WEIGHTED HYPERSURFACES WITH EITHER ASSIGNED VOLUME OR MANY VANISHING PLURIGENERA

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Abstract. In this paper we construct, for every n, smooth varieties of general type of dimension n with the first $\lfloor \frac{n-2}{3} \rfloor$ plurigenera equal to zero. Hacon-McKernan, Takayama and Tsuji have recently shown that there are numbers r_n such that $\forall r \geq r_n$, the $r-$canonical map of every variety of general type of dimension n is birational. Our examples show that r_n grows at least quadratically as a function of n. Moreover they show that the minimal volume of a variety of general type of dimension n is smaller than $\frac{3^{n+1}}{(n-1)^n}$.

In addition we prove that for every positive rational number q there are smooth varieties of general type with volume q and dimension arbitrarily big.

1. Introduction

Let X be a smooth variety of general type (we always intend projective over $\mathbb{C}$). Since the canonical divisor K_X is intrinsically associated to X, the study of the pluricanonical systems $|rK_X|$, of the induced maps ϕ_r, and of the canonical ring $R(X) := \oplus H^0(X, mK_X)$ is a classical and important matter. Further these objects are birational invariants.

It is natural to ask how "small" can the Hilbert function of $R(X)$ be. There are many different possible definition of "small"; we are mainly interested in two of them. First, we would like to consider varieties of small canonical volume. Recall that $vol(X) := \limsup_{m\to+\infty} \frac{n! h^0(X,mK_X)}{m^n}$, so making the volume small is the same as making small the asymptotical behaviour of the Hilbert function. Second, we would like to understand which plurigenera $P_m := h^0(X, mK_X)$ may be zero.

For curves of general type $vol(X) \geq 2$ and K_X is effective; for surfaces $vol(X) \geq 1$ and $P_2 \neq 0$, while there are surfaces of general type with $P_1 = 0$. For threefolds the record from both points of view is attained by an example of Iano-Fletcher (see 15.1 of [If00]) with volume $\frac{1}{420}$ and $P_1 = P_2 = P_3 = 0$. See also [CC10a] and [CC10b] for related results in dimension 3.

In higher dimension we are able to prove the following.

Theorem 1. *Let $n \geq 5$ be an integer. There exists a smooth variety of general type X of dimension n such that $H^0(X, mK_X) = 0$ for $0 < m < \lfloor \frac{n+1}{3} \rfloor$ and $vol(X) < \frac{3^{n+1}}{(n-1)^n}$.*

Recently the following result, generalisation of a famous theorem of [Bo73], was proven in [HM06], [Tak06] and [Tsu07].

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Theorem. *For any positive integer n there exists an integer r_n such that if X is a smooth variety of general type and dimension n, then*

$$\phi_r : X \dashrightarrow \mathbb{P}(H^0(X, rK_X))$$

is birational onto its image for all $r \geq r_n$.

A consequence of this theorem is that for any smooth variety of general type X of dimension n we have

$$\mathrm{vol}(K_X) \geq \frac{1}{r_n^n}.$$

and that the set of the volumes of the manifolds of general type of dimension n has a minimum $v_n > 0$.

An instant consequence of Theorem 1 is the following

Corollary 1. *Let v_n be the minimal volume of an n-dimensional smooth variety of general type. Then $\lim_{n\to\infty} v_n = 0$.*

A natural problem is to estimate r_n. It is well known that $r_1 = 3$ and $r_2 = 5$. By the mentioned example of Iano-Fletcher we know that $r_3 \geq 27$ (see 15.1 of [If00]). Let x'_n be the minimal positive integer such that for every n-dimensional smooth variety X of general type there is an integer $t \leq x'_n$ such that ϕ_t is generically finite; obviously $r_n \geq x'_n$.

The examples of Theorem 1 provide a lower bound for x'_n (and therefore for r_n) which is quadratic in n. More precisely

Theorem 2. *For any integer $n \geq 7$ we have*

$$r_n \geq x'_n \geq \frac{n(n-3)}{9},$$

In particular

$$\lim_{n\to+\infty} r_n = \lim_{n\to+\infty} x'_n = +\infty$$

The canonical system of these varieties is not ample. In view of Fujita's conjecture, smooth varieties with ample canonical system should not give anything better than a linear bound. We show

Theorem 3. *For any positive integer $n > 0$ there is a smooth variety X of dimension n such that K_X is ample and $\phi_{|tK_X|}$ is not birational for $t < n+3$ if n is even or $t < n+2$ if n is odd.*

The idea of this example is taken from [Ka00], Example 3.1 (2).

These bounds are optimal up to dimension 3 (for the three dimensional case see [CCZ07]). Note that the bound for n even is the same predicted by Fujita's conjecture for the very ampleness, while for n odd is one less.

Let r'_n be the minimal positive integer such that for every n-dimensional smooth variety X of general type there is an integer $r \leq r'_n$ such that $|rK_X|$ induces a birational map. Let x_n be the minimal integer such that for every n-dimensional smooth variety of general type and every integer $t \geq x_n$ the map induced by $|tK_X|$ is generically finite. Of course $r_n \geq r'_n \geq x'_n$, $r_n \geq x_n \geq x'_n$. It is also natural to study the behaviour of these numbers.

Taking $X = Y \times C$ with C a smooth curve of genus 2 and Y a smooth variety of general type we get $r_n \geq r_{n-1}$ and $r'_n \geq r'_{n-1}$ for all $n \geq 2$.

Question 1. *Is $r_{n+1} > r_n$ for all n? Is $r'_{n+1} > r'_n$ for all n?*

All our examples satisfy $h^i(X, \mathcal{O}_X) = 0$ for all $1 \leq i \leq n-1$. Manifolds X with the property $h^i(X, \mathcal{O}_X) = 0$ for all $1 \leq i \leq n-1$ are very special, but it may be worthwhile to fix the integer $q := h^1(X, \mathcal{O}_X)$ and study the integers $r_n(q)$, $r'_n(q)$, $x_n(q)$, $x'_n(q)$ ($r_n(\geq q)$, $r'_n(\geq q)$, $x_n(\geq q)$, $x'_n(\geq q)$ and $r_n(\leq q)$, $r'_n(\leq q)$, $x_n(\leq q)$, $x'_n(\leq q)$) obtained taking only manifolds with irregularity q (resp. $\geq q$, resp. $\leq q$). Taking $X = Y \times D$ with D a curve of genus $x \geq 2$ we get $r_n(q) \geq r_{n-1}(q-x)$ for all integers $n \geq 2$ and q, x such that $2 \leq x \leq q$ (and similarly for the other integers $r'_n(q)$).

Our last result is that every positive rational number is a canonical volume:

Theorem 4. *Let $q = r/s$ be a rational number with $r, s > 0$ and $(r, s) = 1$. There are infinite positive integers n such that there is a smooth variety of dimension n with*

$$vol(X) = \frac{r}{s}.$$

This research started from a question of R. Ghiloni on the asymptotical behaviour of r_n: we thank him heartily.

2. The proofs

We will need the following lemma.

Lemma 1. *Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ be a well-formed weighted projective space. If the coordinate points are canonical singularities then all the singularities of $\mathbb{P}$ are canonical.*

Proof. Let P be a singular point of $\mathbb{P}$. Define $U := \{0, 1, \ldots, n\}$ and let $S \subset U$ be the subset of the variables nonzero at P. By hypothesis, if $\#S = 1$, then P is a canonical singularity. Assume then $\#S > 1$.

We denote by h_S the highest common factor of the set $\{a_i | i \in S\}$. Then $h_S > 1$ and, chosen a $k \in S$, P is a cyclic quotient singularity of type

$$\frac{1}{h_S}(a_0, \ldots, \hat{a}_k, \ldots, a_n).$$

By the criterium of [Re87], page 376, P is a canonical singularity if and only if

$$\frac{1}{h_S} \sum_{i=0}^{n} \overline{ja_i}^S \geq 1 \qquad \text{for all } 1 \leq j \leq h_S - 1$$

where $\overline{a}^S$ denotes the smallest (non negative) residue of a mod h_S.

We argue by contradiction. Suppose

$$\frac{1}{h_S} \sum_{i=0}^{n} \overline{ja_i}^S < 1 \qquad \text{for some } 1 \leq j \leq h_S - 1.$$

Take a $k \in S$. Then $a_k = mh_S$ for a positive integer m. Note that $mj < a_k$. Then we have

$$\frac{1}{a_k} \sum_{i=0}^{n} \overline{mja_i}^{\{k\}} = \frac{1}{mh_S} \sum_{i=0}^{n} \overline{mja_i}^{\{k\}} = \frac{1}{h_S} \sum_{i=0}^{n} \overline{ja_i}^S < 1,$$

which contradicts the hypothesis. □

Proposition 1. *Let $k \geq 2$ and $l \geq 0$ be integers. Consider the weighted projective space*

$$\mathbb{P} := \mathbb{P}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}).$$

Then the general hypersurface X_d in $\mathbb{P}$ of degree $d := (l+3)k(k+1)$ has at worst canonical singularities, $K_{X_d} \sim \mathcal{O}_{X_d}(1)$, $\dim X_d = 3k + l - 1$ and

$$vol(X_d) = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}.$$

Proof. The weighted projective space $\mathbb{P}$ is well-formed since $k \geq 2$. We use the criterium of [Re87], page 376, to control that the singularities of $\mathbb{P}$ are canonical. By the previous lemma it is enough to look at the coordinates points. They are of three types.

(1) The singularities of type
$$\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}).$$
We have to check that
$$\frac{1}{k}(2k-1)\overline{j(k+1)} \geq 1$$
for $1 \leq j \leq k-1$, where $\bar{}$ denotes the smallest (non negative) residue mod k. This is trivial since $\overline{j(k+1)} = j \geq 1$ for $1 \leq j \leq k-1$.

(2) The singularities of type
$$\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)}).$$
We have to check that
$$\frac{1}{k+1}(k+2)\overline{jk} \geq 1$$
for $1 \leq j \leq k$, where $\bar{}$ denotes the smallest (non negative) residue mod $k+1$. This is trivial since $\overline{jk} \geq 1$ for $1 \leq j \leq k$.

(3) The singularities of type
$$\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)}),$$
when $l \geq 1$. We have to check that
$$\frac{1}{k(k+1)}((k+2)\overline{ik} + (2k-1)\overline{i(k+1)}) \geq 1$$
for $1 \leq i \leq k(k+1)-1$, where $\bar{}$ denotes the smallest (non negative) residue mod $k(k+1)$. This follows since $k \not| j$ then $\overline{j(k+1)} \geq k+1$ and if $(k+1) \not| j$ then $\overline{jk} \geq k$.

Now note that $\mathcal{O}_{\mathbb{P}}(d)$ is base point free (since d is a multiple of every weight) and locally free (by Lemma 1.3 of [Mo75]).

Then we can apply a Kollár-Bertini theorem (Proposition 7.7 of [Ko97], see also Theorem 1.3 of [Re80]) to conclude that the general hypersurface X_d of degree d is canonical (and obviously well-formed and quasi-smooth, cf. [If00]).

Finally, by adjunction (6.14 of [If00]), $K_{X_d} \sim \mathcal{O}_X(1)$ and so

$$\text{vol}(X_d) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}.$$

□

Proof of theorem 1. Write $n = 3k + l - 1$, with integers $k \geq 2$ and $0 \leq l \leq 2$, so $k = \lfloor \frac{n+1}{3} \rfloor$. We can apply the previous proposition to obtain a canonical variety X_d of dimension n in the projective space

$$\mathbb{P} := \mathbb{P}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}).$$

By Theorem 3.4.4 (and the proof of the lemma above) in [Do82] we deduce that

$$H^0(X, mK_{X_d}) = 0$$

for $0 < m < k$.

Moreover

$$\mathrm{vol}(X_d) = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}} < \frac{(l+3)}{k^{n+l}} = \frac{l+3}{3k^l} \cdot \frac{3}{k^n} \leq \frac{3^{n+1}}{(n-1)^n}.$$

Take as X any desingularization of X_d. □

Proof of theorem 2. Let $n = 3k + l - 1$, with integers $k \geq 2$ and $2 \leq l \leq 4$ so $k = \lfloor \frac{n-1}{3} \rfloor$. We can apply the proposition 1 to obtain a canonical variety X_d of dimension n in the projective space

$$\mathbb{P} := \mathbb{P}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}).$$

We denote the coordinates of this space z_i for $0 \leq i \leq n+1$. Recall that $K_{X_d} \sim \mathcal{O}_{X_d}(1)$.

Fix $t < k(k+1)$. Thanks to Proposition 3.3 of [Mo75] we get

$$|\mathcal{O}_X(t)| = |\mathcal{O}_{\mathbb{P}}(t)|,$$

but the last two variable z_n and z_{n+1} can't appear in an element of $|\mathcal{O}_{\mathbb{P}}(t)|$ for degree's reasons.

Take as X any desingularization of X_d. Then the map induced by $|tK_X|$ is not generically finite. In particular

$$x'_n \geq k(k+1) \geq \frac{n(n-3)}{9}.$$

□

Proof of theorem 3. We first consider the case n even. Let $d = n+3$ and let $\mathbb{P}$ the weighted projective space $\mathbb{P}(1^{(n)}, 2, d)$. The coherent sheaf $\mathcal{O}_{\mathbb{P}}(2d)$ is a line bundle ([Mo75], Lemma 1.5). We call the coordinates of this space z_i for $0 \leq i \leq n+1$. Since d is odd the general weighted hypersurface X of degree $2d$ do not meet the singularities of $\mathbb{P}$ and it is smooth. Note that its equation is of the form

$$z_{n+1}^2 = P(z_0, \ldots, z_n)$$

where P is a polynomial of weighted degree $2d$. Moreover we have $K_X \sim \mathcal{O}_X(1)$.

If we take a positive integer $t < d$, then the linear system $|tK_X|$ does not induce a birational map. Indeed by Proposition 3.3 of [Mo75] we have

$$|\mathcal{O}_X(t)| = |\mathcal{O}_{\mathbb{P}}(t)|,$$

but the variable z_{n+1} can't appear in an element of $|\mathcal{O}_{\mathbb{P}}(t)|$ for degree's reasons and so the induced map has at least degree 2.

If $n = 1$ we use a smooth curve of genus 2. If n is odd and $n \geq 3$ we consider a manifold Y of dimension $n-1$ such that $r_n \geq n+2$ and define $X := Y \times C$ where

C is smooth curve of genus 2. Alternatively, for n odd, take $d = n+2$ and a general hypersurface of degree $2d$ in $\mathbb{P}(1^{(n+1)}, d)$. $\square$

Proof of theorem 4. Let b be a positive integer such that

$$br \equiv 1 \quad mod\ s,$$

and write

$$br - 1 = ts.$$

Let a be a positive integer such that $(a, s) = (a, b) = 1$.

We set

$$n := rab + 1 - a - s - b, \qquad d := n - 1 + a + s + b = rab$$

$$\mathbb{P} = \mathbb{P}(1^{(n-2)}, a, s, b).$$

Observe that choosing a and b we can have n arbitrarily large. Hence we may assume $n \geq 3$.

Note that $(a, s) = (a, b) = (s, b) = 1$, hence the only singularities of $\mathbb{P}$ are $P = (0^{(n-2)}, 0, 1, 0)$, $Q = (0^{(n-2)}, 1, 0, 0)$ and $R = (0^{(n-2)}, 0, 0, 1)$.

Now consider a general weighted hypersurface X_d of degree d in $\mathbb{P}$. The sheaf $\mathcal{O}_{\mathbb{P}}(1)$ is locally free and spanned outside P. By Bertini's theorem applied to $\mathbb{P}\backslash\{P\}$ we get that X_d is smooth outside P. Since $s \not| d$, $a|d$ and $b|d$, so $P \in X_d$ while $Q, R \notin X_d$. Since $n \geq 3$ and X_d has a unique singular point, it is well-formed in the sense of [If00].

We will show that for a and b large enough, X_d is a (well-formed) quasi-smooth variety with at most a terminal singularity in P. Then we would have finished. Indeed note that by adjunction (6.14 of [If00]) $K_X \sim \mathcal{O}_X(1)$ is ample and therefore

$$\mathrm{vol}(X_d) = K_X^n = \mathcal{O}_X(1)^n = \frac{d}{asb} = \frac{r}{s}.$$

To control the quasi-smoothness we use criterium 8.1 of [If00].

Let $z_0, \dots, z_{n+1}$ be the coordinates of $\mathbb{P}$. For every $I \subset \{0, \dots, n+1\}$ except $I = \{n\}$ the condition 2.a of 8.1 in [If00] is satisfied because the only variable whose degree doesn't divide d is z_n. In the case $I = \{n\}$ we can use 2.b since $d = tas + a$ and we can take the monomial

$$z_n^{ta} z_{n-1}.$$

Since X_d is quasi-smooth its singularities are induced by those of $\mathbb{P}$ and so we have only to control that X_d is terminal in P.

Let $f = 0$ be an equation of X_d. We can write

$$f = z_n^{ta} z_{n-1} + \dots$$

We consider the affine piece $(z_n = 1)$. The point $P \in X_d$ looks like

$$(\tilde{f} = f(z_0, \dots, z_{n-1}, 1, z_{n+1}) = z_{n-1} + \dots = 0) \subset \mathbb{A}^{n+1}/\epsilon$$

where ϵ is a primitive s-th root of unity and acts via

$$z_i \mapsto \epsilon z_i \quad 0 \leq i \leq n-2,$$

$$z_{n-1} \mapsto \epsilon^a z_{n-1}$$

and

$$z_{n+1} \mapsto \epsilon^b z_{n+1}.$$

Note that $\partial f/\partial z_{n-1} \neq 0$ in P, hence, by the Inverse Function Theorem, z_i are local coordinates for P in X_d for $i \neq n-1, n$. This gives a quotient singularity of type

$$\frac{1}{s}(1^{(n-2)}, b).$$

By the criterium of [Re87], page 376, if $n-2 \geq s$ then P is terminal.

Now you can simply take a desingularization of X_d. □

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