ON QUASI-ÉTALE QUOTIENTS OF A PRODUCT OF TWO CURVES

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Abstract. A quasi-étale quotient of a product of two curves is the quotient of a product of two curves by the action of a finite group which acts freely out of a finite set of points. A quasi-étale surface is the minimal resolution of the singularities of a quasi-étale quotient. They have been successfully used in the last years by several authors to produce several interesting new examples of surfaces.

In this paper we describe the principal results on this class of surfaces, and report the full list of the minimal quasi-étale surfaces of general type with geometric genus equal to the irregularity ≤ 2.

1. Introduction

Throughout this paper a "surface" (resp. curve) is a smooth complex algebraic surface (resp. curve); these are compact complex manifolds of dimension 2 (resp. 1) with an algebraic structure.

We are interested in the birational geometry of surfaces; in other words we look at surfaces modulo the equivalence relation generated by the blow-up in a point.

For sake of simplicity, we will restrict to projective surfaces, so we assume that the surface can be algebraically embedded in a projective space. Most of the statement in this section are classical results in complex algebraic geometry; a good reference is the classical book [BPV84].

For each surface S, we will denote

- by Ω^q_S the sheaf of algebraic q-forms
- by $\mathcal{O}_S$ the structure sheaf Ω^0_S
- by K_S a canonical divisor, that is the divisor of the zeroes and of the poles of a meromorphic algebraic 2-form
- for each divisor D in S, by $\mathcal{O}_S(D)$ the invertible sheaf associated to it; so $\mathcal{O}_S(K_S) \cong \Omega^2_S$
- for each sheaf $\mathcal{F}$, by $h^q(\mathcal{F})$ the dimension of the $q-$th Cech cohomology group $H^q(\mathcal{F})$

The "classical" birational invariants are

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- the topological fundamental group $\pi_1(S)$
- the geometric genus
$$p_g(S) := h^{2,0}(S) := h^0(\Omega^2_S) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$$
- the irregularity
$$q(S) := h^{1,0}(S) := h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$$
- the Euler characteristic
$$\chi(\mathcal{O}_S) := 1 - q + p_g$$
- the plurigenera
$$P_n(S) := h^0(\mathcal{O}_S(nK_S))$$
- the Kodaira dimension $\kappa(S)$, which is the smallest number κ such that $\frac{P_n}{n^\kappa}$ is bounded from above, known to be at most 2

Definition 1.1. A surface S is of *general type* if $\kappa(S) = 2$.

The Enriques-Kodaira classification provides a relatively good understanding of the surfaces of *special* type, which are those with Kodaira dimension $\kappa(S) \neq 2$ (this includes rational surfaces, K3 surfaces, Enriques surfaces, Abelian surfaces, elliptic surfaces...). The class of the surfaces of general type may be considered as the class of the surfaces... we do not understand, and therefore the most interesting, at least in the opinion of the author. It is worth mentioning that the method we are going to discuss has been very recently applied also to a different (very interesting) class of surfaces, the K3 surfaces, see [GP13].

Definition 1.2. A surface S is minimal if K_S is nef, that is if the intersection of K_S with every curve is nonnegative.

In the birational class of a surface of general type S there is exactly one minimal surface, say $\bar{S}$, which is *its minimal model.* The self intersection of the canonical divisor K^2_S, which is not a birational invariant, measures in some sense the distance among S and $\bar{S}$; more precisely S is obtained by $\bar{S}$ by a sequence of exactly $K^2_{\bar{S}} - K^2_S$ blow ups.

If S is of general type, then the Riemann-Roch formula computes all plurigenera $P_n(S)$ from $\chi(\mathcal{O}_S)$ and $K^2_{\bar{S}}$; so knowing $p_g, q, K^2_{\bar{S}}$ and the topological fundamental group is enough to have all the birational invariants mentioned above.

This leads us to a famous picture, known as *the geography* of the surfaces of general type. By some famous inequality (Noether, Bogomolov-Miyaoka-Yau, ...) the possible values of the pair $(\chi, K^2_{\bar{S}})$ are in the green region of the Figure 1.

It is still unknown if all integral points in the green region of the Figure 1 can be obtained by surfaces of general type, although we know we can fill "most" of it: this is an example of *geographical question.* More generally, one would like to know all possible values of the 4-tuple of topological invariants $(p_g(S), q(S), K^2_{\bar{S}}, \pi_1(S))$; we are very far from that, but one can hope to answer this question at least for

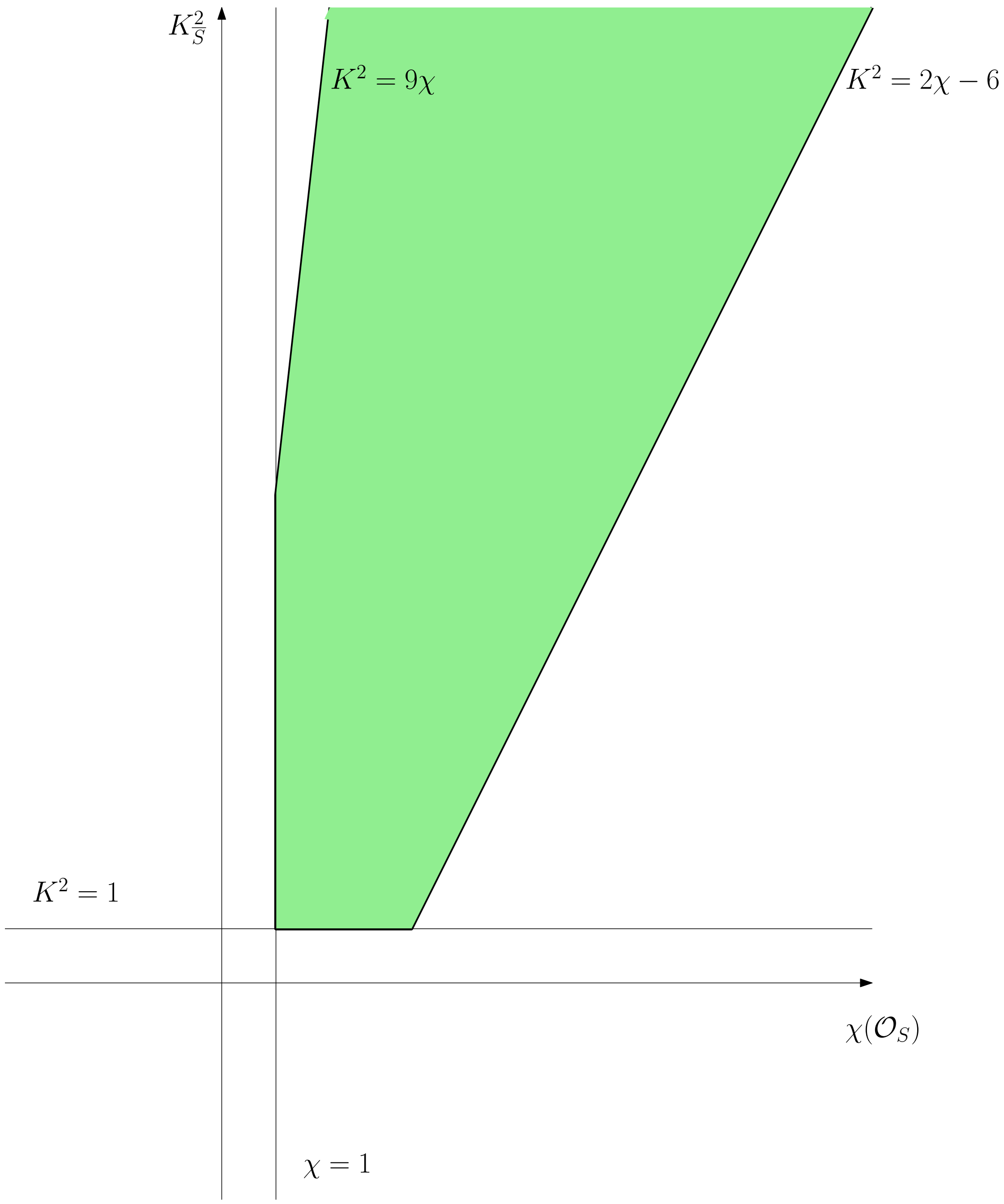


FIGURE 1. The geography of the surfaces of general type

some portion of the green region in Figure 1. Some of the most interesting surfaces of general type, for reasons we are not going to explain here (see e.g. [BCP06], [BCP12]), lie on the vertical "boundary" line $\chi(\mathcal{O}_S) = 1$; we will be back to this line at the end of the paper.

To answer geographical questions it is obviously very important to have some tools to construct examples, as the one we are going to explain.

2. Quasi-étale surfaces

The idea starts from the construction of Beauville in [Be83], which gives the name to the Beauville surfaces which are the center of this book. This is a minimal surface of general type with $p_g = q = 0$ and $K^2 = 8$, quotient of the product of two curves of genus 6 by a free action of the Abelian group $\mathbb{Z}^2_{/5\mathbb{Z}}$.

This example led to the following definition

Definition 2.1. A surface is *isogenous to a product* if it is the quotient of a product of two curves by a free action of a finite group. If both curves have genus at least 2, then it is isogenous to a *higher* product.

Beauville surfaces are isogenous to a higher product. A surface of general type isogenous to a product is automatically isogenous to a higher product, so we will drop the word 'higher' in the following.

Several authors (*e.g.* [MP01, Par03, BC04, Pol06, BCG08, Pol08, CP09]) constructed then new examples of surfaces of general type as surfaces isogenous to a product, in particular surfaces with $p_g = q = 0$ as Beauville example. All surfaces of general type isogenous to a product are minimal with $K^2 = 8\chi$, which forces them in a line of the 2-dimensional Figure 1. This is a strong limitation from the point of view of the birational geometry of surfaces of general type, since it shows that the construction of surfaces isogenous to a product can answer only very particular geographical questions.

This suggested to weaken Definition 2.1, to get something which is as simple to construct, but not limited by $K^2 = 8\chi$. A possibility is the following:

Definition 2.2. A *quasi-étale quotient* is the quotient of a product of two curves by the action of a finite group G acting freely out of a finite set of points. A *quasi-étale surface* is the minimal resolution of the singularities of a quasi-étale quotient.

Indeed, a quasi-étale quotient is smooth if and only if the action is free, so the surface is isogenous to a product. Each point in the product of the two curves stabilized by a non trivial subgroup of G maps onto a singular point of the quotient. We will always denote by X the singular quotient, and by S the smooth resolution of its singularities.

We have in this case the additional problem to study the singularities of X and their minimal resolution. The advantage is that these surfaces get out of the line $K^2 = 8\chi$. Indeed, it is easy to prove that $K^2 \leq 8\chi$, but apparently no other constraint applies ([Pol10]) proves $K^2_S \neq 8\chi - 1$ for a quasi-étale unmixed surface, but this does not extend to the mixed case) and we may hope to fill the whole yellow region in Figure 2.

Quasi-étale surfaces splits naturally in two classes. Indeed (see [Cat00]), if C_1 and C_2 are two algebraic curves

- either C_1 and C_2 are not isomorphic, in which case $\mathrm{Aut}(C_1 \times C_2) \cong \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$
- or $C_1 \cong C_2 \cong C$: in this case $\mathrm{Aut}(C \times C) = (\mathrm{Aut}\, C)^2 \rtimes \mathbb{Z}_2$, where the involution generating the (non normal) subgroup on the right is the exchange of the factors.

This leads to distinguish the finite group actions on a product of two curves in two classes, depending if G is a subgroup of $\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$ or not.

Definition 2.3. We set $G^{(0)} = G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$, and say that the action is

unmixed if $G < \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$, equivalently if $G = G^{(0)}$.

mixed if $G \neq G^{(0)}$, in which case we have an exact sequence

$$1 \to G^{(0)} \to G \to \mathbb{Z}_2 \to 1 \tag{1}$$

We will say that a quasi-étale quotient, resp. a quasi-étale surface is mixed if the action defining it is mixed. Similarly, if the action is unmixed, we will say that the induced quasi-étale quotient/surface is unmixed. Unmixed quasi-étale surfaces have been also called *product-quotient surfaces* and *standard isotrivial fibrations.*

By a remark of Catanese ([Cat00]), every quasi-étale quotient can be constructed by a minimal action, that means that $G^{(0)}$ acts faithfully on both factors. So without loss of generality we can and will always assume that both maps $G^{(0)} \to \mathrm{Aut}\, C_i$ are injective.

In this language, the quasi-étale condition has a completely algebraic surprisingly simple description.

Theorem 2.4 ([Fra11])**.** *Consider a minimal action of a finite group G on a product of two curves. Then G do not act freely out of a finite set of points if and only if the action is unmixed and the exact-sequence (1) splits.*

In other words G acts freely out of a finite set of points if and only if either the action is unmixed or the exact sequence (1) does not split.

In other words, the only minimal actions which violate the quasi-étale condition are the mixed actions in which $G \setminus G^{(0)}$ contains an involution.

Remark 2.5. Indeed, if $G \setminus G^{(0)}$ contains an involution, then one can assume (up to automorphisms) that the involution is $(x, y) \mapsto (y, x)$, which fixes the diagonal, and therefore the action is not free out of a finite set of points. Theorem 2.4 says that this is the only case in which the quasi-étale condition fail. In this last case, $(C \times C)/G$ is dominated by the symmetric product $C^{(2)}$ of the curve C. It would be interesting to extend to this case all results we have for the quasi-étale case.

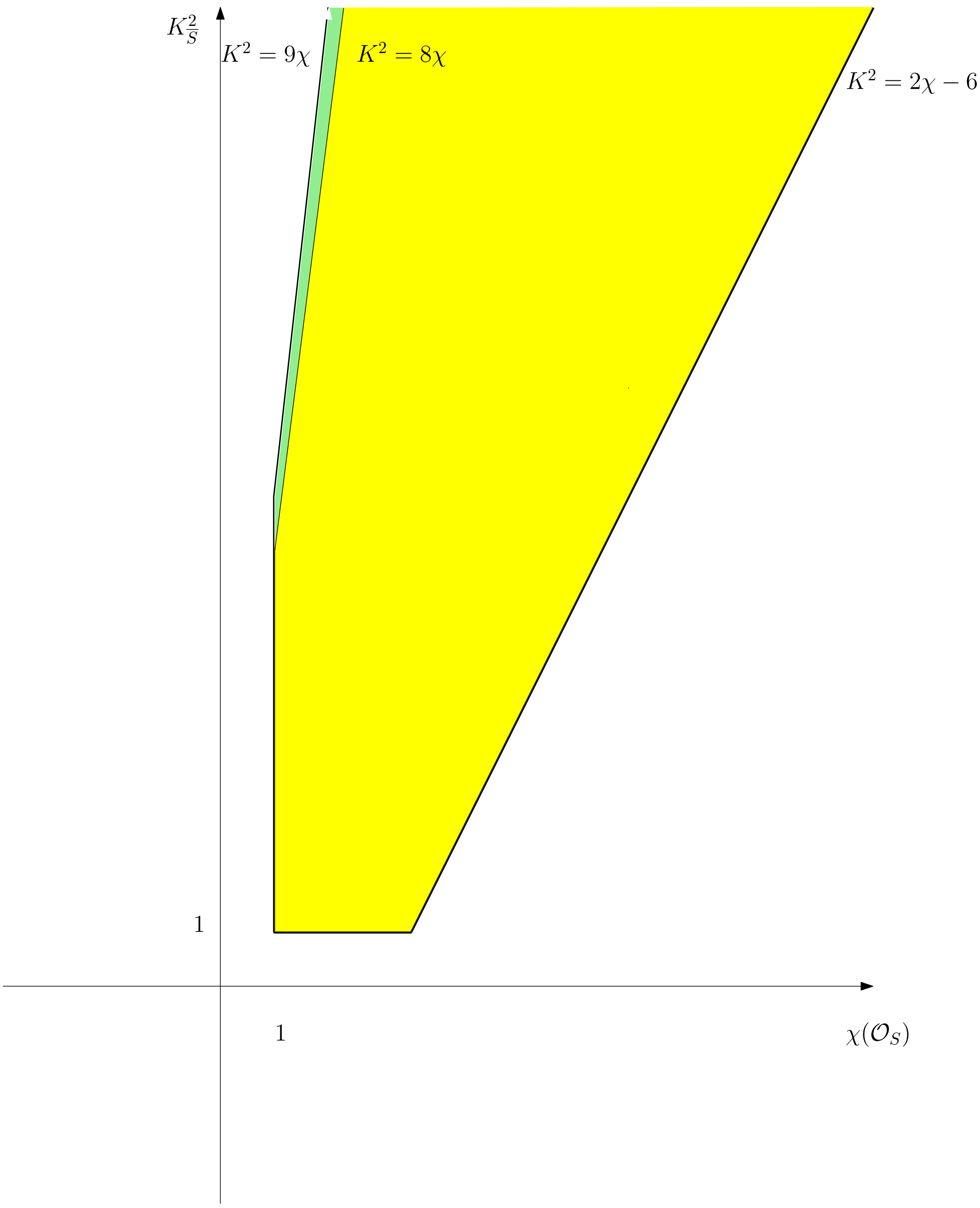


FIGURE 2. The region of the geography we expect to be able to fill with quasi-étale surfaces

3. Constructing curves with a finite group action

In order to construct a quasi-étale surface, we need curves with several automorphisms; indeed the general curve of genus $g > 2$ has no nontrivial automorphisms at all. It is worth mentioning here that if $g \geq 2$, which is the case we are interested in, $\operatorname{Aut} C$ is finite by the classical Schwarz Theorem. Anyway the method we are going to describe works also for $g \leq 1$, assuming the order of the group to be finite.

The method for constructing it, based on the classical Riemann Existence Theorem, is the following.

Let C be a smooth algebraic curve, and let $G^{(0)}$ be a finite subgroup of $\operatorname{Aut} C$. Then $C' := C/G$ is a smooth algebraic curve of smaller genus g' and the projection $C \to C'$ has finitely many critical values, say $p_1, \dots, p_r$. Let us fix one of these p_i; each of its preimages is stabilized by a cyclic subgroup of $G^{(0)}$, and all these subgroups lie in the same conjugacy class; in particular they all have the same cardinality, say m_i.

Removing the p_i from C' and their preimages from C, we remain with a regular topological cover $C^0 \to C' \setminus \{p_i\}$; these are determined by their monodromy map $\pi_1(C' \setminus \{p_i\}) \to G^{(0)}$. The Riemann Existence Theorem shows that each such topological cover may be uniquely extended to a map among compact complex curves $C \to C'$.

This gives a way to construct all pairs $(C, G^{(0)})$ where C is a smooth compact complex curve and $G^{(0)}$ is a finite subgroup of $\operatorname{Aut} C$.

Theorem 3.1 (Consequence of the Riemann Existence Theorem)**.** *Given*

a) *a compact complex curve* C'
b) *a finite set* $\{p_i\} \subset C'$
c) *a surjective homomorphism* $\pi_1(C' \setminus \{p_i\}) \to G^{(0)}$

there is, up to automorphisms, a unique curve C*, and a unique inclusion of* $G^{(0)}$ *in* $\operatorname{Aut} C$ *such that* $C' = C/G^{(0)}$*, the critical values of the quotient map* $C \to C'$ *belong to the set* $\{p_i\}$*, and such that, removing* $\{p_i\}$ *from* C' *and its preimage from* C*, we get the topological cover whose monodromy map is the map in c).*

The key point here is the homomorphism c). The more effective way is to construct a *generating vector* (see, *e.g.*, [Pol10]) of $G^{(0)}$. One chooses in a "standard" way a set of generators of $\pi_1(C' \setminus \{p_i\})$, and then gives the map c) by giving their images in $G^{(0)}$; we need then a set of generators (to ensure the surjectivity) of $G^{(0)}$ respecting some relations reflecting the relations among the chosen generators of $\pi_1(C' \setminus \{p_i\})$.

These relations are summarized in the definition of *generating vector* of *signature* $(g'; m_1, \dots, m_r)$. We do not repeat here this definition, referring the interested reader, *e.g.*, to [FP13]; just repeat here that g' is the genus of C', r the number of critical values of the map $C \to C'$, m_i the order of the stabilizer of each preimage of the critical value p_i.

By Hurwitz formula, the genus of C can be computed by the signature of the generating formula, namely

$$2g(C) - 2 = |G| \left(2g' - 2 + \sum_i \left(1 - \frac{1}{m_i} \right) \right) \tag{2}$$

4. Constructing quasi-étale quotients

In the case of unmixed actions (where, as seen in Theorem 2.4, the quasi-étale condition is empty), we need to give two curves C_1, C_2 and inclusions of $G^{(0)}$ in both $\operatorname{Aut}(C_i)$. Following the strategy in Section 3 we need to give two generating vectors of $G^{(0)}$, look at their signatures, say $(g'_1; m_1, \dots, m_r)$ and $(g'_2; n_1, \dots, n_s)$, and then choose two curves C'_1, C'_2 of respective genera g'_1, g'_2, and two finite subsets $\{p_1, \dots, p_r\} \subset C'_1$, $\{q_1, \dots, q_s\} \subset C'_2$.

In the mixed case, the construction is even simpler; one needs only one generating vector of $G^{(0)}$, look at its signature, say $(g'; m_1, \dots, m_r)$, then choose a curve C' of genus g' and a finite subset $\{p_1, \dots, p_r\} \subset C'$, and finally choose a degree 2 extension G of $G^{(0)}$ as in (1).

These data determine (see [Cat00, Fra11]) a mixed action on the product of two curves as follows. First of all they give, as in section 3, a curve C and an inclusion of $G^{(0)}$ in $\operatorname{Aut} C$. We choose an element $\tau' \in G \setminus G^{(0)}$, and notice that for each element $g \in G \setminus G^{(0)}$ there is a unique $g_0 \in G^{(0)}$ such that $g = \tau' g_0$. Then we give the mixed action on $C \times C$ as follows: we note that $(\tau')^2 \in G^{(0)}$ and set, $\forall g_0 \in G^{(0)}$, $\forall x, y \in C$

$$\begin{cases} g_0(x,y) = & (g_0 x, \tau' g_0 {\tau'}^{-1} y) \\ \tau' g_0(x,y) = & (\tau' g_0 {\tau'}^{-1} y, {\tau'}^2 g_0 x) \end{cases}$$

In the mixed case, since we are only interested in the quasi-étale surfaces, we will assume from now on that the extension (1) is unsplit.

4.1. Singularities.

We have a recipe to construct quasi-étale quotient; to obtain the quasi-étale surfaces, we need to understand their singularities.

Let then $X := (C_1 \times C_2)/G$ be a quasi-étale quotient. Then ([MP10, BP12, Fra11, FP13]) the singularities of X are the images of all points in $C_1 \times C_2$ which are stabilized by some nontrivial subgroup of G; these can be computed by the generating vectors, see, *e.g.*, [BP12] for the unmixed case and [FP13] for the mixed case. We recall here the analytic type of singularities one can find.

Definition 4.1. For each rational number $0 < \frac{q}{n} < 1$ ($\gcd(q,n) = 1$) a singularity of type $C_{n,q}$ (also called of type $\frac{1}{n}(1,q)$, or of type $\frac{q}{n}$) is an isolated singularity locally isomorphic to the singularity obtained by quotienting $\mathbb{C}^2$ by the action of the diagonal matrix with eigenvalues $e^{\frac{2\pi i}{n}}$ and $e^{\frac{2q\pi i}{n}}$.

The exceptional divisor of the minimal resolution of a singularity of type $C_{n,q}$ is a chain of rational curves $A_1, \dots, A_k$, each intersecting only the previous and the next one transversally in a single point, with respective self intersections $-b_1, \dots, -b_k$ where the b_i are given by the continued fraction

$$\frac{n}{q} = [b_1, \dots, b_k] = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}.$$

In other words, the dual graph of the exceptional divisor is

In the mixed case, we have an intermediate unmixed quotient $Y := C^2/G^{(0)}$ which has then only cyclic quotient singularities, and a double cover $Y \to X$, so we can see X as the quotient of Y by an involution.

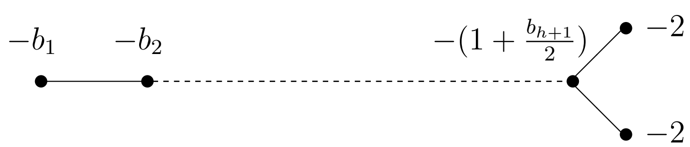
Since the square of each element of $G \setminus G^{(0)}$ is a nontrivial element of $G^{(0)}$, the points of $C \times C$ stabilized by a nontrivial subgroup of $G^{(0)}$ are the same stabilized by a nontrivial subgroup of G; therefore the singular locus of X is the image of the singular locus of Y and, once we have computed the cyclic quotient singularities of Y, we only have to describe how the involution acts on them.

There are two possibilities. If the involution exchange two singular points, then they are isomorphic, and we get one singularity of the same type $C_{n,q}$ on the quotient. In the other case

Theorem 4.2 ([FP13]). *Let P be a singular point of Y which is fixed by the involution. Then P is a singular point of type $C_{n,q}$ with $q^2 \equiv 1$ mod n; in other words the dual graph of the minimal resolution of the singularity of P is a symmetric string*

Moreover the number of vertices of the graph (which is the number of components of the exceptional divisor) is odd, say $2h+1$, and the lift of the involution to the resolution exchanges the extremal curves.

The exceptional divisor of the minimal resolution of the corresponding singular point of X has $h+3$ component, all rational, and its dual graph is the following



Definition 4.3. We will say that such a singular point of X is of type $D_{n,q}$.

4.2. **Invariants.** Once we have computed all singular points of X, which means that we know exactly how many singular points of each type has our X, we can compute some of the characteristic numbers of the constructed quasi-étale surface S.

We give here the formulas ([MP10, BP12, FP13]) in terms of the construction data described at the beginning of this section, and of the singularities of X. Recall that the signatures of the generating vectors determine the genera g_i of the induced covers by the formula (2).

In the following, for each singular point of type $C_{n,q}$, we will denote by q' the only integer $0 < q' < n$ with $qq' \equiv 1 \mod n$, and by $b_1, \ldots, b_k$ the coefficient of the continued fraction of $\frac{n}{q}$. Then $\frac{n}{q'} = [b_k, \ldots, b_1]$ and therefore a singularity of type $C_{n,q}$ is also of type $C_{n,q'}$.

For each singular point of type $C_{n,q}$ (*e.g.* [BP12])

$$k_x := -2 + \frac{2+q+q'}{n} + \sum_1^k (b_i - 2) \qquad B_x := \frac{q+q'}{n} + \sum_1^k b_i$$

whereas, for singular points of type $D_{n,q}$ ([FP13])

$$k_x := -1 + \frac{2+q+q'}{2n} + \sum_1^k \frac{b_i - 2}{2} \qquad B_x := 6 + \frac{q+q'}{2n} + \sum_1^k \frac{b_i}{2}$$

All k_x, B_x are nonnegative. Then

$$K_S^2 = \frac{8(g_1-1)(g_2-1)}{|G|} - \sum_{x \in \operatorname{Sing} X} k_x$$

$$\chi(\mathcal{O}_S) = \frac{K_S^2}{8} + \frac{\sum_{x \in \operatorname{Sing} X} B_x}{24}$$

Finally, for the irregularity ([Ser96]), in the unmixed case

$$q(S) = g_1' + g_2'$$

whereas in the mixed case

$$q(S) = g_1' = g_2'$$

The topological fundamental group, finally, can be computed directly by a method due to Armstrong [Arm65, Arm68]. The computation is rather complicated, but can easily performed by a computer. For example, in [BCGP12] there is an implementation for that in the MAGMA [BCP97] language.

So, if the surface is minimal ($S = \overline{S}$), we know all invariants of the constructed surfaces. As we will see in the next sections, this happens often, but not always, and determining $K_{\overline{S}}^2$ is sometimes a challenging task.

5. The minimality problem

In the previous sections we have given a method to construct quasi-étale surfaces and to compute, from the construction of S, $\pi_1(S)$, K_S^2, $p_g(S)$ and $q(S)$. If S is minimal, then $K_S^2 = K_{\bar{S}}^2$ and we can compute by Riemann-Roch Theorem all the plurigenera of S, and locate its position in the "geography".

This is often the case. Indeed K_X (which is the Weil divisor defined as closure of the canonical divisor of the open subset of the smooth points of X) pulls back to $K_{C_1 \times C_2}$ and therefore, if C_1 and C_2 have genus at least 2, it is big and nef. Therefore, if the action of G is free, then $X = S$ is minimal. On the other hand, if the action is not free, the singularities of X induce a discrepancy among K_S and the pull-back of K_X which is big and nef; if this discrepancy is big enough, the surface S may become non minimal or even not of general type. The first examples of non minimal unmixed quasi-étale surfaces of general type have been produced in [MP10], two surfaces with $p_g = q = 1$. The first example of an unmixed regular non minimal quasi-étale surfaces of general type has been produced in [BP12], a surface with $p_g = q = 0$; some more examples have been constructed in [BP13].

It is unclear if there exists a non minimal mixed quasi-étale surface of general type. Indeed we have the following

Theorem 5.1 ([FP13]). *Every irregular mixed quasi-étale surface of general type is minimal.*

The argument of the proof can't be extended to the regular case. Indeed [FP13] shows that, if S is a mixed quasi-étale surface of general type containing a smooth rational curve E with self intersection (-1), then the image of E on X pulls back to a curve of $C \times C$ whose image in $C' \times C'$ is a rational curve (here as before $C' := C/G^{(0)}$). Then, since the genus of C' equals the irregularity of the surface, we get a contradiction in the irregular case ($C' \times C'$ does not contain any rational curve), but nothing can be deduced in the regular case.

The unmixed irregular case, as (previously) shown in [MP10], is not much more complicated. Indeed if both curves C_1' and C_2' are irregular, then $C_1' \times C_2'$ does not contain any rational curve and one concludes as above (indeed this remark in [MP10] inspired the proof of Theorem 5.1). Else, up to exchanging C_1 and C_2, C_1' is rational, C_2' has genus $q > 0$ and all rational curves on S are contracted by the natural fibration $f_2 \colon S \to C_2'$. On the other hand, the only fibers of f_2 which are not isomorphic to C_1 are those whose image on X contains some singular points: it is not difficult then to compute explicitly their decomposition in irreducible components, and therefore describe all rational curves on S: in this way [MP10] could determine the minimal model of all the surfaces with $p_g = q = 1$ they constructed.

For example: [MP10] constructed one surface with $K_S^2 = 1$ whose quotient model had three singular points of respective type $\frac{1}{7}, \frac{2}{7}, \frac{2}{7}$ all in the image of the same fiber of f_2. That fiber depicted in Figure 3, on the left, contains then all rational curves

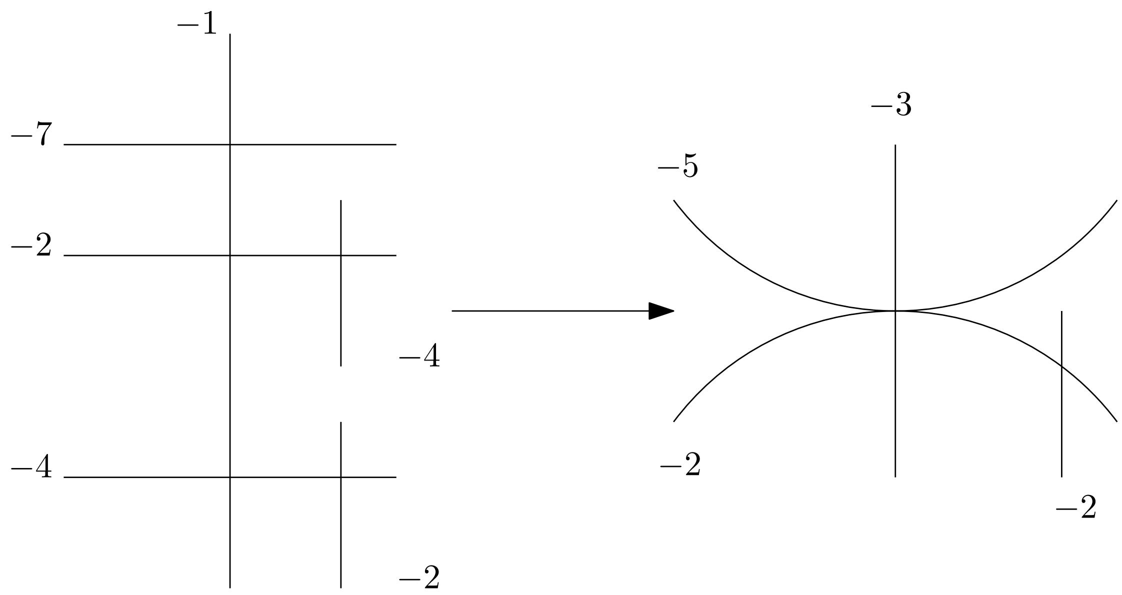


FIGURE 3. The rational curves in the example by Polizzi and Mistretta

of S. It has six irreducible components, all smooth rational curves, 5 forming the exceptional divisor of the resolution of the singularities, and the sixth having self-intersection -1. Contracting the (-1)-curve and the (-2)-curve transversal to it one gets a surface $\bar{S}$ with $K^2_{\bar{S}} = 3$ with exactly 4 rational curves, depicted on the right of Figure 3. The minimality follows since none has self intersection -1.

Determining the minimal model in the regular case may me much more challenging, since the rational curves are not forced to stay in the fibers of any of the two fibrations $f_i \colon S \to C_i/G$; indeed we are not able, in the regular case, to compute all rational curves on S. The surface constructed in [BP12], named "fake Godeaux" there, is a surface with $K^2_S = 1$, $p_g = q = 0$ whose quotient model has exactly the same configuration of singularities, $\frac{1}{7}, \frac{2}{7}, \frac{2}{7}$, of the previous example. [BP12] constructed two rational curves on S with self-intersection -1 intersecting the exceptional divisor of the resolution $S \to X$ as in Figure 4, on the left. Contracting both we get a surface $\overline{S}$ with $K^2 = 3$, $p_g = q = 0$ with the funny configuration of rational curves on the right of Figure 4.

Still, we know that ([BP12, Section 4] and [FP13, Corollary 4.7])

Proposition 5.2. *Let $\mathbb{P}^1 \to X$ be a rational curve on the quotient model of a quasi-étale surface such that C_1 and C_2 have both genus at least 2 (that's a necessary condition for X to be of general type). Then there are at least three distinct points of $\mathbb{P}^1$ mapped to singular point of X.*

This forces every further rational curve in $\overline{S}$ to intersect in at least three distinct points the configuration of curves on the right of Figure 4. On the other hand ([FP13, Corollary 4.8])

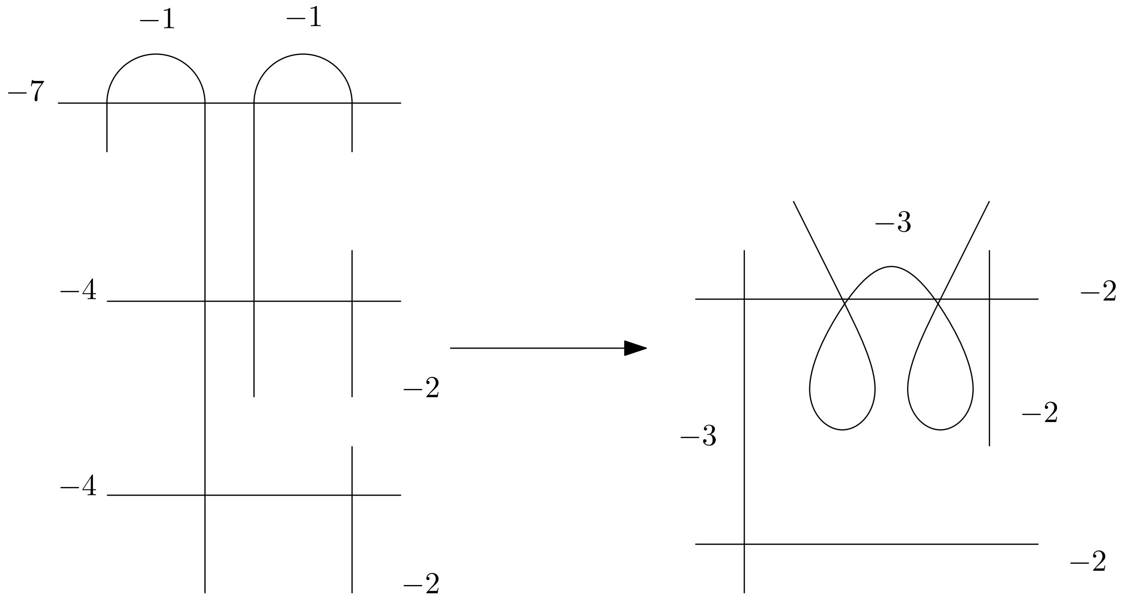


FIGURE 4. The rational curves in the fake Godeaux surfaces

Proposition 5.3. *Let E be a rational curve with self intersection -1 on a surface of general type, F be a reduced divisor whose support is made by rational curves of self-intersection either -2 or -3. Then $EF \leq 2$.*

We know that $\overline{S}$ is of general type since by the Enriques-Kodaira classification a regular surface with $K^2 > 0$ is either of general type or rational, all rational surfaces are simply connected, and we can compute its fundamental group, which is $\mathbb{Z}_{/6\mathbb{Z}}$. Looking at the configuration of curves on the right of Figure 4, by Propositions 5.2 and 5.3 we conclude that $\overline{S}$ is minimal.

A similar arguments works for proving the minimality of many of the surfaces constructed with this method, although one may need to substitute Proposition 5.3 with similar statements involving different configurations of rational curves. Some examples are in [BP12, Proposition 2.7].

We have not mentioned here how the two (-1)-curves on S are constructed; that's a rather complicated construction whose details goes beyond the scopes of this paper; the interested reader may read [BP12, Section 5]. We have recently found ([BP13]) few more examples of non minimal unmixed quasi-étale surfaces of general type with $p_g = q = 0$, and in all cases (up to now) we were able to determine their minimal model by constructing all the (-1)-curves on them. Still we have to run a different construction in each case: it would be nice to have a general method.

TABLE 1. Minimal unmixed quasi-étale surfaces of general type with $p_g = q = 0$

Sing X	K_S^2	Sign.		G	#fam
	8	$2^3,3$	5^3	$\langle 60,5\rangle$	1
	8	2^5	2^6	$\langle 8,5\rangle$	1
	8	2^5	2^5	$\langle 16,14\rangle$	1
	8	2^5	$3^2,5$	$\langle 60,5\rangle$	1
	8	$2^2,4^2$	$2^2,4^2$	$\langle 16,3\rangle$	1
	8	3^4	3^4	$\langle 9,2\rangle$	1
	8	2^6	$3,4^2$	$\langle 24,12\rangle$	1
	8	$2^2,4^2$	$2^3,4$	$\langle 32,27\rangle$	1
	8	$2,5^2$	3^4	$\langle 60,5\rangle$	1
	8	$2,4,6$	2^6	$\langle 48,48\rangle$	1
	8	$2^3,4$	2^6	$\langle 16,11\rangle$	1
	8	5^3	5^3	$\langle 25,2\rangle$	2
$\frac{1}{2}^2$	6	$2,4,6$	$2,4,10$	$\langle 240,189\rangle$	1
$\frac{1}{2}^2$	6	$2^3,4$	$2^4,4$	$\langle 16,11\rangle$	1
$\frac{1}{2}^2$	6	$2,4,6$	$2^4,4$	$\langle 48,48\rangle$	1
$\frac{1}{2}^2$	6	$2,3^3$	$2,5^2$	$\langle 60,5\rangle$	1
$\frac{1}{2}^2$	6	$2,7^2$	$3^2,4$	$\langle 168,42\rangle$	2
$\frac{1}{2}^2$	6	$2,5^2$	$3^2,4$	$\langle 360,118\rangle$	2
$\frac{1}{2}^4$	4	$2^2,3^2$	$2^2,3^2$	$\langle 18,4\rangle$	1
$\frac{1}{2}^4$	4	$2^3,4$	$2^3,4$	$\langle 32,27\rangle$	1
$\frac{1}{2}^4$	4	$2^2,4^2$	$2^2,4^2$	$\langle 8,2\rangle$	1
$\frac{1}{2}^4$	4	2^5	2^5	$\langle 8,5\rangle$	1
$\frac{1}{2}^4$	4	$2,4,6$	2^5	$\langle 48,48\rangle$	1
$\frac{1}{2}^4$	4	$2,5^2$	$2^2,3^2$	$\langle 60,5\rangle$	1
$\frac{1}{2}^4$	4	2^5	$3,4^2$	$\langle 24,12\rangle$	1
$\frac{1}{2}^4$	4	$2,4,6$	$2^2,4^2$	$\langle 48,48\rangle$	1
$\frac{1}{2}^4$	4	$2^3,4$	2^5	$\langle 16,11\rangle$	1
$\frac{1}{2}^4$	4	$2,4,5$	$3,6^2$	$\langle 120,34\rangle$	1
$\frac{1}{2}^4$	4	$2^2,3^2$	$3,6^2$	$\langle 18,3\rangle$	1

Sing X	K_S^2	Sign.		G	#fam
$\frac{1}{2}^6$	2	$2^3,4$	$2^3,4$	$\langle 16,11\rangle$	1
$\frac{1}{2}^6$	2	$2,4,6$	$2^3,4$	$\langle 48,48\rangle$	1
$\frac{1}{2}^6$	2	$2,4,5$	$2,6^2$	$\langle 120,34\rangle$	1
$\frac{1}{2}^6$	2	$2,3,7$	4^3	$\langle 168,42\rangle$	2
$\frac{1}{2}^6$	2	4^3	4^3	$\langle 16,2\rangle$	1
$\frac{1}{2}^6$	2	$2,5^2$	$2^3,3$	$\langle 60,5\rangle$	1
$\frac{1}{2}^6$	2	$2,6^2$	$2^3,3$	$\langle 36,10\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	$2^3,3$	$3,4^2$	$\langle 96,227\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	$2^4,3$	$3,4^2$	$\langle 24,12\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	$2^3,3$	$4^2,6$	$\langle 48,48\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	$2^3,3$	$3,5^2$	$\langle 60,5\rangle$	2
$\frac{1}{3},\frac{2}{3}$	5	$2,5,6$	$3,4^2$	$\langle 120,34\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	$2,4,6$	$2^4,3$	$\langle 48,48\rangle$	1
$\frac{1}{3},\frac{1}{2}^2,\frac{2}{3}$	3	$2,4,6$	$2^2,3,4$	$\langle 48,48\rangle$	1
$\frac{1}{3}^2,\frac{2}{3}^2$	2	$2,6^2$	$2^2,3^2$	$\langle 24,13\rangle$	1
$\frac{1}{3}^2,\frac{2}{3}^2$	2	$3^2,5$	$3^2,5$	$\langle 75,2\rangle$	2
$\frac{1}{3}^2,\frac{2}{3}^2$	2	$2^3,3$	$3^2,5$	$\langle 60,5\rangle$	1
$\frac{1}{3}^2,\frac{2}{3}^2$	2	$2^2,3^2$	$3,4^2$	$\langle 24,12\rangle$	1
$\frac{1}{3},\frac{1}{2}^4,\frac{2}{3}$	1	$2^3,3$	$3,4^2$	$\langle 24,12\rangle$	1
$\frac{1}{3},\frac{1}{2}^4,\frac{2}{3}$	1	$2,4,6$	$2^3,3$	$\langle 48,48\rangle$	1
$\frac{1}{3},\frac{1}{2}^4,\frac{2}{3}$	1	$2,3,7$	$3,4^2$	$\langle 168,42\rangle$	1
$\frac{1}{4},\frac{1}{2}^2,\frac{3}{4}$	2	$2,4,7$	$3^2,4$	$\langle 168,42\rangle$	2
$\frac{1}{4},\frac{1}{2}^2,\frac{3}{4}$	2	$2,4,5$	$3^2,4$	$\langle 360,118\rangle$	2
$\frac{1}{4},\frac{1}{2}^2,\frac{3}{4}$	2	$2,4,5$	$3,4,6$	$\langle 120,34\rangle$	2
$\frac{2}{5}^2$	4	$2^3,5$	$3^2,5$	$\langle 60,5\rangle$	1
$\frac{2}{5}^2$	4	$2,4,5$	$3^2,5$	$\langle 360,118\rangle$	1
$\frac{2}{5}^2$	4	$2,4,5$	$4^2,5$	$\langle 160,234\rangle$	3
$\frac{1}{5},\frac{4}{5}$	3	$2^3,5$	$3^2,5$	$\langle 60,5\rangle$	1
$\frac{1}{5},\frac{4}{5}$	3	$2,4,5$	$3^2,5$	$\langle 360,118\rangle$	1
$\frac{1}{5},\frac{4}{5}$	3	$2,4,5$	$4^2,5$	$\langle 160,234\rangle$	3

6. SURFACES OF GENERAL TYPE WITH $\chi(\mathcal{O}_S) = 1$

This strategy has been used in the last year by several authors to construct new surfaces of general type, mainly for the "special" region of the geography of the surfaces with $\chi = 1$ (equivalently $p_g = q$). Thank to the contribution of, among others, [MP01, Par03, BC04, BCG08, BCGP12, BP12, Fra11, FP13, Pol06, Pol08, CP09, Pol09, MP10, Pol10, Zuc03, Pen10, Pen12], we have now the complete list of all the minimal quasi-étale surfaces of general type with $\chi = 1$ and $K_S^2 > 0$, which includes all the minimal surfaces.

Table 2. Minimal mixed quasi-étale surfaces of general type with $p_g = q = 0$

Sing X	K_S^2	Sign.	$G^{(0)}$	G	#fam
	8	2^5	$\langle 32, 46\rangle$	$\langle 64, 92\rangle$	1
	8	4^3	$\langle 128, 36\rangle$	$\langle 256, 3678\rangle$	3
	8	4^3	$\langle 128, 36\rangle$	$\langle 256, 3679\rangle$	1
$\frac{1}{2}^4$	4	2^5	$\langle 16, 11\rangle$	$\langle 32, 7\rangle$	1
$\frac{1}{2}^4$	4	2^5	$\langle 16, 14\rangle$	$\langle 32, 22\rangle$	1
$\frac{1}{2}^4$	4	4^3	$\langle 64, 23\rangle$	$\langle 128, 836\rangle$	1
$\frac{1}{2}^6$	2	2^5	$\langle 8, 5\rangle$	$\langle 16, 3\rangle$	1
$\frac{1}{2}^6$	2	4^3	$\langle 32, 2\rangle$	$\langle 64, 82\rangle$	1
$\frac{1}{3}^2, \frac{2}{3}^2$	2	$3^2, 4$	$\langle 384, 4\rangle$	$\langle 768, 1083540\rangle$	1
$\frac{1}{3}^2, \frac{2}{3}^2$	2	$3^2, 4$	$\langle 384, 4\rangle$	$\langle 768, 1083541\rangle$	1
$\frac{1}{2}^3, \frac{1}{4}^2$	2	$2^3, 4$	$\langle 64, 73\rangle$	$\langle 128, 1535\rangle$	1
$\frac{3}{8}, \frac{5}{8}$	3	$2^3, 8$	$\langle 32, 39\rangle$	$\langle 64, 42\rangle$	1
$\frac{1}{2}^2, D_{2,1}^2$	1	$2^3, 4$	$\langle 16, 11\rangle$	$\langle 32, 6\rangle$	1
$\frac{1}{2}^2, D_{2,1}^2$	2	$2^3, 4$	$\langle 32, 27\rangle$	$\langle 64, 32\rangle$	1
$\frac{1}{2}^2, D_{2,1}^2$	2	$2^2, 3^2$	$\langle 18, 4\rangle$	$\langle 36, 9\rangle$	1

The surfaces of general type with $p_g = q \geq 3$ are classified (see [BCP06] for a more precise account), so we only consider here the case $q \leq 2$. There are only three nonminimal surfaces , which we have partially discussed in the previous section: the interested reader will find them in [MP10, BP12].

The minimal quasi-étale surfaces of general type with $p_g = q \leq 2$ form few hundreds of families, whose construction is spread among several papers, sometimes using different notations. We take the opportunity offered by this paper to collect all of them in the same place with a coherent notation, reporting the full list in the tables 1, 2, 3, 4, 5 and 6.

A short explanation of the notation:

- the column Sing X gives the singularities of the quasi-étale quotient: we use the notation $\frac{q}{n}$ for the cyclic quotient singularities, $D_{n,q}$ for the singularities, in the mixed case, which are branching points of the double cover $Y \to X$. We use exponents for multiplicities: for example $\frac{1}{2}^2, D_{2,1}^2$ means that X has 4 singular points, 2 ordinary nodes and 2 of type $D_{2,1}$ (in particular Y has 6 nodes, and the fixed locus of the involution is given by two of them). In the case of surfaces isogenous of a product, equivalently if X is smooth, we leave this field blank.

- the column K_S^2 is self-explanatory.

- the columns Sign. give the involved signatures, two in the unmixed case, one in the mixed case. We use here, for short, exponents for representing the multiplicity, and omit g' when equal to zero. So $2^3, 3$ is a shortcut for $0; 2, 2, 2, 3$.

- the columns G and $G^{(0)}$ (the latter only in the mixed case) give the corresponding group in the MAGMA/GAP4 notation. So $\langle 60, 5\rangle$, for example, is the

TABLE 3. Minimal unmixed quasi-étale surfaces of general type with $p_g = q = 1$

Sing X	K_S^2	g_{alb}	Sign.		G	#fam
	8	3	2^6	$1;2^2$	$\langle 4,2\rangle$	1
	8	3	2^5	$1;2^2$	$\langle 8,5\rangle$	1
	8	3	$2^2,4^2$	$1;2^2$	$\langle 8,2\rangle$	2
	8	3	$2,8^2$	$1;2^2$	$\langle 16,5\rangle$	1
	8	3	$2^2,4^2$	$1;2^2$	$\langle 8,3\rangle$	1
	8	3	$2^3,6$	$1;2^2$	$\langle 12,4\rangle$	1
	8	3	$2^3,4$	$1;2^2$	$\langle 16,11\rangle$	1
	8	3	$2,4,12$	$1;2^2$	$\langle 24,5\rangle$	1
	8	3	$2,6^2$	$1;2^2$	$\langle 24,13\rangle$	1
	8	3	$3,4^2$	$1;2^2$	$\langle 24,12\rangle$	1
	8	3	$2,4,8$	$1;2^2$	$\langle 32,9\rangle$	1
	8	3	$2,4,6$	$1;2^2$	$\langle 48,48\rangle$	1
	8	4	2^6	$1;3$	$\langle 6,1\rangle$	1
	8	4	2^5	$1;3$	$\langle 12,4\rangle$	1
	8	4	$2^2,3^2$	$1;3$	$\langle 18,3\rangle$	2
	8	4	$3,6^2$	$1;3$	$\langle 18,3\rangle$	1
	8	4	$2^3,4$	$1;3$	$\langle 24,12\rangle$	1
	8	4	$2,6^2$	$1;3$	$\langle 36,10\rangle$	1
	8	4	$2,6^2$	$1;3$	$\langle 36,12\rangle$	1
	8	4	$2,4^2$	$1;3$	$\langle 36,9\rangle$	2
	8	4	$2,5^2$	$1;3$	$\langle 60,5\rangle$	1
	8	4	$2,3,12$	$1;3$	$\langle 72,42\rangle$	1
	8	4	$2,4,5$	$1;3$	$\langle 120,34\rangle$	1
	8	5	2^6	$1;2$	$\langle 8,3\rangle$	1
	8	5	3^4	$1;2$	$\langle 12,3\rangle$	2
	8	5	$2^2,4^2$	$1;2$	$\langle 16,3\rangle$	3
	8	5	$2^2,3^2$	$1;2$	$\langle 24,13\rangle$	2
	8	5	$3,6^2$	$1;2$	$\langle 24,13\rangle$	1
	8	5	$2,8^2$	$1;2$	$\langle 32,5\rangle$	1
	8	5	$2,8^2$	$1;2$	$\langle 32,7\rangle$	1
	8	5	4^3	$1;2$	$\langle 32,2\rangle$	1
	8	5	4^3	$1;2$	$\langle 32,6\rangle$	1
	8	5	$2,6^2$	$1;2$	$\langle 48,49\rangle$	1
	8	5	$2,4,8$	$1;2$	$\langle 64,32\rangle$	2
	8	5	$2,5^2$	$1;2$	$\langle 80,49\rangle$	2

Sing X	K_S^2	g_{alb}	Sign.		G
$\frac{1}{2}^2$	6	3	$2,5^2$	$1;2$	$\langle 24,3\rangle$
$\frac{1}{2}^2$	6	3	$2,5^2$	$1;2$	$\langle 32,9\rangle$
$\frac{1}{2}^2$	6	3	$2,5^2$	$1;2$	$\langle 32,11\rangle$
$\frac{1}{2}^2$	6	3	$2,5^2$	$1;2$	$\langle 48,33\rangle$
$\frac{1}{2}^2$	6	3	2^6	$1;2^2$	$\langle 48,3\rangle$
$\frac{1}{2}^2$	6	3	2^5	$1;2^2$	$\langle 168,42\rangle$
$\frac{1}{2}^2$	6	4	$2^2,4^2$	$1;2^2$	$\langle 8,3\rangle$
$\frac{1}{2}^2$	6	4	$2,8^2$	$1;2^2$	$\langle 12,3\rangle$
$\frac{1}{2}^2$	6	4	$2^2,4^2$	$1;2^2$	$\langle 24,10\rangle$
$\frac{1}{2}^2$	6	4	$2^3,6$	$1;2^2$	$\langle 36,11\rangle$
$\frac{1}{2}^2$	6	4	$2^3,4$	$1;2^2$	$\langle 72,40\rangle$
$\frac{1}{2}^2$	6	4	$2,4,12$	$1;2^2$	$\langle 120,34\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	$2,6^2$	$1;2^2$	$\langle 6,1\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	$3,4^2$	$1;2^2$	$\langle 12,1\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	$2,4,8$	$1;2^2$	$\langle 12,4\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	$2,4,6$	$1;2^2$	$\langle 24,5\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	2^6	$1;3$	$\langle 24,12\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	2^5	$1;3$	$\langle 48,48\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	$2^2,3^2$	$1;3$	$\langle 96,64\rangle$
$\frac{1}{3},\frac{2}{3}$	5	3	$3,6^2$	$1;3$	$\langle 168,42\rangle$
$\frac{1}{2}^4$	4	2	$2^3,4$	$1;3$	$\langle 4,2\rangle$
$\frac{1}{2}^4$	4	2	$2,6^2$	$1;3$	$\langle 6,2\rangle$
$\frac{1}{2}^4$	4	2	$2,6^2$	$1;3$	$\langle 6,1\rangle$
$\frac{1}{2}^4$	4	2	$2,4^2$	$1;3$	$\langle 8,3\rangle$
$\frac{1}{2}^4$	4	2	$2,5^2$	$1;3$	$\langle 12,5\rangle$
$\frac{1}{2}^4$	4	2	$2,3,12$	$1;3$	$\langle 12,4\rangle$
$\frac{1}{2}^4$	4	2	$2,4,5$	$1;3$	$\langle 16,8\rangle$
$\frac{1}{2}^4$	4	2	2^6	$1;2$	$\langle 24,8\rangle$
$\frac{1}{2}^4$	4	2	3^4	$1;2$	$\langle 48,29\rangle$
$\frac{1}{2}^4$	4	3	$2^2,4^2$	$1;2$	$\langle 8,3\rangle$
$\frac{1}{2}^4$	4	3	$2^2,3^2$	$1;2$	$\langle 12,3\rangle$
$\frac{1}{2}^4$	4	3	$3,6^2$	$1;2$	$\langle 16,6\rangle$
$\frac{1}{2}^4$	4	3	$3,6^2$	$1;2$	$\langle 16,4\rangle$
$\frac{1}{2}^4$	4	3	$2,8^2$	$1;2$	$\langle 24,13\rangle$
$\frac{1}{2}^2,\frac{1}{3},\frac{2}{3}$	3	2	$2,8^2$	$1;2$	$\langle 24,8\rangle$
$\frac{1}{2}^2,\frac{1}{3},\frac{2}{3}$	3	2	4^3	$1;2$	$\langle 48,29\rangle$
$\frac{1}{2}^2,\frac{1}{4},\frac{3}{4}$	2	2	4^3	$1;2$	$\langle 16,8\rangle$
$\frac{1}{2}^2,\frac{1}{4},\frac{3}{4}$	2	2	4^3	$1;2$	$\langle 24,3\rangle$
$\frac{1}{3}^2,\frac{2}{3}^2$	2	2	$2,6^2$	$1;2$	$\langle 6,1\rangle$
$\frac{1}{3}^2,\frac{2}{3}^2$	2	2	$2,4,8$	$1;2$	$\langle 12,1\rangle$
$\frac{1}{3}^2,\frac{2}{3}^2$	2	2	$2,5^2$	$1;2$	$\langle 12,4\rangle$
$\frac{1}{2}^6$	2	2	$2,5^2$	$1;2$	$\langle 8,4\rangle$
$\frac{1}{2}^6$	2	2	$2,5^2$	$1;2$	$\langle 8,3\rangle$

TABLE 4. Minimal mixed quasi-étale surfaces of general type with $p_g = q = 1$

Sing X	K_S^2	g_{alb}	Sign.	$G^{(0)}$	G	#fam
	8	5	$1;2^2$	$\langle 8,2\rangle$	$\langle 16,6\rangle$	1
	8	5	$1;2^2$	$\langle 8,3\rangle$	$\langle 16,8\rangle$	1
	8	5	$1;2^2$	$\langle 8,5\rangle$	$\langle 16,3\rangle$	1
$\frac{1}{2}^2$	6	3	1;2	$\langle 24,13\rangle$	$\langle 48,30\rangle$	1
$\frac{1}{2}^2$	6	7	1;2	$\langle 24,13\rangle$	$\langle 48,31\rangle$	1
$\frac{1}{2}^4$	4	3	$1;2^2$	$\langle 4,1\rangle$	$\langle 8,1\rangle$	1
$\frac{1}{2}^4$	4	3	$1;2^2$	$\langle 4,2\rangle$	$\langle 8,2\rangle$	1
$\frac{1}{2}^4$	4	2	1;2	$\langle 16,3\rangle$	$\langle 32,29\rangle$	1
$\frac{1}{2}^4$	4	3	1;2	$\langle 16,4\rangle$	$\langle 32,13\rangle$	1
$\frac{1}{2}^4$	4	3	1;2	$\langle 16,4\rangle$	$\langle 32,14\rangle$	1
$\frac{1}{2}^4$	4	2	1;2	$\langle 16,4\rangle$	$\langle 32,32\rangle$	1
$\frac{1}{2}^4$	4	2	1;2	$\langle 16,4\rangle$	$\langle 32,35\rangle$	1
$\frac{1}{2}^4$	4	3	1;2	$\langle 16,5\rangle$	$\langle 32,15\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	3	1;3	$\langle 12,1\rangle$	$\langle 24,4\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5	3	1;3	$\langle 12,4\rangle$	$\langle 24,5\rangle$	1
$\frac{3}{5}$	6	5	1;5	$\langle 10,1\rangle$	$\langle 20,3\rangle$	1
$\frac{1}{2},D_{2,1}^2$	2	2	$1;2^2$	$\langle 2,1\rangle$	$\langle 4,1\rangle$	1
$\frac{1}{2},D_{2,1}^2$	2	2	1;2	$\langle 8,3\rangle$	$\langle 16,8\rangle$	1
$\frac{1}{2},D_{2,1}^2$	2	2	1;2	$\langle 8,4\rangle$	$\langle 16,9\rangle$	1

5^{th} group of order 60 in the MAGMA/GAP4 database of finite groups: this is the alternating group in 5 elements $\mathfrak{A}_5$.

- g_{alb} is the genus of the general fibre of the Albanese map, which is very important for the classification of the irregular surfaces. The column is missing for $q = 0$ since there is no Albanese map in that case. When, for $q = 2$, we leave it blank, it means that the Albanese map is not a fibration.

- in few cases there are 2 or 3 different families for which all the previous data coincide: instead of putting more identical rows, we used only one row for all of them, and add a last column, #fam, counting the number of families corresponding to the row. The reader will not find the column #fam in the latter part of table 3; indeed in that cases the number of families corresponding to each row is not known, at least to the author.

It is worth noticing that all possible values of K_S^2 in the yellow region of Figure 2 are obtained already by the table 1, so by surfaces with $p_g = q = 0$. In the irregular case, since an inequality of Debarre shows that for a minimal irregular surface of general type $K_S^2 \geq 2p_g$, also the quasi-étale surfaces with $p_g = q = 1$ and $p_g = q = 2$ reach all values of K_S^2 attained by minimal surfaces of general type: this supports our claim that this method could possibly fill the yellow region.

Table 5. Minimal unmixed quasi-étale surfaces of general type with $p_g = q = 2$

Sing X	K_S^2	g_{alb}	Sign.		G	#fam
	8	2	2^6	2;	$\langle 2,1\rangle$	1
	8	2	3^4	2;	$\langle 3,1\rangle$	1
	8	2	2^5	2;	$\langle 4,2\rangle$	2
	8	2	$2^2,4^2$	2;	$\langle 4,1\rangle$	1
	8	2	5^3	2;	$\langle 5,1\rangle$	1
	8	2	$2^2,3^2$	2;	$\langle 6,2\rangle$	1
	8	2	$3,6^2$	2;	$\langle 6,2\rangle$	1
	8	2	$2,8^2$	2;	$\langle 8,1\rangle$	1
	8	2	$2,5,10$	2;	$\langle 10,2\rangle$	1
	8	2	$2,6^2$	2;	$\langle 12,5\rangle$	2
	8	2	$2^2,3^2$	2;	$\langle 6,1\rangle$	1
	8	2	4^3	2;	$\langle 8,4\rangle$	1
	8	2	$2^3,4$	2;	$\langle 8,3\rangle$	2
	8	2	$2^3,3$	2;	$\langle 12,4\rangle$	2
	8	2	$3,4^2$	2;	$\langle 12,1\rangle$	1
	8	2	$2,4,8$	2;	$\langle 16,8\rangle$	1
	8	2	$2,4,6$	2;	$\langle 24,8\rangle$	2
	8	2	$3^2,4$	2;	$\langle 24,3\rangle$	1
	8	2	$2,3,8$	2;	$\langle 48,29\rangle$	1
	8		$1;2^2$	$1;2^2$	$\langle 4,2\rangle$	1
	8		$1;3$	$1;2^2$	$\langle 6,1\rangle$	1
	8		$1;2$	$1;2^2$	$\langle 8,3\rangle$	1
$\frac{1}{2}^2$	6		$1;2$	$1;2$	$\langle 12,3\rangle$	1
$\frac{1}{2}^4$	4		$1;2^2$	$1;2^2$	$\langle 2,1\rangle$	1
$\frac{1}{2}^4$	4		$1;2$	$1;2$	$\langle 8,3\rangle$	1
$\frac{1}{2}^4$	4		$1;2$	$1;2$	$\langle 8,4\rangle$	1
$\frac{1}{3},\frac{2}{3}$	5		$1;3$	$1;3$	$\langle 6,1\rangle$	1

Table 6. Minimal mixed quasi-étale surfaces of general type with $p_g = q = 2$

Sing X	K_S^2	Sign.	$G^{(0)}$	G	#fam
	8	2;	$\langle 2,1\rangle$	$\langle 4,1\rangle$	1

It would be nice to have a complete classification of all unmixed quasi-étale surfaces of general type with $\chi(\mathcal{O}_S) = 1$; the list is complete only in the mixed irregular case by Theorem 5.1.

The methods developped in the mentioned papers allow in principle to construct the whole list of quasi-étale surfaces with a fixed value of the triple $(p_g,\ q,\ K_S^2)$. Since when we blow up a point the value of K_S^2 drops by 1, there is no lower bound for the value of K_S^2 of a surface of general type in terms of birational invariants as p_g and q; in particular these methods can't give a complete classification unless one can prove such a lower bound for quasi-étale surfaces; at the moment we do not have even a reasonable conjecture for that bound.

A different method, which could possibly in the future produce a complete classification of the quasi-étale surfaces with $\chi = 1$, will be presented in [BP13], where K_S^2 is substituted by a different number related to the structure of the Néron-Severi group of a quasi-étale surface.

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