}}(S)$ is a convex polygon P containing $\overline{U}$. By (3.3), the point e is a vertex of P . Summing up, the ellipse C is contained in the convex polygon P and passes through a vertex of it, contradiction. $\square$

We can now complete the proof of Theorem A by showing the inequality $\rho(S) \leq n(f) + 2$. Our argument is a refinement of the proof of a weaker inequality due to Rabindranath [35, Theorem 12 p. 36].

Theorem 3.4. *Let $f: S \rightarrow C$ be a fibration of a smooth complex projective surface S onto a smooth curve C . If $\overline{\text{Eff}}(S)$ is polyhedral, then equality (1.1) holds, namely*

$$\rho(S) = n(f) + 2.$$

Proof. By Lemma 3.2 it is equivalent to prove that, if $\rho(S) > n(f) + 2$, then $\overline{\text{Eff}}(S)$ is non-polyhedral.

Consider the pushforward map $f_*: N^1(S) \cong N_1(S) \rightarrow N_1(C) \simeq \mathbb{R}$. The ample class intersects the fibre e positively, so f_* is non-zero and $\ker f_* \subset N^1(S)$ has codimension 1.

Any irreducible component of a fibre is contracted by f , so we have the inclusion

$$(3.4) \quad \text{span}(e, e_1, \dots, e_m) \subseteq \ker f_*.$$

By Lemma 3.2, if $\rho(S) > n(f) + 2$ then the inclusion in (3.4) is strict, because

$$\dim \ker f_* = \rho(S) - 1 > n(f) + 1 = \dim_{\mathbb{R}} \text{span}(e, e_1, \dots, e_m).$$

This means that we can find $v \in \ker f_*$ such that

$$(3.5) \quad v \notin \text{span}(e, e_1, \dots, e_m).$$

Since $v \in \ker f_*$, we have $ev = 0$. Let us fix an ample divisor h ; then, up to adding to v a suitable multiple of e , we may assume $hv = 0$.

Let us consider now the line $e + \mathbb{R}v \subset \ker f_*$. A result by Debarre-Jiang-Voisin [16, Thm. 5.1] implies that

$$\overline{\text{Eff}}(S) \cap \ker f_* = \mathbb{R}_{\geq 0} \cdot e + \mathbb{R}_{\geq 0} \cdot e_1 + \dots + \mathbb{R}_{\geq 0} \cdot e_m,$$

and so, because of (3.5), we have

$$(e + \mathbb{R}v) \cap \overline{\text{Eff}}(S) = \{e\}.$$

Summing up, the three classes $e, v, h \in N^1(S)$ satisfy all the assumptions in Proposition 3.3, hence $\overline{\text{Eff}}(S)$ is non-polyhedral. $\square$

Corollary B follows now easily:

Corollary 3.5. *Let S be a smooth projective surface endowed with a fibration $f: S \rightarrow C$. If*

$$\rho(S) > n(f) + 2$$

then $\overline{\text{Eff}}(S)$ is non-polyhedral. In particular, S is not a Mori dream space.

Proof. Theorem 3.4 together with the assumption $\rho(S) > n(f) + 2$ implies that $\overline{\text{Eff}}(S)$ is non-polyhedral. The last claim follows from Theorem 2.2. $\square$

Remark 3.6. There are surfaces such that $\rho(S) = n(f) + 2$ and nevertheless $\overline{\text{Eff}}(S)$ is non-polyhedral. One example is the Fermat quartic surface $S_4 \subset \mathbb{P}^3$, see Remark 6.6.

We conclude with a direct consequence of Theorem 3.4 regarding the finiteness of the Mordell-Weil group of the generic fibre of an elliptic surface.

Corollary 3.7. *Let S be a smooth projective surface with $q(S) = 0$ and let $f: S \rightarrow C$ be an elliptic fibration with a section. If $\overline{\text{Eff}}(S)$ is polyhedral, then the Mordell-Weil group of the generic fibre of f is finite.*

Proof. Shioda–Tate formula [38, Corollary 6.13] implies that, under our assumptions, the difference $\rho(S) - (n(f) + 2)$ equals the rank of the Mordell–Weil group of the generic fibre of f . $\square$

While we were finalizing this article, a related result for Jacobian elliptic surfaces of Kodaira dimension 1 was announced on the arXiv [29, Proposition 3.1].

4. IRREGULAR EXAMPLES OF NON-POLYHEDRAL $\overline{\text{EFF}}(S)$: SELF-PRODUCTS OF CURVES AND DIAGONAL DOUBLE KODAIRA SURFACES

As a first application of the previous theory, we get the following result due to Rabindranath [36, Theorem 1.1].

Proposition 4.1. *Let C be a smooth curve with $g(C) \geq 1$. Then $\overline{\text{Eff}}(C \times C)$ is non-polyhedral.*

Proof. If $f: C \times C \rightarrow C$ is the projection onto one of the factors, then $n(f) = 0$. Moreover, we have $\rho(C \times C) \geq 3$, since the two fibres and the diagonal $\Delta \subset C \times C$ are linearly independent in $N^1(C \times C)$ for $g(C) \geq 1$. So the result follows from Corollary B. $\square$

We can use the same technique to prove that some double Kodaira surfaces (i.e., surfaces endowed with two smooth, non-isotrivial fibrations, see [13]) also have a non-polyhedral pseudo-effective cone. The following example can be found in [33], see also [34] for related constructions. Let C be a smooth curve with $g(C) = 2$ and let $\Delta \subset C \times C$ be the diagonal. Then there exists a G -cover

$$\pi: S \rightarrow C \times C,$$

where G is an extra-special group of order $2^5 = 32$, which is branched precisely over Δ , with branching order 2. Composing π with the two natural projections of $C \times C$, we get two smooth, non-isotrivial fibrations $f_i: S \rightarrow C$. The surface S is called a *diagonal double Kodaira surface*.

Proposition 4.2. *Let S be a diagonal double Kodaira surface as described above. Then the cone $\overline{\text{Eff}}(S)$ is non-polyhedral.*

Proof. Writing F_1, F_2 for the fibres of the two fibrations on S , and calling D the ramification locus of π , namely, the smooth curve $D \subset S$ such that $\pi^* \Delta = 2D$, we get

$$F_1^2 = F_2^2 = 0, \quad D^2 = -16, \quad F_1 D = F_2 D = 16, \quad F_1 F_2 = 32.$$

Thus the Gram matrix of the restriction of the intersection form to the subspace of $N^1(S)$ spanned by F_1, F_2, D is the matrix

$$\begin{pmatrix} 0 & 32 & 16 \\ 32 & 0 & 16 \\ 16 & 16 & -16 \end{pmatrix},$$

whose determinant is 32768. So F_1, F_2, D are linearly independent in $N^1(S)$, hence we get $\rho(S) \geq 3$. On the other hand, since $f_1: S \rightarrow C$ is a smooth fibration, we have $n(f_1) = 0$. Thus the claim follows from Corollary B. $\square$

Similar examples can be constructed using a smooth curve C of any genus $g \geq 2$ and an extra-special group G of order p^{2g+1} , with p dividing $g + 1$; see [14] for more details.

5. DOUBLE COVERS OF RULED SURFACES AND HORIKAWA SURFACES

In this section we show how to apply Corollary B to double covers of ruled surfaces. Let n be a non-negative integer and consider the Hirzebruch ruled surface $\mathbb{F}_n$. We have

$$N^1(\mathbb{F}_n) = \mathbb{R}[F] \oplus \mathbb{R}[C_0],$$

where F is the fibre of the ruling and C_0 is a section with $C_0^2 = -n$ (which is unique as soon as $n \geq 1$). Let us consider a double cover

$$\pi: S \rightarrow \mathbb{F}_n,$$

branched over a smooth curve $B \subset \mathbb{F}_n$. Let us also consider the fibration $f: S \rightarrow \mathbb{P}^1$ obtained composing $\pi: S \rightarrow \mathbb{F}_n$ with the ruling of $\mathbb{F}_n$.

We want to make the Picard number of S big enough, by applying the following

Lemma 5.1. *Assume that the branch divisor $B \subset \mathbb{F}_n$ is everywhere tangent to C_0 , namely, that the restriction $B|_{C_0}$ yields an effective divisor (possibly trivial) which is 2-divisible in $\text{Div}(C_0)$. If $C_0 B = 0$, assume in addition $n > 0$. Then $\rho(S) \geq 3$.*

Proof. The assumption that B is everywhere tangent to C_0 implies that the normalization of the restriction of $\pi: S \rightarrow \mathbb{F}_n$ to C_0 is étale, hence disconnected as $C_0 \simeq \mathbb{P}^1$ has trivial fundamental group. Then we get $\pi^*(C_0) = C_1 + C_2$, where C_1, C_2 are smooth rational curves in S and $C_1 C_2 = \frac{1}{2} C_0 B$.

The double cover involution on S exchanges C_1 and C_2 , so that

$$C_1^2 = C_2^2 = \frac{1}{2}(C_1 + C_2)^2 - C_1 C_2 = C_0^2 - \frac{1}{2}C_0 B.$$

Calling $e = \pi^* F$ the class of a fibre of f , we have

$$\pi^*(C_0) \cdot e = C_0 \cdot \pi_*(e) = C_0 \cdot (2F) = 2,$$

hence $C_1 \cdot e = C_2 \cdot e = 1$. Thus the Gram matrix of the restriction of the intersection form to the subspace of $N^1(S)$ spanned by C_1, C_2, e is the matrix

$$\begin{pmatrix} C_0^2 - \frac{1}{2}C_0 B & \frac{1}{2}C_0 B & 1 \\ \frac{1}{2}C_0 B & C_0^2 - \frac{1}{2}C_0 B & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

whose determinant is $2(C_0 B - C_0^2) = 2(C_0 B + n) > 0$. Therefore C_1, C_2, e are linearly independent, and so $\rho(S) \geq 3$. $\square$

Lemma 5.2. *Let k be an algebraically closed field with $\text{char}(k) = 0$, and let $\alpha, \beta \in k$ such that $(\alpha, \beta) \neq (0, 0)$. For all $b \geq 2$, the polynomial*

$$p(x) = x^{2b} + \alpha x^{2b-1} + \beta \in k[x]$$

has at least one simple root in k .

Proof. If $\beta = 0$ then $p(x) = x^{2b} + \alpha x^{2b-1} = x^{2b-1}(x + \alpha)$ has $-\alpha$ as a simple root. If $\alpha = 0$ then $p(x) = x^{2b} + \beta$ which has only simple roots. Thus we can assume $\alpha \neq 0$ and $\beta \neq 0$. The derivative of $p(x)$ is the polynomial

$$p'(x) = 2bx^{2b-1} + (2b-1)\alpha x^{2b-2} = x^{2b-2}(2bx + (2b-1)\alpha),$$

which has the two distinct roots 0 (with multiplicity $2b-2$) and $\gamma := \frac{(1-2b)\alpha}{2b}$ (with multiplicity 1).

As $p(0) = \beta \neq 0$, the only possible multiple root of $p(x)$ is γ . Moreover, since γ is a simple root of $p'(x)$, it can have at most multiplicity 2 as a root of $p(x)$. But the degree of $p(x)$ is $2b \geq 4$, so $p(x)$ has at least two simple roots. $\square$

In the sequel, we identify $\mathbb{F}_n$ with a toric variety $\mathbb{C}^4 // (\mathbb{C}^*)^2$ as follows: we take coordinates t_0, t_1, x_0, x_1 on $\mathbb{C}^4$, and we let $(\mathbb{C}^*)^2$ act by the weight matrix

$$(5.1) \quad \begin{pmatrix} t_0 & t_1 & x_0 & x_1 \\ 1 & 1 & 0 & n \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and we choose $(t_0, t_1) \cap (x_0, x_1)$ as the irrelevant ideal.

This identifies the vector spaces $H^0(\mathbb{F}_n, \mathcal{O}(aF + bC_0))$ with the homogeneous polynomials in the variables t_0, t_1, x_0, x_1 of bidegree (a, b) with respect to the grading indicated by the weight matrix (5.1): t_0 and t_1 have bidegree $(1, 0)$, whereas x_0 has bidegree $(0, 1)$ and x_1 has bidegree $(n, 1)$.

For simplicity we denote $H^0(\mathbb{F}_n, \mathcal{O}(aF + bC_0))$ by $H^0(\mathbb{F}_n, \mathcal{O}(a, b))$: the ruling is given by $H^0(\mathbb{F}_n, \mathcal{O}(1, 0)) = \text{span}(t_0, t_1)$, whereas C_0 is defined by the vanishing of $x_0 \in H^0(\mathbb{F}_n, \mathcal{O}(0, 1))$.

Lemma 5.3. *It is possible to choose the branch curve $B \subset \mathbb{F}_n$ so that both conditions below are satisfied:*

- (1) *B is smooth and everywhere tangent to C_0 ;*
- (2) *every fibre of $f: S \rightarrow \mathbb{P}^1$ is irreducible.*

Proof. We will consider suitable polynomials $p_B \in H^0(\mathbb{F}_n, \mathcal{O}(2a + 2bn, 2b))$ of the form

$$(5.2) \quad p_B := x_1^{2b}(t_0 - c_1 t_1)^2(t_0 - c_2 t_1)^2 \cdots (t_0 - c_a t_1)^2 + x_0 g(t_0, t_1; x_0, x_1),$$

where $a \geq 0$, $b \geq 2$ and the c_i are fixed complex numbers. The zero locus B of p_B is a divisor which is 2-divisible in $\text{Pic}(\mathbb{F}_n)$, so there is a (unique, since $\text{Pic}(\mathbb{F}_n)$ has no torsion) double cover of $\mathbb{F}_n$ branched on B .

As $g(t_0, t_1; x_0, x_1)$ varies, the polynomials of the form (5.2) define an affine subspace $\mathcal{L}$ of a linear system on $\mathbb{F}_n$, such that every $B \in \mathcal{L}$ is everywhere tangent to C_0 at the same points

$$[t_0 : t_1 : x_0 : x_1] = [c_i : 1 : 0 : 1] \quad 1 \leq i \leq a,$$

and the general B is smooth there. As these points are the unique base points of $\mathcal{L}^1$, by Bertini's theorem the general polynomial of the form (5.2) defines a smooth curve B . So (1) is satisfied.

In order to verify (2), it suffices to show that a general curve B in the linear system $\mathcal{L}$ has at least one point of transversal intersection with every fibre of the ruling. In fact, an easy local computation shows that $\pi^* F$ is smooth at the preimage on S of such an intersection point. Since B is the branch locus of $f: S \rightarrow \mathbb{F}_n$, if $\pi^* F$ consists of two components, they would necessarily intersect at that preimage, contradicting the smoothness of $\pi^* F$.

The locus of pairs $(B, t) \in \mathcal{L} \times \mathbb{P}^1$ such that the restriction of p_B to F_t has no simple zero is closed. Since the projection $\mathcal{L} \times \mathbb{P}^1 \rightarrow \mathcal{L}$ is proper, its image is closed as well. We will now construct an explicit member $B \in \mathcal{L}$ that does not belong to this image, and this will show that the desired property holds on a non-empty open subset of $\mathcal{L}$.

More precisely, we will prove that the curve B defined by the polynomial

$$p_B := x_1^{2b}(t_0 - c_1 t_1)^2(t_0 - c_2 t_1)^2 \cdots (t_0 - c_a t_1)^2 + t_1^{2a+n} x_0 x_1^{2b-1} + t_0^{2a+2bn} x_0^{2b},$$

where all c_i are pairwise distinct and different from zero, intersects transversally in at least one point every fibre of the ruling of $\mathbb{F}_n$.

In fact, for a fixed $[\bar{t}_0 : \bar{t}_1] \in \mathbb{P}^1$, let us consider the specialization

$$p_B(\bar{t}_0, \bar{t}_1; x_0, x_1) = \tau x_1^{2b} + \alpha x_0 x_1^{2b-1} + \beta x_0^{2b} \in \mathbb{C}[x_0, x_1],$$

where

$$\tau := \prod_{i=1}^a (\bar{t}_0 - c_i \bar{t}_1)^2, \quad \alpha := \bar{t}_1^{2a+n}, \quad \beta := \bar{t}_0^{2a+2bn}.$$

¹If $a = 0$ then B is disjoint from C_0 , and $\mathcal{L}$ is base-point free.

Since all c_i are different from zero, at most one among α, β and τ may vanish.

It remains to prove that, for all $[\bar{t}_0 : \bar{t}_1] \in \mathbb{P}^1$, the polynomial $p_B(\bar{t}_0, \bar{t}_1; x_0, x_1)$ has a linear factor appearing with multiplicity 1. If $\beta = 0$ the desired linear factor is $\tau x_1 + \alpha x_0$, thus we may assume that $\beta \neq 0$. Dehomogenizing with respect to x_0 , i.e. setting $x := x_1/x_0$, we get the polynomial

$$p(x) = \tau x^{2b} + \alpha x^{2b-1} + \beta \in \mathbb{C}[x]$$

If $\tau \neq 0$, up to rescaling we can suppose $\tau = 1$, and since $b \geq 2$, then $p(x)$ has at least one simple root by Lemma 5.2. Otherwise $\tau = 0$, $(\alpha, \beta) \neq (0, 0)$ and $p(x)$ has only simple roots. This proves (2). $\square$

We can now state the main result of this section.

Theorem 5.4. *Assume $a, n \geq 0$, $a + n \geq 2$, and $b \geq 2$. Then the double cover $\pi: S \rightarrow \mathbb{F}_n$, branched on a curve $B \in |2(a + bn)F + 2bC_0|$ whose defining polynomial is general among those of the form (5.2), is a smooth minimal surface whose pseudo-effective cone $\overline{\text{Eff}}(S)$ is non-polyhedral.*

Proof. Since $b \geq 2$, by Lemma 5.3 every fibre of $f: S \rightarrow \mathbb{P}^1$ is irreducible, hence $n(f) = 0$. Moreover, Lemma 5.1 applies and we deduce $\rho(S) \geq 3$. From Corollary B, it follows that the pseudo-effective cone of S is non-polyhedral. Finally, S is minimal because K_S is numerical equivalent to the pullback of $((a - 2) + (b - 1)n)F + (b - 2)C_0$, which is nef for $a + n \geq 2$ and $b \geq 2$. $\square$

Let us recall that a *Horikawa surface* is a minimal surface S of general type with $K_S^2 = 2p_g(S) - 4$. Horikawa surfaces satisfy $p_g(S) \geq 3$ and $q(S) = 0$; we refer to [3, Chapter VII] and to the reference given therein for more details.

Corollary 5.5. *For all $p_g \geq 3$, there exist Horikawa surfaces with that p_g and which are not Mori dream spaces, as their pseudo-effective cone is non-polyhedral.*

Proof. Choosing a, n as in the assumptions of Theorem 5.4 and $b = 3$, we get a minimal surface S with a non-polyhedral pseudo-effective cone. Such a surface has $p_g(S) = 3n + 2a - 2$ and $K_S^2 = 2p_g(S) - 4$, where K_S is big when $3n + 2a \geq 5$.

Thus, it is sufficient to choose $n \in \{0, 1\}$ to obtain Horikawa surfaces, see also [22, Theorem 1.6], with every possible $p_g \geq 3$ and such that $\overline{\text{Eff}}(S)$ is non-polyhedral. $\square$

6. SURFACES IN $\mathbb{P}^3$ CONTAINING LINES AND FERMAT SURFACES

A natural class of surfaces to which one may try to apply Corollary B is the class of surfaces S in $\mathbb{P}^3$ containing a line, since the pencil of planes containing the line defines a fibration $f: S \rightarrow \mathbb{P}^1$.

In fact, let H_S be its hyperplane section. Setting $d := \deg S$, by adjunction we have $K_S = (d - 4)H_S$. Denoting by ℓ a line in S , $K_S \cdot \ell = d - 4$ and the genus formula gives $\ell^2 = 2 - d$, so that

$$(H_S - \ell)^2 = (H_S)^2 - 2H_S \cdot \ell + \ell^2 = d - 2 + (2 - d) = 0.$$

This means that the residual pencil $|H_S - \ell|$ is without base points, and so defines a fibration $f: S \rightarrow \mathbb{P}^1$, whose general fibre is a plane curve of degree $d - 1$.

We are interested in counting the irreducible components of these plane curves, which are always reduced as observed in the following

Remark 6.1. The fibers of f are always reduced because S is a smooth surface of degree $d \geq 2$. Indeed, suppose by contradiction that a fiber of f , obtained by cutting S with a hyperplane H containing the line ℓ , contains a multiple irreducible component Γ . For any smooth point p of Γ , the hyperplane H must coincide with the tangent plane to S at p . Consequently, H would be tangent to S along the entire curve Γ . This implies that the Gauss map $\gamma: S \rightarrow (\mathbb{P}^3)^*$ contracts Γ to a point, a contradiction with the fact that γ is a finite morphism [43].

Definition 6.2. Let C be a reduced plane curve. Consider its topological Euler characteristic $\chi_{\text{top}}(C)$ and set

$$\mu(C) := \chi_{\text{top}}(C) + (\deg C) \cdot (\deg C - 3).$$

By the standard theory of plane curves, $\mu(C) \geq 0$ and C is smooth if and only if $\mu(C) = 0$. The number $\mu(C)$ measures the difference between the Euler characteristic of the curve and that of a smooth curve of the same degree. The next lemma shows a useful relation between μ and the number of components of C .

Lemma 6.3. *For a reduced plane curve C*

$$(6.1) \quad \mu(C) \geq \frac{k-1}{2} \deg C,$$

where k is the number of the irreducible components of C .

Proof. Write $C = \sum_{i=1}^k C_i$. Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C and set for all $p \in C$, $r_p := \#\nu^{-1}(p)$. Obviously

$$\chi_{\text{top}}(C) = \sum_{i=1}^k (2 - 2g_i) - \sum_{p \in \text{Sing} C} (r_p - 1)$$

where g_i is the genus of the normalization of C_i .

The natural inclusion $\mathcal{O}_C \hookrightarrow \nu_* \mathcal{O}_{\tilde{C}}$ has cokernel sheaf $\mathcal{Q}$ supported on $\text{Sing} C$, so we can write $\mathcal{Q} = \bigoplus_{p \in \text{Sing} C} \mathcal{Q}_p$ with $\mathcal{Q}_p$ supported on p . Setting $\delta_p := \text{length } \mathcal{Q}_p$, by the short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \nu_* \mathcal{O}_{\tilde{C}} \rightarrow \mathcal{Q} \rightarrow 0$$

we obtain the formulas

$$\delta_p \geq r_p - 1$$

$$(6.2) \quad p_a(C) = p_a(\tilde{C}) + \sum_{p \in \text{Sing} C} \delta_p = 1 - k + \sum_{i=1}^k g_i + \sum_{p \in \text{Sing} C} \delta_p$$

Then

$$(6.3) \quad \mu(C) = \chi_{\text{top}}(C) - (2 - 2p_a(C)) = \sum_{p \in \text{Sing } C} (2\delta_p - r_p + 1) \geq \sum_{p \in \text{Sing } C} \delta_p$$

Set now $d := \deg C$ so that $p_a(C) = \frac{(d-1)(d-2)}{2}$. Setting $d_i := \deg C_i$ so that $d = \sum_1^k d_i$, by (6.2) we get

$$(6.4) \quad \begin{aligned} \sum_{p \in \text{Sing } C} \delta_p &= \frac{(d-1)(d-2)}{2} - \sum_{i=1}^k g_i + k - 1 \geq \\ &\geq \frac{(d-1)(d-2)}{2} - \sum_{i=1}^k \frac{(d_i-1)(d_i-2)}{2} + k - 1 = \frac{1}{2} \sum_{i \neq j} d_i d_j \end{aligned}$$

Finally we observe

$$(6.5) \quad \frac{1}{2} \sum_{i \neq j} d_i d_j = \frac{1}{2} \sum_{i=1}^k d_i (d - d_i) \geq \frac{1}{2} \sum_{i=1}^k d_i (k-1) = \frac{1}{2} d(k-1)$$

and then (6.1) follows by (6.3), (6.4) and (6.5). $\square$

Our method gives now the following

Theorem 6.4. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 2$ containing a line ℓ . If $\rho(S) > 2 \frac{(d-2)^2(d+2)}{d-1} + 2$ then S is not a Mori dream space, as its pseudo-effective cone is non-polyhedral.*

Proof. We consider the fibration $f: S \rightarrow \mathbb{P}^1$ given by $|H_S - \ell|$. For all $t \in \mathbb{P}^1$ set F_t for the corresponding fibre. All F_t are plane curves of degree $d-1$.

By the Zeuthen-Segre formula, see [3, Chapter III, Proposition 11.4 and Remark 11.5]

$$\sum_{t \in \mathbb{P}^1} \mu(F_t) = d(d^2 - 4d + 6) - 2(2 - (d-2)(d-3)) = (d-2)^2(d+2).$$

Furthermore, all F_t are reduced plane curves from Remark 6.1, so we can apply Lemma 6.3 obtaining

$$\sum_{t \in \mathbb{P}^1} \mu(F_t) \geq \frac{n(f)}{2} (d-1)$$

and then $n(f) \leq 2 \frac{(d-2)^2(d+2)}{d-1}$. The claim now follows from Corollary B. $\square$

We now apply Theorem 6.4 to the *Fermat surface* $S_d \subset \mathbb{P}^3$, defined by

$$x_0^d - x_1^d + x_2^d - x_3^d = 0.$$

We take the line $\ell \subset S_d$, of equation

$$x_0 - x_1 = x_2 - x_3 = 0$$

and we consider the corresponding fibration

$$f_d: S_d \longrightarrow \mathbb{P}^1,$$

whose general fibre F_t consists of the movable part of the linear system cut out on S_d by the pencil of planes

$$x_0 - x_1 = t(x_2 - x_3).$$

Theorem 6.5. *All Fermat surfaces $S_d \subseteq \mathbb{P}^3$ of degree $d \geq 4$ are not Mori dream spaces, as their pseudo-effective cones are non-polyhedral.*

Proof. By [41, formula 9] we have

$$\rho(S_d) \geq 3(d-1)(d-2) + 1.$$

By Theorem 6.4, if

$$(6.6) \quad 3(d-1)(d-2) + 1 > 2 \frac{(d-2)^2(d+2)}{d-1} + 2$$

then the pseudo-effective cone of S_d is non-polyhedral.

Multiplying by $d-1$ (6.6) becomes

$$(d-3)(d^2 - 5d + 7) > 0$$

which is true for all $d \geq 4$. □

Note that Theorem 6.5 is sharp, as the Fermat surfaces S_2 and S_3 are del Pezzo surfaces, hence they are Mori dream spaces.

Remark 6.6. It was already known that the pseudo-effective cone $\overline{\text{Eff}}(S_4)$ of the Fermat quartic $S_4 \subset \mathbb{P}^3$ is non-polyhedral. In fact, as S_4 is a K3 surface with Picard number 20, from a result of Shioda and Inose [42, Theorem 5 in §5] (see also [39, §§17–18]), its automorphism group is infinite. As S_4 is projective, by a result of Pjateckiĭ–Šapiro–Šafarevič, and Sterk [24, Chapter 8, §4, Corollary 4.7] S_4 contains infinitely many (-2) -curves, which provide infinitely many extremal rays for its pseudo-effective cone.

There is another interesting fibration $g: S_4 \longrightarrow \mathbb{P}^1$ defined by the rational function

$$t = \frac{x_0^2 - x_1^2}{x_2^2 - x_3^2} = -\frac{x_2^2 + x_3^2}{x_0^2 + x_1^2}.$$

This is an elliptic fibration having six singular fibres, each a cycle of four lines of Kodaira type I_4 [35, Example p. 44]. Thus, we have

$$n(g) + 2 = 6 \cdot (4 - 1) + 2 = 20 = \rho(S_4).$$

This shows at once two interesting things:

- (1) the equality $\rho(S) = n(f) + 2$ in Theorem 3.4 is necessary, but not sufficient, for the polyhedrality of $\overline{\text{Eff}}(S)$;

- (2) if $\overline{\text{Eff}}(S)$ is non-polyhedral then S may admit two distinct fibrations with different values of $n(f)$: in fact $n(g) = 18$ whereas $n(f_4) \leq 16$. This is not possible if $\overline{\text{Eff}}(S)$ is polyhedral by Theorem 3.4.

7. PRODUCT-QUOTIENT SURFACES

If C is a smooth curve with $g(C) \geq 1$, we have seen (Proposition 4.1) that $\overline{\text{Eff}}(C \times C)$ is non-polyhedral. However, since $q(C \times C) = 2g(C) > 0$, we do not need to look at the pseudo-effective cone to conclude that $C \times C$ is not a Mori dream space.

To include regular surfaces in our analysis, we can slightly change the setting and investigate quotients of product of curves by the diagonal action of a finite group. These surfaces have been extensively studied in the literature [4–7, 11, 19–21, 28, 30, 31] and are known as *product-quotient surfaces*.

Definition 7.1. Let C_1, C_2 be two smooth projective curves and let G be a finite group acting faithfully on both of them. A *product-quotient surface* is the minimal resolution of singularities $\lambda: S \rightarrow X$ of the *quotient model* $X = (C_1 \times C_2)/G$, where G acts diagonally on the product.

The surface S comes naturally with two fibrations

$$f_1: S \rightarrow C_1/G, \quad f_2: S \rightarrow C_2/G,$$

whose general fibre we denote by F_1 and F_2 , respectively.

We refer to [40, Theorem 2.1], where the author studied the fibres of f_1 and f_2 .

In particular, one can easily observe that

$$F_1 \simeq C_2, \quad F_2 \simeq C_1 \quad \text{and} \quad F_1 F_2 = |G|,$$

and both fibrations f_1 and f_2 are *isotrivial*. Moreover

$$n(f_1) = n(f_2) = l(X)$$

where $l(X)$ is the number of irreducible curves contracted by λ , see [8, Definition 2.3]. Moreover, by [40, Proposition 2.2] we get

Proposition 7.2. *The irregularity of a product-quotient surface equals the sum $g(C_1/G) + g(C_2/G)$.*

7.1. Surfaces isogenous to a product, of unmixed type. We first consider the case where the action of G on $C_1 \times C_2$ is *free*. In this situation X is smooth and $S = X = (C_1 \times C_2)/G$ is called a *surface isogenous to product, of unmixed type*. If $g(C_1) \geq 2$ and $g(C_2) \geq 2$, then S is a minimal surface of general type.

Proposition 7.3. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product, of unmixed type. If $\rho(S) \geq 3$, then $\overline{\text{Eff}}(S)$ is non-polyhedral. In particular, S is not a Mori dream space.*

Proof. Since the action of G on $C_1 \times C_2$ is free, by [40, Theorem 2.1] all the singular fibres of f_i are multiple of smooth curves, hence $n(f_i) = 0$ and the claim follows from Corollary B. $\square$

Recall that a *non-trivial correspondence* in $C_1 \times C_2$ is a (possibly reducible) curve $\Gamma \subset C_1 \times C_2$ that dominates the bases of both natural projections $C_1 \times C_2 \rightarrow C_i$.

Proposition 7.4. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product, of unmixed type, and assume that there exists a non-trivial G -invariant correspondence $\Gamma \subset C_1 \times C_2$ such that $\Gamma^2 \leq 0$. Then $\overline{\text{Eff}}(S)$ is non-polyhedral.*

Proof. Let $\pi: C_1 \times C_2 \rightarrow S$ be the natural projection and set $B = \pi(\Gamma)$. Since Γ is G -invariant and π is unramified, we get $\pi^*(B) = \Gamma$. We claim that the three classes

$$F_1, F_2, B$$

are linearly independent in $N^1(S)$. In fact, suppose that there exists a linear relation

$$\lambda_1 F_1 + \lambda_2 F_2 = \mu B,$$

where $\lambda_1, \lambda_2, \mu \in \mathbb{R}$ and we can assume $\mu \geq 0$. Pulling back on $C_1 \times C_2$ via π , we get

$$\tau_1 C_2 + \tau_2 C_1 = \mu \Gamma,$$

where $\tau_1 = \lambda_1 \cdot |G|$, $\tau_2 = \lambda_2 \cdot |G|$. Intersecting with C_1 we get $\tau_1 \geq 0$, and intersecting with C_2 we get $\tau_2 \geq 0$. Thus, intersecting with Γ we get

$$0 \leq (\tau_1 C_2 + \tau_2 C_1) \Gamma = \mu \Gamma^2 \leq 0.$$

Then $(\tau_1 C_2 + \tau_2 C_1) \Gamma = 0$. As Γ is non-trivial, both products $C_i \Gamma$ are positive, and then $\tau_1 = \tau_2 = \mu = 0$ and so $\lambda_1 = \lambda_2 = 0$. This proves our claim, hence $\rho(S) \geq 3$ and we can conclude by using Proposition 7.3. $\square$

Remark 7.5. In Proposition 7.4, the G -invariance condition for Γ is an essential one. In fact, if $C_1 = C_2 = C$, then the diagonal $\Delta \subset C \times C$ is a correspondence having self-intersection $\Delta^2 = 2 - 2g(C)$, which is negative as soon as $g(C) \geq 2$. However, there are examples of diagonal G -actions where Δ is not G -invariant and $S = (C \times C)/G$ is a Mori dream space; one of them is Beauville's surface described in Remark 7.8.

We now give some examples of surfaces to which Proposition 7.4 may be applied. The first example is due to J. Starr, see [32].

Example (J. Starr) 7.6. Let $p \geq 7$ be a prime number and consider the finite group $G = \mathbb{Z}_p \times \mathbb{Z}_p$. The three elements

$$g_1 = (1, 0), \quad g_2 = (0, -1), \quad g_3 = (-1, 1)$$

generate G and satisfy $g_1 + g_2 + g_3 = 0$, hence there is an associated G -cover

$$\pi: C \rightarrow \mathbb{P}^1,$$

branched over $0, 1, \infty$ such that

- the stabilizer at every preimage of 0 is $\langle g_1 \rangle$;
- the stabilizer at every preimage of 1 is $\langle g_2 \rangle$;

- the stabilizer at every preimage of ∞ is $\langle g_3 \rangle$.

By the Riemann-Hurwitz formula, $g(C) = \frac{1}{2}(p-1)(p-2)$. Now, let us consider the automorphism $\phi: G \rightarrow G$ given by the matrix

$$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

We have

$$\phi(g_1) = (2, 1), \quad \phi(g_2) = (3, 2), \quad \phi(g_3) = (-5, -3)$$

and so, since $p \geq 7$,

$$\langle g_i \rangle \cap \langle \phi(g_j) \rangle = \{(0, 0)\} \quad \text{for all } i, j.$$

Therefore the action of G on $C \times C$ defined by

$$g \cdot (x, y) := (g \cdot x, \phi(g) \cdot y)$$

is free, and we obtain a surface isogenous to a product, of unmixed type

$$S = (C \times C)/G,$$

which is minimal, of general type and satisfies $q(S) = 0$ by Proposition 7.2.

Moreover (and this will be relevant in the sequel) the element $1 \in \mathbb{Z}_p$ is an eigenvalue for ϕ , and we have

$$\ker(\phi - \mathrm{id}) = \langle (3, 1) \rangle, \quad \mathrm{im}(\phi - \mathrm{id}) = \langle (1, 1) \rangle.$$

For any $g \in G$, we can consider the graph of g

$$\Gamma_g = \{(x, g \cdot x) \mid x \in C\} \subset C \times C.$$

It is a smooth divisor isomorphic to C and such that $\Gamma_g^2 = 2 - 2g(C)$. Note that $\Gamma_{(0,0)}$ is the diagonal $\Delta \subset C \times C$, and that

$$(7.1) \quad \Gamma_{h_1} \cap \Gamma_{h_2} \neq \emptyset \quad \text{if and only if} \quad h_1 h_2^{-1} \in \langle g_1 \rangle \cup \langle g_2 \rangle \cup \langle g_3 \rangle.$$

Moreover, for all $g \in G$, we have

$$\begin{aligned} g \cdot \Delta &= \{(g \cdot x, \phi(g) \cdot x) \mid x \in C\} \\ &= \{(y, \phi(g)g^{-1} \cdot y) \mid y \in C\} = \Gamma_{\phi(g)g^{-1}}. \end{aligned}$$

Theorem 7.7. *For all prime numbers $p \geq 7$, the surface S constructed above is not a Mori dream space, as its pseudo-effective cone is non-polyhedral.*

Proof. Let us consider the correspondence

$$\Gamma := \sum_{h \in \mathrm{im}(\phi - \mathrm{id})} \Gamma_h.$$

It is G -invariant since

$$\sum_{g \in G} g \cdot \Delta = \sum_{g \in G} \Gamma_{\phi(g)g^{-1}} = p\Gamma.$$

Since $\text{im}(\phi - \text{id})$ intersects each stabilizer trivially, for distinct $h_1, h_2 \in \text{im}(\phi - \text{id})$ relation (7.1) gives $\Gamma_{h_1} \Gamma_{h_2} = 0$. Thus we infer

$$\Gamma^2 = \left(\sum_{h \in \text{im}(\phi - \text{id})} \Gamma_h \right)^2 = \sum_{h \in \text{im}(\phi - \text{id})} \Gamma_h^2 = p(2 - 2g(C)) < 0,$$

and the claim follows from Proposition 7.4. $\square$

Remark 7.8. For $p \leq 5$ it is not possible to find a matrix $A \in \text{GL}_2(\mathbb{Z}_p)$ fulfilling all the conditions in the example above. In fact, for $p \leq 3$ we can't even make the diagonal action free. For $p = 5$ this is instead possible, and we get the famous example with $p_g = q = 0$, $K^2 = 8$ presented by Beauville in [9, Ex. X.13(4)]. However, for such a surface 1 is not an eigenvalue of the matrix A , so the construction in the example does not apply. Indeed, it was proved that Beauville's example is a Mori dream space, see [25, Prop. 3.4].

More examples of surfaces to which Proposition 7.4 may be applied can be constructed as follows.

Proposition 7.9. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product, of unmixed type, let H be a normal subgroup of G and set Q for the group quotient G/H .*

Assume that the two Q -covers

$$\eta_i: C_i/H \longrightarrow C_i/G, \quad i = 1, 2$$

are isomorphic, and $g(C_i/H) \geq 1$. Then S is not a Mori dream space, as its pseudo-effective cone is non-polyhedral.

Proof. After choosing an isomorphism of the two Q -covers, we identify $C = C_1/H = C_2/H$ and $\eta = \eta_1 = \eta_2$. By assumption there are two factorizations

$$\begin{array}{ccc} C_1 & \xrightarrow{\rho_1} & C \\ & \searrow \varphi_1 & \swarrow \eta \\ & C_1/G & \end{array} \quad \begin{array}{ccc} C_2 & \xrightarrow{\rho_2} & C \\ & \searrow \varphi_2 & \swarrow \eta \\ & C_2/G & \end{array}$$

of the G -covers $\varphi_i: C_i \longrightarrow C_i/G$. These factorizations induce in turn a commutative diagram

$$\begin{array}{ccc} C_1 \times C_2 & \xrightarrow{\rho} & C \times C \\ \pi \downarrow & & \downarrow \bar{\pi} \\ S = (C_1 \times C_2)/G & \xrightarrow{\bar{\rho}} & (C \times C)/Q, \end{array}$$

where $\rho = \rho_1 \times \rho_2$ is an $(H \times H)$ -cover and Q acts diagonally on $C \times C$. Since the action of Q on the two factors of $C \times C$ is the same, the diagonal $\Delta \subset C \times C$ is Q -invariant. Moreover, the surface $(C \times C)/Q$ has only isolated cyclic quotient

singularities, hence it is $\mathbb{Q}$ -factorial and the pull-back of its Weil divisors is well-defined. Setting

$$\Gamma = \rho^*(\Delta) \subset C_1 \times C_2, \quad R = \bar{\pi}(\Delta) \subset (C \times C)/Q,$$

we get

$$\Gamma = \rho^*(\Delta) = \rho^*(\bar{\pi}^*R) = \pi^*(\bar{\rho}^*R),$$

that is, the correspondence Γ is G -invariant. Finally, since $g(C) \geq 1$ we have $\Delta^2 \leq 0$ and consequently $\Gamma^2 \leq 0$, so we can conclude using Proposition 7.4. $\square$

We use it to construct a family of more examples: the case $p = 3$ was explained to us by W. Sawin in [32].

Example (W. Sawin) 7.10. Let p be a prime number and take three homogeneous polynomials $f(x_0, x_1), g(x_0, x_1), h(x_0, x_1)$ of degree p such that the product fgh has $3p$ distinct roots in $\mathbb{P}^1$.

Take $\mathbb{P}^3$ with homogeneous coordinates x_0, x_1, y_0, y_1 and let $C_1 \subset \mathbb{P}^3$ be the curve

$$\begin{cases} y_0^p = f \\ y_1^p = g \end{cases}$$

The curve C_1 is smooth by the assumption on fgh . We consider the faithful action of $G := \mathbb{Z}_p \times \mathbb{Z}_p$ on C_1 induced by the following action on $\mathbb{P}^3$:

$$(7.2) \quad (a, b) \cdot (x_0, x_1, y_0, y_1) = (x_0, x_1, \xi^a y_0, \xi^b y_1)$$

where $\xi = e^{2\pi i/p}$. The induced G -cover $\varphi_1: C_1 \rightarrow C_1/G \simeq \mathbb{P}^1$ is simply given by $(x_0, x_1, y_0, y_1) \mapsto (x_0, x_1)$.

Take now the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ with weighted coordinates x_0, x_1, z_0, z_1 : here z_1 is the variable of weight 2. Let $C_2 \subset \mathbb{P}(1, 1, 1, 2)$ be the curve

$$\begin{cases} z_0^p = h \\ z_1^p = fg \end{cases}$$

The curve C_2 is smooth by the assumption on fgh . We consider the action of G on C_2 induced by the following action on $\mathbb{P}(1, 1, 1, 2)$:

$$(7.3) \quad (a, b) \cdot (x_0, x_1, z_0, z_1) = (x_0, x_1, \xi^{a-b} z_0, \xi^{a+b} z_1)$$

Here we need to assume $p \geq 3$ to ensure that the action (7.3) is faithful, obtaining a further G -cover $\varphi_2: C_2 \rightarrow C_2/G \simeq \mathbb{P}^1$ also given by (x_0, x_1) . Note that if $p = 2$ then the element $(1, 1) \in G$ would act trivially on C_2 .

Set H for the cyclic subgroup of G generated by $(1, -1)$.

The subring of the elements of $\mathbb{C}[x_0, x_1, y_0, y_1]/(y_0^p - f, y_1^p - g)$ that are H -invariant by the action (7.2) is generated by x_0, x_1 , and $y_0 y_1$. Setting $y := y_0 y_1$ we can write C_1/H as the smooth curve $\{y^p = fg\} \subset \mathbb{P}(1, 1, 2)$ and the cover $\eta_1: C_1/H \rightarrow \mathbb{P}^1$ is given again by $(x_0, x_1, y_0, y_1) \mapsto (x_0, x_1)$.

The subring of the elements of $\mathbb{C}[x_0, x_1, z_0, z_1]/(z_0^p - h, z_1^p - fg)$ that are H -invariant with respect to the action (7.3) is generated by x_0, x_1 and z_1 : we can write C_2/H as the smooth curve $\{z_1^p = fg\} \subset \mathbb{P}(1, 1, 2)$ and the cover $\eta_2: C_2/H \rightarrow \mathbb{P}^1$ is given again by $(x_0, x_1, y_0, y_1) \mapsto (x_0, x_1)$. This shows that the G/H -covers η_1 and η_2 are isomorphic.

Theorem 7.11. *For all prime numbers $p \geq 3$, consider the actions of G on C_1 and C_2 in (7.2) and (7.3), and let S be the surface $S = (C_1 \times C_2)/G$ given by the diagonal action. Then S is a regular surface isogenous to a product, of unmixed type, which is not a Mori dream space, as its pseudo-effective cone is non-polyhedral.*

Proof. It is easy to deduce from (7.2), as x_0 and x_1 do not vanish simultaneously on C_1 , that the elements of G fixing some point of C_1 are exactly those of the form $(a, 0)$ or $(0, b)$. By (7.3), the only element of this form fixing a point of C_2 is $(0, 0)$; thus the diagonal action of G on $C_1 \times C_2$ is free and S is isogenous to a product of unmixed type. Moreover, $q(S) = 0$ by Proposition 7.2.

So we have shown that the G/H -covers $\eta_i: C_i/H \rightarrow C_i/G$ are isomorphic. The result now follows by Proposition 7.9, since $g(C_i/H) = (p-1)^2 \geq 4$. $\square$

7.2. Product-quotient surfaces with geometric genus zero. In [8, Definitions 2.1-2.3] the authors associate to every surface X whose singular points are cyclic quotient singularities a rational number $\gamma(X)$ that only depends on the singularities of X . They prove the following, see [8, Prop. 4.2],

Proposition 7.12. *If X is the quotient model of a product-quotient surface S , then $\gamma(X)$ is integral with $p_g(S) + \gamma(X) \geq 0$ and*

$$h^{1,1}(S) - n(f_i) - 2 = 2(p_g(S) + \gamma(X)).$$

Thus, in this case Theorem A has the following reformulation.

Corollary 7.13. *Let S be a product-quotient surface with quotient model X . If $\overline{\text{Eff}}(S)$ is polyhedral, then*

$$h^{1,1}(S) - \rho(S) = 2(p_g(S) + \gamma(X)).$$

If moreover $p_g(S) = 0$, by the Lefschetz $(1, 1)$ -theorem we have $\rho(S) = h^{1,1}(S)$. Thus by Corollary B we obtain

Theorem 7.14. *Let S be a product-quotient surface with $p_g(S) = 0$. If $\gamma(X) > 0$ then $\overline{\text{Eff}}(S)$ is non-polyhedral, hence S is not a Mori dream space.*

In [8, Tables 2–5] one can find more than 100 families of surfaces with $p_g = q = 0$ and $\gamma > 0$. By Theorem 7.14, these surfaces are not Mori dream spaces. Moreover, none of them is of general type [8, Subsection 7.3].

To the best of our knowledge, only four examples of product-quotient surfaces of general type with $p_g = q = 0$ and $\gamma > 0$ are known to exist; they are described in Table 1. By Theorem 7.14, they are not Mori dream spaces.

Note that these examples are necessarily non-minimal (Remark 7.15). It is an interesting open problem to determine whether their minimal models are Mori dream spaces.

The last two examples, namely those with $\gamma = 2$, contain precisely two (-1) -curves E_1, E_2 , such that E_i is contracted by $f_i: S \rightarrow C_i/G$. Contracting either E_1 or E_2 we get a surface S' with $p_g(S') = q(S') = 0$, $K_{S'}^2 = 0$ which still satisfies the assumption of Corollary 3.5.² So S' is not a Mori dream space, either.

γ	K_S^2	$K_{\tilde{S}}^2$	G	$\text{Sing}(X)$	t_1	t_2	Reference
1	1	3	$\text{PSL}(2, 7)$	$1/7, 2/7, 4/7$	3, 3, 7	2, 4, 7	[7, Sec. 5]
1	0	1	$\text{GL}(2, 4)$	$2 \times 1/6, 2 \times 1/2, 2/3$	4, 4, 6	2, 4, 6	[8, Sec. 7.1].
2	-1	1	$\mathbb{Z}_5 \times \mathbb{Z}_5$	$5 \times 1/5$	5, 5, 5	5, 5, 5	[8, Sec. 7.2].
2	-1	1	$\mathbb{Z}_5 \times \mathbb{Z}_5$	$5 \times 1/5$	5, 5, 5	5, 5, 5	[20, Thm. 5.5]

TABLE 1. Known examples of product-quotient surfaces of general type with $p_g = q = 0$ and $\gamma > 0$. Here $\tilde{S}$ is the minimal model of S , whereas t_i denotes the signatures of the cover $f_i: C_i \rightarrow C_i/G \simeq \mathbb{P}^1$.

Remark 7.15. A product-quotient surface S of general type with $p_g(S) = q(S) = 0$ is minimal if and only if $\gamma(X) = 0$, see [8, Theorem 6.2]. These surfaces are listed in [7, Tables 1 and 2], and it is an interesting problem to classify those which are Mori dream spaces.

8. THE CRAIGHERO-GATTAZZO SURFACE

The Craighero-Gattazzo surface V is a particular numerical Godeaux surface, namely, a minimal surface of general type with

$$p_g(V) = q(V) = 0, \quad K_V^2 = 1.$$

It was first constructed by Craighero and Gattazzo [15]. Later, it was studied by Dolgachev and Werner [17] who proved, among other things, the vanishing of $H_1(V, \mathbb{Z})$ and claimed that V is simply connected. The proof of this last result contained a gap [18], and the fact that V is simply connected has been proved only subsequently by Rana, Tevelev and Urzúa [37], using different methods.

The aim of this section is to show that the Craighero-Gattazzo surface is not a Mori dream space by applying Corollary B to a special genus 2 pencil on V , studied in [17]. We will use the notation of that paper throughout.

The Craighero-Gattazzo surface V is defined as the minimal resolution of singularities of a quintic surface $S \subset \mathbb{P}^3$, having four elliptic singularities, see [17, Section 2]. The quintic S is invariant with respect to the order 4 automorphism of

²Given a fibration $f: S \rightarrow C$ of a surface to a curve, and a (-1) -curve $E \subset S$ contracted by f , then $f = f' \circ c$ where $c: S \rightarrow S'$ is the contraction of E and f' is a fibration on S' . As $\rho(S) = \rho(S') + 1$ and $n(f) = n(f') + 1$ we see that $\rho(S) > n(f) + 2$ if and only if $\rho(S') > n(f') + 2$.

$\mathbb{P}^3$

$$\sigma: [x:y:z:t] \mapsto [t:x:y:z]$$

which has the four fixed points

$$\begin{aligned} P_0 &= [1:1:1:1], & P_1 &= [1:i:-1:-i], \\ P_2 &= [1:-i:-1:i], & Q_0 &= [1:-1:1:-1]. \end{aligned}$$

The set of fixed points of

$$\sigma^2: [x:y:z:t] \mapsto [z:t:x:y]$$

is equal to the union of the two lines

$$\begin{aligned} r &= \{x+z=y+t=0\} = \langle P_1, P_2 \rangle \\ r' &= \{x-z=y-t=0\} = \langle P_0, Q_0 \rangle, \end{aligned}$$

and the elliptic singularities of S are at the four reference points

$$a_1 = [1:0:0:0], \quad a_2 = [0:1:0:0], \quad a_3 = [0:0:1:0], \quad a_4 = [0:0:0:1].$$

Let us now consider the minimal resolution of singularities

$$\pi: V \longrightarrow S$$

and let $E_i := \pi^{-1}(a_i)$. Then the E_i are four elliptic curves such that

$$E_i^2 = -1, \quad K_V E_i = 1$$

and, by adjunction,

$$K_V = \pi^* H - E_1 - E_2 - E_3 - E_4,$$

where H is a hyperplane section of the quintic surface S .

The reader can find in [17, Section 2] a simple proof that V is of general type with $K_V^2 = 1$, $p_g(V) = q(V) = 0$. Moreover, we will later need that V does not contain any (-2) -curve, [17, Theorem 6.2]: in other words, K_V is ample.

The automorphisms σ and σ^2 on S both lift to automorphisms on V , which we denote by the same symbols. The automorphism $\sigma: V \longrightarrow V$ cyclically permutes the exceptional elliptic curves E_1, E_2, E_3, E_4 , hence

$$(8.1) \quad \sigma^2(E_1) = E_3, \quad \sigma^2(E_2) = E_4.$$

The line r is contained in the quintic S , and we call

$$R = \pi^{-1}(r)$$

the inverse image of r in V . This is a smooth rational curve, fixed by the involution $\sigma^2: V \longrightarrow V$ and such that

$$K_V R = 1, \quad R^2 = -3.$$

The isolated fixed points of σ^2 on V form a set $\Sigma = \{Q_0, Q_1, Q_2, Q_3, Q_4\}$, whose five elements are the preimages of the five points where r' intersects S .

Now we consider the complete linear system

$$|3K_V - R|$$

on V . It is a pencil without fixed part [17, Proposition 3.2]. Since $F^2 = (3K_V - R)^2 = 0$, such pencil has no base points at all, so it defines a morphism

$$f: V \longrightarrow \mathbb{P}^1.$$

By the adjunction formula, the general fibre of f is a smooth curve of genus 2, since $(3K_V - R)K_V = 2$. We are interested in the reducible fibres of f . As K_V is ample, a fibre F of $f: V \longrightarrow \mathbb{P}^1$ has at most $K_V F = 2$ irreducible components. In particular, if F is reducible, we have $F = A + A'$, with $K_V A = K_V A' = 1$. Moreover, since $(A + A')A = (A + A')A' = 0$, and $0 \leq p_a(A) = \frac{1}{2}(3 - AA')$, we get only two numerical possibilities:

- (a) $AA' = -A^2 = -(A')^2 = 1$;
- (b) $AA' = -A^2 = -(A')^2 = 3$.

Lemma 8.1. *Case (b) does not occur.*

Proof. We have seen that the fixed locus of the involution $\sigma^2: V \longrightarrow V$ consists of the disjoint union of the smooth rational curve R and the five isolated points in Σ . Then, using the topological fixed point formula [27, Theorem 1] and recalling that $q(V) = 0$ implies $H^1(V, \mathbb{Q}) = H^3(V, \mathbb{Q}) = 0$, we get

$$\begin{aligned} 2 + 5 &= \chi_{\text{top}}(\text{Fix}(\sigma^2)) = \sum (-1)^q \text{trace}((\sigma^2)^* | H^q(V, \mathbb{Q})) \\ &= 2 + \text{trace}((\sigma^2)^* | H^2(V, \mathbb{Q})), \end{aligned}$$

that is, $\text{trace}((\sigma^2)^* | H^2(V, \mathbb{Q})) = 5$.

On the other hand, since $p_g(V) = q(V) = 0$ and $\chi_{\text{top}}(V) = 11$, we infer $b_2(V) = 9$.

Thus, the invariant and anti-invariant subspaces $H^2(V, \mathbb{Q})^+$ and $H^2(V, \mathbb{Q})^-$ for the involution $(\sigma^2)^*: H^2(V, \mathbb{Q}) \longrightarrow H^2(V, \mathbb{Q})$ have respective dimensions

$$\dim H^2(V, \mathbb{Q})^+ = 7, \quad \dim H^2(V, \mathbb{Q})^- = 2.$$

By (8.1), a basis for $H^2(V, \mathbb{Q})^-$ consists of

$$\{E_1 - E_3, E_2 - E_4\}.$$

Assume now by contradiction that there exists a reducible fibre $A + A'$ as in (b). Then

- $A + A'$ does not contain any of the five isolated fixed points of σ^2 , as they lie on fibres of type (a) as shown in [17, Lemma 3.5, Corollary 3.6].³
- $A^2 = (A')^2 = -3$ is odd, hence A and A' are not σ^2 -invariant [17, Lemma 3.4];
- $A + A'$ is σ^2 -invariant, as all members of $|3K_V - R|$ are so [17, p. 742].

³As K_V is ample, the cases (ii) and (iii) in [17, Lemma 3.5] do not occur; cf. [17, p. 751 and Remark 3.9].

Thus $A' = \sigma^2(A)$, that is $A - A' \in H^2(V, \mathbb{Q})^-$. Therefore there exist $\lambda, \mu \in \mathbb{Q}$ such that, in $H^2(V, \mathbb{Q})$:

$$A - A' = \lambda(E_1 - E_3) + \mu(E_2 - E_4).$$

Since $\lambda = (A - A')E_3$ and $\mu = (A - A')E_4$, both λ and μ are integers. Then

$$2(\lambda^2 + \mu^2) = -(A - A')^2 = -(A^2 - 2AA' + (A')^2) = 12$$

provides the desired contradiction, because the equation $\lambda^2 + \mu^2 = 6$ has no integral solutions. $\square$

So, all reducible fibres of f are of type (a). We are now in a position to prove the main result of this section.

Theorem 8.2. *The Craighero-Gattazzo surface V is not a Mori dream space, as its pseudo-effective cone is non-polyhedral.*

Proof. All reducible fibres of f are of type (a), hence they are not 2-connected.

By the theory of genus 2 fibrations [12, Theorem 4.13, Remark 4.14], the number of fibres of f that are not 2-connected is at most $K_V^2 - 2\chi(\mathcal{O}_V) + 6 = 5$. However [17, Lemma 3.5, Corollary 3.6] shows that the five fibres containing a point of Σ are of type (a): then f has exactly five reducible fibres and $n(f) = 5$.

On the other hand, since $p_g(V) = q(V) = 0$, we get

$$\rho(V) = h^{1,1}(V) = b_2(V) = 9,$$

so the result follows from Corollary B. $\square$

ACKNOWLEDGEMENTS

All authors were partially supported by INdAM-GNSAGA. The first author held a research grant from INdAM, Istituto Nazionale di Alta Matematica. The second author thanks C. Fontanari for first pointing out the paper [25] to him and for some interesting discussions on the problems raised there. We thank A. Rabinathan for useful discussions. We are also indebted to W. Sawin and J. Starr, who answered our questions on MathOverflow [32].

REFERENCES

- [1] M. Artebani, J. Hausen, and A. Laface, *On Cox rings of K3 surfaces*, Compos. Math. **146** (2010), no. 4, 964–998, DOI 10.1112/S0010437X09004576. [↑2,2](#)
- [2] M. Artebani and A. Laface, *Cox rings of surfaces and the anticanonical Iitaka dimension*, Adv. Math. **226** (2011), no. 6, 5252–5267, DOI 10.1016/j.aim.2011.01.007. [↑2](#)
- [3] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., Vol. 4, Springer-Verlag, Berlin, 2004. [↑3, 5, 6](#)
- [4] I. Bauer and F. Catanese, *Some new surfaces with $p_g = q = 0$* , The Fano Conference, Univ. Torino, Turin, 2004, pp. 123–142. [↑7](#)
- [5] I. Bauer, F. Catanese, and F. Grunewald, *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves*, Pure Appl. Math. Q. **4** (2008), no. 2, Special Issue: In honor of Fedor Bogomolov., 547–586, DOI 10.4310/PAMQ.2008.v4.n2.a10. [↑7](#)

- [6] I. Bauer, F. Catanese, F. Grunewald, and R. Pignatelli, *Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups*, Amer. J. Math. **134** (2012), no. 4, 993–1049, DOI 10.1353/ajm.2012.0029. ↑7
- [7] I. Bauer and R. Pignatelli, *The classification of minimal product-quotient surfaces with $p_g = 0$* , Math. Comp. **81** (2012), no. 280, 2389–2418, DOI 10.1090/S0025-5718-2012-02604-4. ↑7, 7.2, 7.15
- [8] ———, *Product-quotient surfaces: new invariants and algorithms*, Groups Geom. Dyn. **10** (2016), no. 1, 319–363, DOI 10.4171/GGD/351. ↑7, 7.2, 7.2, 7.15
- [9] A. Beauville, *Complex algebraic surfaces*, 2nd ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. ↑7.8
- [10] P. Cascini, F. Catanese, Y. Chen, and J. Keum, *Mori dream spaces and $\mathbf{Q}$ -homology quadrics*, Pacific J. Math. **339** (2025), no. 2, 223–242, DOI 10.2140/pjm.2025.339.223. ↑1
- [11] F. Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces*, Amer. J. Math. **122** (2000), no. 1, 1–44. ↑7
- [12] F. Catanese and R. Pignatelli, *Fibrations of low genus. I*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 1011–1049, DOI 10.1016/j.ansens.2006.10.001 (English, with English and French summaries). MR2316980 ↑8
- [13] F. Catanese and S. Rollenske, *Double Kodaira fibrations*, J. Reine Angew. Math. **628** (2009), 205–233, DOI 10.1515/CRELLE.2009.024. ↑4
- [14] A. Causin and F. Polizzi, *Surface braid groups, finite Heisenberg covers and double Kodaira fibrations*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **22** (2021), no. 3, 1309–1352, DOI 10.2422/2036-2145.201908_004. ↑4
- [15] P. C. Craighero and R. Gattazzo, *Quintic surfaces of $\mathbf{P}^3$ having a nonsingular model with $q = p_g = 0$, $P_2 \neq 0$* , Rend. Sem. Mat. Univ. Padova **91** (1994), 187–198. ↑8
- [16] O. Debarre, Z. Jiang, and C. Voisin, *Pseudo-effective classes and pushforwards*, Pure Appl. Math. Q. **9** (2013), no. 4, 643–664, DOI 10.4310/PAMQ.2013.v9.n4.a3. ↑3
- [17] I. Dolgachev and C. Werner, *A simply connected numerical Godeaux surface with ample canonical class*, J. Algebraic Geom. **8** (1999), no. 4, 737–764. ↑8, 8, 8, 3, 8
- [18] ———, *Erratum to: “A simply connected numerical Godeaux surface with ample canonical class”* [J. Algebraic Geom. **8** (1999), no. 4, 737–764], J. Algebraic Geom. **10** (2001), no. 2, 397. ↑8
- [19] F. Fallucca, *Examples of surfaces with canonical maps of degree 12, 13, 15, 16 and 18*, Ann. Mat. Pura Appl. (4) **203** (2024), no. 3, 1015–1024, DOI 10.1007/s10231-023-01363-6. ↑7
- [20] ———, *On the classification of product-quotient surfaces with $q = 0$, $p_g = 3$ and their canonical map*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **35** (2024), no. 4, 529–596, DOI 10.4171/rfm/1051. ↑7, 7.2
- [21] F. Fallucca and C. Gleissner, *Some surfaces with canonical maps of degrees 10, 11, and 14*, Math. Nachr. **296** (2023), no. 11, 5063–5069, DOI 10.1002/mana.202200450. ↑7
- [22] E. Horikawa, *Algebraic surfaces of general type with small c_1^2 . I*, Ann. of Math. (2) **104** (1976), no. 2, 357–387, DOI 10.2307/1971050. ↑5
- [23] Y. Hu and S. Keel, *Mori dream spaces and GIT.*, Mich. Math. J. **48** (2000), 331–348, DOI 10.1307/mmj/1030132722. ↑1, 2
- [24] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR3586372 ↑6.6
- [25] J. Keum and K.-S. Lee, *Examples of Mori dream surfaces of general type with $p_g = 0$* , Adv. Math. **347** (2019), 708–738, DOI 10.1016/j.aim.2019.02.028. ↑1, 7.8, 8
- [26] S. L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344, DOI 10.2307/1970447. ↑3
- [27] S. Kwasik and R. Schultz, *Homological properties of periodic homeomorphisms of 4-manifolds*, Duke Math. J. **58** (1989), no. 1, 241–250, DOI 10.1215/S0012-7094-89-05812-2. ↑8

- [28] E. Mistretta and F. Polizzi, *Standard isotrivial fibrations with $p_g = q = 1$. II*, J. Pure Appl. Algebra **214** (2010), no. 4, 344–369, DOI 10.1016/j.jpaa.2009.05.010. [↑7](#)
- [29] A. Laface, S. Li, and J. Liu, *Mori dream Jacobian elliptic surfaces of Kodaira dimension one*, arXiv:2608.09002 (2026). [↑3](#)
- [30] F. Polizzi, *Numerical properties of isotrivial fibrations*, Geom. Dedicata **147** (2010), 323–355, DOI 10.1007/s10711-010-9457-z. [↑7](#)
- [31] ———, *Isotrivially fibred surfaces and their numerical invariants*, Beauville surfaces and groups, Springer Proc. Math. Stat., vol. 123, Springer, Cham, 2015, pp. 171–183, DOI 10.1007/978-3-319-13862-6_11. [↑7](#)
- [32] ———, *G-invariant correspondence in a product of curves*, MathOverflow question 511245 (2026). [↑7.1](#), [7.1](#), [8](#)
- [33] F. Polizzi and P. Sabatino, *Extra-special quotients of surface braid groups and double Kodaira fibrations with small signature*, Geom. Dedicata **216** (2022), no. 6, DOI 10.1007/s10711-022-00720-8. Id/No 65. [↑4](#)
- [34] ———, *Groups of order 64 and non-homeomorphic double Kodaira fibrations with the same biregular invariants*, Ann. Mat. Pura Appl., posted on 2026, DOI 10.1007/s10231-026-01691-3. [↑4](#)
- [35] A. Rabindranath, *Pseudoeffective Cones and Morphisms of Projective Varieties*, Ph.D. Thesis, University of Michigan, Michigan, 2018. [↑3](#), [6.6](#)
- [36] ———, *Some surfaces with non-polyhedral nef cones*, Proc. Amer. Math. Soc. **147** (2019), no. 1, 15–20, DOI 10.1090/proc/13296. [↑3](#), [4](#)
- [37] J. Rana, J. Tevelev, and G. Urzúa, *The Craighero-Gattazzo surface is simply connected*, Compos. Math. **153** (2017), no. 3, 557–585, DOI 10.1112/S0010437X16008125. [↑8](#)
- [38] M. Schütt and T. Shioda, *Elliptic surfaces*, Algebraic geometry in East Asia—Seoul 2008, Adv. Stud. Pure Math., vol. 60, Math. Soc. Japan, Tokyo, 2010, pp. 51–160, DOI 10.2969/aspm/06010051. [↑3](#)
- [39] B. Segre, *On the quartic surface $x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$* , Proc. Cambridge Philos. Soc. **40** (1944), 121–145. MR0011003 [↑6.6](#)
- [40] F. Serrano, *Isotrivial fibred surfaces*, Ann. Mat. Pura Appl. (4) **171** (1996), 63–81, DOI 10.1007/BF01759382. [↑7](#), [7.1](#)
- [41] T. Shioda, *On the Picard number of a Fermat surface*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 725–734 (1982). [↑6](#)
- [42] T. Shioda and H. Inose, *On singular K3 surfaces*, Complex analysis and algebraic geometry, Iwanami Shoten Publishers, Tokyo, 1977, pp. 119–136. MR0441982 [↑6.6](#)
- [43] F. L. Zak, *The structure of Gauss mappings*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 1, 39–50 (Russian). MR0888013 [↑6.1](#)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, VIA SOMMARIVE 14, I-38123 TRENTO, ITALY.

Email address: federico.fallucca@unitn.it

Email address: fallucca@altamatematica.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, VIA SOMMARIVE 14, I-38123 TRENTO, ITALY.

Email address: Roberto.Pignatelli@unitn.it

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "RENATO CACIOPPOLI", UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II, VIA CINTIA, MONTE S.ANGELO, I-80126 NAPOLI, ITALY.

Email address: francesco.polizzi@unina.it