RIGID BUT NOT INFINITESIMALLY RIGID COMPACT COMPLEX MANIFOLDS

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ABSTRACT. In this paper the authors give for each dimension $d \geq 2$ an infinite series of rigid compact complex manifolds which are not infinitesimally rigid, hence giving a complete answer to a problem of Morrow and Kodaira stated in the famous book *Complex manifolds*.

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INTRODUCTION

In the famous book *Complex manifolds* by J. Morrow and K. Kodaira the following problem is posed

Problem. [MK71, p. 45] *Find an example of a (compact complex manifold) M which is rigid, but $H^1(M, \Theta) \neq 0$. (Not easy?)*

A compact complex manifold is *rigid* if it has no nontrivial (small) deformations (cf. Definition 1.1). Moreover, recall that a complex manifold M is called *infinitesimally rigid*, if $H^1(M, \Theta_M) = 0$, and that, by Kuranishi theory, infinitesimal rigidity implies rigidity (cf. [MK71, Theorem 3.2]).

The above Problem asks for examples of compact complex manifolds which are rigid, but not infinitesimally rigid, showing that the converse of [MK71, Theorem 3.2] does not hold.

To our knowledge this problem is up to now unsolved and the aim of our article is to give an infinite series of such examples for each dimension $d \geq 2$.

In [BC18] several different notions of rigidity (cf. Definition 1.1, where we repeat the notions which are relevant for our purposes) have been recalled and newly introduced and their relations have been studied.

It is wellknown that in dimension 1 all concepts of rigidity coincide and that the only rigid curve is $\mathbb{P}^1$.

For complex surfaces, in [BC18] the following was proven (in the slightly more general context of compact complex surfaces):

Theorem. *Let S be a smooth projective surface, which is rigid. Then either*

(1) S is a minimal surface of general type, or
(2) S is a Del Pezzo surface of degree $d \geq 5$.

Del Pezzo surfaces are infinitesimally rigid, and rigid surfaces of general type are also globally rigid due to the existence of a moduli space.

The above result seems to suggest that the property of rigidity puts strong restrictions on the Kodaira dimension of the manifold X, but if we go to higher dimensions this is no longer true. In fact, in [BC18] the following is shown:

Theorem. *For each $n \geq 3$ and for each $k = -\infty, 0, 2, \ldots n$ there is a rigid projective variety X of dimension n and Kodaira dimension $\mathrm{kod}(X) = k$.*

The above stated result on rigid surfaces shows therefore that the problem of classifying rigid surfaces reduces to the same question for surfaces of general type, and the list of known rigid surfaces of general type is rather short. Again we refer to [BC18] for a detailed account of the status of the art.

Among the several questions raised in [BC18] there is the following [BC18, Question 1.5. B], special case of the problem of Morrow and Kodaira:

Question. *Does there exist a rigid, but not infinitesimally rigid surface of general type?*

This means that the moduli space of such surfaces consists of a single non reduced point.

On one hand, the existence of such surfaces is expected in view of Murphy's law for moduli spaces (cf. [Vak06]), which says that however bad a singularity is it appears as singular locus of some moduli space and since there are known

examples of everywhere non reduced moduli spaces (cf. [Cat89]). Still the proofs of these results rely on constructions where the moduli spaces have to be positive dimensional.

On the other hand, showing rigidity can be quite difficult, and usually there are only techniques which allow to show the rigidity of a surface of general type proving the vanishing of $H^1(\Theta_S)$. This probably is the reason that the problem of Morrow and Kodaira remained open for more than 45 years.

In this paper we give an infinite series of rigid regular surfaces of general type with unbounded invariants (p_g, K^2). More precisely, our first main result is:

Theorem. *For every even $n \geq 8$ such that $3 \nmid n$ there is a minimal regular surface S_n of general type with*

$$K_{S_n}^2 = 2(n-3)^2, \quad p_g(S_n) = \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right),$$

such that S_n is rigid, but not infinitesimally rigid.

This gives a positive answer to the question of Morrow and Kodaira in dimension 2.

The surface S_n is constructed as the minimal resolution of singularities of a so-called *product-quotient surface* whose singular model X_n has six nodes. More precisely, X_n is the quotient of a product of two algebraic curves $C_1 \times C_2$ by the faithful action of a finite group G, such that G acts on each factor and the quotient map $C_i \to C_i/G \cong \mathbb{P}^1$ is branched in three points (i.e., each of the two curves C_1, C_2 is a so-called *triangle curve*).

The nodes are the key for obtaining an obstructed moduli space, as noticed first, to our knowledge, by Burns and Wahl ([BW74]). Thanks to their results *one readily constructs examples (e.g., those of Segre) of surfaces of general type with obstructed deformations* by constructing suitable nodal surfaces. Indeed, since S_n has six nodes, by [BW74, Corollary 1.3] $h^1(\Theta_{S_n})(:= \dim H^1(\Theta_{S_n})) \geq 6$ and it suffices to show that S_n is in fact rigid.

Once we have a product-quotient surface coming from two triangle curves it is immediate that the equisingular deformations of the canonical model are trivial. Then it has to be shown that none of the local deformations of the singularities lift to deformations of the canonical model. This can be deduced by the linear independence of certain elements of $H^2(\Theta_{S_n})$ (see Theorem 1.3, condition 2) that have a simple explicit description in local coordinates due to Kas (cf. [Kas77]).

In this paper we use only very special product-quotient surfaces: they are regular, their group G is the Abelian group $(\mathbb{Z}/n\mathbb{Z})^2$, where $n \geq 8$, even and not divisible by 3, and their singular models have only nodal singularities, but are indeed singular.

Product-quotient surfaces have been extensively studied, especially for low invariants (like in the limit case $p_g = q = 0$) and partial classification results are obtained in a long series of papers. We refer to [BC04], [BCG08], [BCGP12], [BP12], [BP16] and the literature there quoted.

For constructing the higher dimensional rigid, but not infinitesimally rigid examples, we take the product of S_n's with a rigid manifold.

More precisely, the second main result is:

Theorem. *Let $n \geq 8$ be an even integer such that $3 \nmid n$, and let X be a compact complex rigid manifold.*

Then $S_n \times X$ is rigid, but not infinitesimally rigid.

In particular there are rigid, but not infinitesimally rigid, manifolds of dimension d and Kodaira dimension κ for all possible pairs (d, κ) with $d \geq 5$ and $\kappa \neq 0, 1, 3$ and for $(d, \kappa) = (3, -\infty), (4, -\infty), (4, 4)$.

The paper is organized as follows.

In the first section we collect some background material on deformation theory which will be used in the rest of the paper. We recall the notions of rigidity which are relevant for our purposes and give a criterion for the minimal resolution of the singularities of a nodal surface to be rigid (cf. Theorem 1.3).

The second section is dedicated to Abelian covers and in particular to the proof of formulae for the character decomposition of direct images of canonical and bicanonical sheaves. In the next section we give the construction of the infinite series of product-quotient surfaces S_n and calculate their invariants.

The fourth section is dedicated to the proof of our first main theorem, whereas the last section is dedicated to the higher dimensional examples, i.e., the proof of our second main result.

1. A CRITERION TO PROVE RIGIDITY

In this section we shall recall the definitions of different concepts of rigidity of compact complex varieties, which were introduced and discussed in [BC18] and which are relevant for our paper. Then we briefly review results by Burns-Wahl ([BW74]), Kas ([Kas77]), Pinkham ([Pin81]) and Catanese ([Cat89]) which allow us to prove a criterion for rigidity (cf. Theorem 1.3).

Recall that two compact complex manifolds X and X' are said to be *deformation equivalent* if and only if there is a proper smooth holomorphic map

$$f \colon \mathfrak{X} \to \mathcal{B}$$

where $\mathcal{B}$ is a connected (possibly not reduced) complex space and there are points $b_0, b_0' \in \mathcal{B}$ such that the fibres $X_{b_0} := f^{-1}(b_0), X_{b_0'} := f^{-1}(b_0')$ are respectively isomorphic to X, X' $(X_{b_0} \cong X, X_{b_0'} \cong X')$.

For the convenience of the reader we recall part of the notions of rigidity given in [BC18, Definition 2.1]:

Definition 1.1.

(1) A compact complex manifold X is said to be *globally rigid* if for any compact complex manifold X', which is deformation equivalent to X, we have an isomorphism $X \cong X'$.

(2) A compact complex manifold X is said to be *infinitesimally rigid* if
$$h^1(X, \Theta_X) = 0,$$
where Θ_X is the sheaf of holomorphic vector fields on X.

(3) A compact complex manifold X is said to be *(locally) rigid* (or just *rigid*) if for each deformation of X,
$$f \colon (\mathfrak{X}, X) \to (\mathcal{B}, b_0)$$
there is an open neighbourhood $U \subset \mathcal{B}$ of b_0 such that $X_t := f^{-1}(t) \cong X$ for all $t \in U$.

Remark 1.2. Observe that a globally/infinitesimally rigid compact complex manifold is (locally) rigid. If $X = S$ is a surface of general type, then S is rigid if and only if S is globally rigid due to the existence of the Gieseker moduli space for canonical models of surfaces of general type.

Let X be a nodal surface, i.e., a compact complex variety of dimension 2 with r singular points, all of type A_1.

Let $S \to X$ be the minimal resolution of singularities of X and let $E = E_1 + \cdots + E_r$ be its exceptional locus. By [BW74, Corollary 1.3] the local cohomology group $H^1_E(\Theta_S) = \oplus^r_1 H^1_{E_i}(\Theta_S)$ embeds in $H^1(\Theta_S)$.

By [BW74, Proposition 1.10], for each $1 \le i \le r$, $H^1_{E_i}(\Theta_S)$ has dimension 1. Let now $\nu_i \in X$ be the node, image of E_i, and let θ_i be a generator of $H^1_{E_i}(\Theta_S)$ seen as element in $H^1(\Theta_S)$. The primary obstruction $[\theta_i, \theta_i]$ belongs to $H^2(\Theta_S)$, which is by Serre duality isomorphic to the dual of $H^0(\Omega^1_S \otimes \Omega^2_S)$.

Therefore we can see $[\theta_i, \theta_i]$ as a map $\alpha_{\nu_i} \colon H^0(\Omega^1_S \otimes \Omega^2_S) \to \mathbb{C}$ which is explicitly described in [Kas77] as follows: a small neighbourhood U_i of ν_i in X is isomorphic to the quotient of a small disc $\Delta \subset \mathbb{C}^2$, with coordinates (z_1, z_2), by the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$. This gives an inclusion of $H^0(\Omega^1_{U_i} \otimes \Omega^2_{U_i})$ into the invariant subspace
$$H^0(\Omega^1_\Delta \otimes \Omega^2_\Delta)^+ \subset H^0(\Omega^1_\Delta \otimes \Omega^2_\Delta),$$
and thus every $\eta \in H^0(\Omega^1_S \otimes \Omega^2_S)$ can be locally written as
$$\eta = (f_1 dz_1 + f_2 dz_2) \otimes (dz_1 \wedge dz_2).$$

Then

(1.1)
$$\alpha_{\nu_i}(\eta) = \left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right)(0,0).$$

This allows to prove the following:

Theorem 1.3. *Let $S \to X$ be the minimal resolution of the singularities of a nodal surface X. Assume that*

(1) $h^1(\Theta_X) = 0$;

(2) *the maps α_{ν_i} associated to the nodes ν_i of X locally described in (1.1) are linearly independent in $H^0(\Omega^1_S \otimes \Omega^2_S)^\vee$.*

Then S is rigid and $h^1(\Theta_S)$ equals the number of nodes of X.

Proof. By [Pin81, Proof of Corollary [3]] we have $H^1(\Theta_S) = H^1(\Theta_X) \oplus H^1_E(\Theta_S)$. Therefore, by condition 1, we can identify $H^1(\Theta_S)$ with $H^1_E(\Theta_S) \cong \mathbb{C}^r$.

Then for every $\theta \in H^1(\Theta_S)$ there are $t_i \in \mathbb{C}$ such that $\theta = \sum_1^r t_i \theta_i$. Because

$$[\theta, \theta] = \left[\sum_1^r t_i \theta_i, \sum_1^r t_i \theta_i \right] = \sum_1^r t_i^2 [\theta_i, \theta_i],$$

the primary obstruction $[\theta, \theta]$ equals, considered as element of $H^0(\Omega^1_S \otimes \Omega^2_S)^\vee$, $\sum_1^r t_i^2 \alpha_{\nu_i}$ and then, by condition 2, it vanishes only for $\theta = 0$. Thus S is rigid, since no infinitesimal deformation of S (corresponding to θ) can be lifted to a local deformation of S. $\qquad\square$

Let S be a minimal surface of general type and let X be its canonical model. Then $\mathrm{Def}(S)$ (respectively $\mathrm{Def}(X)$) denotes the base of the Kuranishi family of deformations of S (respectively of X).

Let G be a finite group acting faithfully on a smooth algebraic surface Z and let $p\colon Z \to Z/G$ be the quotient map. If p is unramified in codimension 1, then by [Cat89, Lemma 4.1] the natural map $(p_*\Theta_Z)^G \to \Theta_X$ is an isomorphism.

We get thus the following special case of the more general [Cat89, Corollary 1.20]

Corollary 1.4. *Let Z be a smooth algebraic surface, let G be a finite group acting on Z in such a way that the quotient map $p\colon Z \to X = Z/G$ is unramified in codimension 1, and the singular locus of X is a set of r nodes. If $h^1(\Theta_Z)^G = 0$ and condition 2 in Theorem 1.3 holds for X, then S is rigid and $h^1(\Theta_S) = r$.*

In particular, $\mathrm{Def}(S)$ is scheme of embedding dimension r supported in a point.

2. CHARACTER DECOMPOSITION OF THE DIRECT IMAGE OF THE BICANONICAL SHEAF OF AN ABELIAN COVER

Let G be a finite Abelian group, acting on a normal complex variety X, such that X/G is smooth, and denote by $\pi\colon X \to X/G =: Y$ the quotient map.

We assume X to be Gorenstein. Then the dualizing sheaf ω_X is invertible.

G acts on $\pi_*\mathcal{O}_X$, $\pi_*\omega_X$, $\pi_*\omega_X^{\otimes 2}$ inducing a direct sum decomposition in eigensheaves according to the characters as follows:

$$\pi_*\mathcal{O}_X = \oplus_{\chi \in G^*} \mathcal{L}_\chi^{-1},$$

$$\pi_*\omega_X = \oplus_{\chi \in G^*} (\pi_*\omega_X)^{(\chi)},$$

$$\pi_*\omega_X^{\otimes 2} = \oplus_{\chi \in G^*} (\pi_*\omega_X^{\otimes 2})^{(\chi)}.$$

In the following we prefer to use the additive notation for the character group G^*.

We need the following mild generalization of a result of Pardini ([Par91]):

Proposition 2.1. $(\pi_*\omega_X)^{(\chi)} \cong \omega_Y \otimes \mathcal{L}_{-\chi}.$

Proof. In ([Par91, Proposition 4.1]) the claim is proved under the hypothesis that X be smooth. Since ω_X is an invertible sheaf the proof of loc.cit. works without modifications. $\qquad\square$

Let R, respectively D, be the ramification, respectively the branch, locus of π. Then D is a Cartier divisor and R is a $\mathbb{Q}$-Cartier divisor. The stabilizer of each irreducible component T of R is a cyclic group H. Moreover, there is a character $\psi \in H^*$ such that for all smooth points p of X contained in T, there are local coordinates $(z_1, \ldots, z_n)$ around p with $T = \{z_n = 0\}$ and such that $\forall g \in H$, $g^* z_i = z_i$, for $1 \le i \le n-1$ and $g^*(z_n) = \psi(g) z_n$. This induces a splitting

$$R = \sum_{H, \psi} R_{H, \psi}.$$

Since two irreducible components of R dominating the same component of D are contained in the same divisor $R_{H, \psi}$ this induces a splitting

$$D = \sum_{H, \psi} D_{H, \psi}.$$

For the details we refer to [Par91].

Proposition 2.2.

$$\left(\pi_* \omega_X^{\otimes 2}\right)^{(\chi)} \cong (\pi_* \omega_X)^\chi \otimes \omega_Y \left(\sum_{\chi_{|H} \ne \psi} D_{H, \psi}\right) \cong \omega_Y^{\otimes 2} \otimes \mathcal{L}_{-\chi} \left(\sum_{\chi_{|H} \ne \psi} D_{H, \psi}\right).$$

Proof. We show that the cokernel of the injective morphism

(2.1) $$(\pi_* \omega_X)^\chi \otimes (\pi_* \omega_X)^G \to (\pi_* \omega_X^{\otimes 2})^\chi$$

is supported on the divisor $\sum_{\chi_{|H} \ne \psi} D_{H, \psi}$, and has multiplicity 1 in each of these irreducible divisors.

For each $q \in X \setminus \operatorname{Sing} X$ let H be the (possibly trivial) stabilizer of q, $m = |H|$. If $m \ge 2$, then there is a ψ such that $\pi(q) \in D_{H, \psi}$. Then there are local coordinates $(z_1, \ldots, z_n)$ in a neighbourhood of q such that for all $g \in H$, $g^* z_i = z_i$, for $1 \le i \le n-1$ and $g^*(z_n) = \psi(g) z_n$. If instead $m = 1$, we can choose any local coordinates in a neighbourhood of q.

A local generator of $(\pi_* \omega_X)^G$ is $z_n^{m-1} dz_1 \wedge \ldots \wedge dz_n$.

Similarly, local generators of $(\pi_* \omega_X^{\otimes k})^\chi$ are

$$z_n^{a_k}(dz_1 \wedge \ldots \wedge dz_n)^{\otimes k}, \ 0 \le a_k \le m - 1.$$

Note that $a_k = 0 \Leftrightarrow \chi_{|H} = \psi^k$. If $a_1 = 0$, equiv. if $\chi_{|H} = \psi$, the tensor product of the given local generators of $(\pi_* \omega_X)^G$ and $(\pi_* \omega_X)^\chi$ maps to the given local generator of $(\pi_* \omega_X^{\otimes 2})^\chi$, and then the map (2.1) is an isomorphism in a neighbourhood of q.

On the other hand, if $a_1 \neq 0$, equiv. if $\chi_{|H} \neq \psi$, the same tensor product maps to z_n^m times the given local generator of $(\pi_* \omega_X^{\otimes 2})^\chi$. Now, z_n^m is the pull-back of a local generator of the ideal of $D_{H,\psi}$ at $\pi(q)$, and this implies the result. $\qquad \square$

3. An infinite series of product-quotient surfaces

The aim of this section is to construct for each even $n \in \mathbb{N}$, such that $3 \nmid n$, a surface X_n of general type having 6 nodes as singularities. Let be $G := (\mathbb{Z}/n\mathbb{Z})^2$. For $n \geq 2$, let $p \colon C^{(n)} \to \mathbb{P}^1$ be the G-Galois cover branched on $\{0, 1, \infty\}$ with local monodromies $g_0 = (1, 0)$ at 0, $g_\infty = (0, 1)$ at ∞ and therefore $g_1 = (-1, -1)$ at 1.

In other words, using the notation of section 2, there are three branch divisors $D_{H,\psi}$ of positive degree, all of degree 1. In fact, each branch point $p \in \{0, 1, \infty\}$ is the branch divisor D_{H_p, ψ_p}, where $H_p = \langle g_p \rangle$ and $\psi \colon H_p \to \mathbb{C}^*$ is the character mapping g_p to $\eta := e^{\frac{2\pi i}{n}}$.

Remark 3.1.

1) We recall that giving a $(\mathbb{Z}/n\mathbb{Z})^2$-Galois cover $p \colon C^{(n)} \to \mathbb{P}^1$ branched on $\{0, 1, \infty\}$ as above is essentially equivalent to give generators g_0, g_1, g_∞ of $(\mathbb{Z}/n\mathbb{Z})^2$ such that $g_0 + g_1 + g_\infty = 0$. For details (in a much more general setting) we refer to [BCGP12, page 1002].

2) A finite Galois cover of $\mathbb{P}^1$ branched on $\{0, 1, \infty\}$ is called a triangle curve.

3) By Hurwitz' formula the genus of $C^{(n)}$ is

$$g(C^{(n)}) = 1 + \frac{n^2}{2} \left(-2 + 3 \frac{n-1}{n} \right) = 1 + \frac{n(n-3)}{2}.$$

Notation 3.2. For describing the characters in G^* we fix a bijection $\mathbb{Z}/n\mathbb{Z} \to \{0, 1, \ldots, n-1\}$, in other words if we write a character χ as (α, β) we automatically assume that $0 \leq \alpha, \beta \leq n - 1$.

Then

$$\forall (a, b) \in G \quad \chi(a, b) = (\alpha, \beta)(a, b) = \eta^{\alpha a + \beta b},$$

whence

$$\chi_{|H_0} = \psi_0^\alpha, \qquad\qquad \chi_{|H_\infty} = \psi_\infty^\beta, \qquad\qquad \chi_{|H_1} = \psi_1^{-\alpha-\beta}.$$

Splitting $p_* \mathcal{O}_{C^{(n)}} = \oplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}$ as sum of line bundles according to the action of G, using [Par91, Theorem 2.1], we get the formula

$$\mathcal{L}_{(\alpha,\beta)}^n \cong \mathcal{O}_{\mathbb{P}^1}(\alpha p_0 + \beta p_\infty + \gamma p_1),$$

where γ is the unique integer $0 \leq \gamma \leq n - 1$ such that $\alpha + \beta + \gamma$ is divisible by n. It follows that $\mathcal{L}_{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}\left(\frac{\alpha+\beta+\gamma}{n} \right)$, in particular:

$$\mathcal{L}_{(0,0)} = \mathcal{O}_{\mathbb{P}^1},$$
$$\mathcal{L}_{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1}(1) \text{ if } 1 \leq \alpha + \beta \leq n,$$
$$\mathcal{L}_{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1}(2) \text{ if } \alpha + \beta \geq n + 1.$$

	0	1	2	3				n-2	n-1
n-1	-1 -1 -1 -1 -1	*	-1 -1 -1 -1 -1
n-2	-1 0 -1 -1 -1	*	-1 -1 -1 -1 -1
	-1 0 0 -1 -1	*	-1 -1 -1 -1 -1
	-1 0 0 0 -1	*	-1 -1 -1 -1 -1
	-1 0 0 0 0	*	-1 -1 -1 -1 -1
	* * * * *	*	* * * * *
	-1 0 0 0 0	*	0 -1 -1 -1 -1
3	-1 0 0 0 0	*	0 0 -1 -1 -1
2	-1 0 0 0 0	*	0 0 0 -1 -1
1	-1 0 0 0 0	*	0 0 0 0 -1
0	-2 -1 -1 -1 -1	*	-1 -1 -1 -1 -1

FIGURE 1. The degrees of $(p_*\omega_{C^{(n)}})^{(\alpha,\beta)}$

Therefore, by Proposition 2.1, and observing that if $\chi = (\alpha, \beta)$, then (if $\alpha, \beta \neq 0$) $-\chi = (n - \alpha, n - \beta)$, we obtain that the summands of $p_*\omega_{C^{(n)}}$ are

$$(p_*\omega_{C^{(n)}})^{(0,0)} = \mathcal{O}_{\mathbb{P}^1}(-2),$$
(3.1)
$$(p_*\omega_{C^{(n)}})^{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1} \text{ if } \alpha, \beta \neq 0, \alpha + \beta \leq n - 1,$$
$$(p_*\omega_{C^{(n)}})^{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1}(-1) \text{ else.}$$

Remark 3.3. This implies in particular

$$H^0(\omega_{C^{(n)}}) = \bigoplus_{\substack{\chi=(\alpha,\beta) \\ \alpha+\beta\leq n-1 \\ \alpha,\beta\geq 1}} \omega^{(\chi)}\mathbb{C}.$$

where $\omega^{(\chi)}$ is a global form such that $\forall g \in G, g^*\omega^{(\chi)} = \chi(g)\omega^{(\chi)}$.

Denoting by $R_{H,\psi}$ the reduced preimage of $D_{H,\psi}$ under p, the divisor of $\omega^{(\chi)}$ is

$$(\omega^{(\chi)}) = (\alpha - 1)R_{H_0,\psi_0} + (\beta - 1)R_{H_\infty,\psi_\infty} + (n - \alpha - \beta - 1)R_{H_1,\psi_1}.$$

Then $(p_*\omega_{C^{(n)}}^{\otimes 2})^{(\chi)}$ can be determined by Proposition 2.2.

Proposition 3.4. *If $n \geq 4$, then*

$$(p_*\omega_{C^{(n)}}^{\otimes 2})^{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1}(-1), \quad \text{if } (\alpha, \beta) \in \{0,1\}^2 \cup \{(0, n-1), (n-1, 0)\}$$
$$\cup \{(1, n-1), (n-1, 1)\} \cup \{(1, n-2), (n-2, 1)\},$$
$$(p_*\omega_{C^{(n)}}^{\otimes 2})^{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1}(1), \quad \text{if } \alpha, \beta \geq 2, \alpha + \beta \leq n - 2,$$
$$(p_*\omega_{C^{(n)}}^{\otimes 2})^{(\alpha,\beta)} = \mathcal{O}_{\mathbb{P}^1}, \quad \text{else.}$$

	0	1	2	3			n-3	n-2	n-1
n-1	-1 -1	0	0	0	*	0	0	0	0 0
n-2	0 -1	0	0	0	*	0	0	0	0 0
n-3	0 0	0	0	0	*	0	0	0	0 0
	0 0	1	0	0	*	0	0	0	0 0
	0 0	1	1	0	*	0	0	0	0 0
	* *	*	*	*	*	*	*	*	* *
	0 0	1	1	1	*	0	0	0	0 0
3	0 0	1	1	1	*	1	0	0	0 0
2	0 0	1	1	1	*	1	1	0	0 0
1	-1 -1	0	0	0	*	0	0	0	-1 -1
0	-1 -1	0	0	0	*	0	0	0	0 -1

FIGURE 2. The degrees of $(p_*\omega^2_{C^{(n)}})^{(\alpha,\beta)}$

Proof. By Proposition 2.2, $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)} \cong (p_*\omega_{C^{(n)}})^{(\alpha,\beta)} \otimes \mathcal{O}_{\mathbb{P}^1}(\delta - 2)$ where δ is the degree of the divisor D with $0 \leq D \leq p_0 + p_1 + p_\infty$ such that

$$p_0 \leq D \Leftrightarrow \alpha \neq 1, \qquad p_\infty \leq D \Leftrightarrow \beta \neq 1, \qquad p_1 \leq D \Leftrightarrow \alpha + \beta \neq n-1.$$

This leads us to consider the three lines $\alpha = 1$, $\beta = 1$ and $\alpha + \beta = n - 1$ and the triangle they form.

In the three vertices $(1,1), (1, n-2), (n-2, 1)$ of the triangle $\delta = 1$. By (3.1) they all have $(p_*\omega_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}$ and therefore $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$.

In the remaining points of these three lines, $\delta = 2$ and then $(p_*\omega_{C^{(n)}})^{(\alpha,\beta)} \cong (p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)}$.

By (3.1), if $\chi \in \{(1,0), (0,1), (0, n-1), (n-1, 0), (1, n-1), (n-1, 1)\}$, then $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, else $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}$.

Finally, outside the three lines we have $\delta = 3$.

If χ is inside the triangle, then by (3.1) $(p_*\omega_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}$ and $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}(1)$.

For $(\alpha, \beta) = (0, 0)$, then $(p_*\omega_{C^{(n)}})^{(0,0)} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ whence $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(0,0)} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$.

In the remaining cases $(p_*\omega_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $(p_*\omega^{\otimes 2}_{C^{(n)}})^{(\alpha,\beta)} \cong \mathcal{O}_{\mathbb{P}^1}$. $\square$

From now on we fix $n \geq 4$, even and $3 \nmid n$ and we denote $C^{(n)}$ simply by C.

Let $Z := C \times C$ and we define the following action of G on $C \times C$: for $(a, b) \in G$, for $(z_1, z_2) \in C \times C$

$$(a, b)(z_1, z_2) := ((a, b)z_1, (a', b')z_2),$$

where

$$A \begin{pmatrix} a' \\ b' \end{pmatrix} := \begin{pmatrix} a \\ b \end{pmatrix} \;, \quad A := \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

Since $3 \nmid n$, $A \in GL\left(2, \mathbb{Z}/n\mathbb{Z}\right)$.

Remark 3.5. In other words, we take two different G-actions on the same curve C, differing by an automorphism of G.

The first one is as before $p_1 \colon C_1 \cong C \to C_1/G \cong \mathbb{P}^1$ branched on $\{0, 1, \infty\}$ with local monodromies $g_0 = (1, 0)$ at 0, $g_\infty = (0, 1)$ at ∞ and $g_1 = (-1, -1)$ at 1, and the second one is $p_2 \colon C_2 \cong C \to C_2/G \cong \mathbb{P}^1$ branched on $\{0, 1, \infty\}$ with local monodromies $h_0 = (1, 2)$ at 0, $h_\infty = (-2, -1)$ at ∞ and $h_1 = (1, -1)$ at 1.

The local monodromies of p_2 are then the images of the local monodromies of p_1 by the matrix A.

Remark 3.6. The line bundles $((p_1)_* \omega_{C_1}^{\otimes k})^\chi$, $k \in \{1, 2\}$, are exactly the line bundles $(p_* \omega_{C^{(n)}})^\chi$ computed in (3.1) and Proposition 3.4.

Instead, for the action on the second factor, we observe that

$$((p_2)_* \omega_{C_2}^{\otimes k})^\chi \cong (p_* \omega_{C^{(n)}})^{\chi'}$$

where, if $\chi = (\alpha, \beta)$, then $\chi' = (\alpha', \beta')$ with

$${}^t A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \quad \mod n.$$

Let be $X_n := (C \times C)/G$ and let $\rho \colon S_n \to X_n$ be the minimal resolution of the singularities of X_n.

Proposition 3.7. *For each even $n \geq 4$, not divisible by 3, X_n has six nodes as only singularities. S_n is a minimal regular surface of general type with invariants:*

$$K_{S_n}^2 = 2(n-3)^2,$$

$$\chi(\mathcal{O}_{S_n}) = \frac{n^2 - 6n + 12}{4},$$

$$p_g(S_n) = \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right).$$

Proof. Note that $\langle g_p \rangle \cap \langle h_q \rangle = \{(0, 0)\}$ for $p \neq q$ whereas $\langle g_p \rangle \cap \langle h_p \rangle \cong \mathbb{Z}/2\mathbb{Z}$. More precisely, $\langle g_p \rangle \cap \langle h_p \rangle = \langle s_p \rangle$ where $s_0 = \left(\frac{n}{2}, 0\right)$, $s_\infty = \left(0, \frac{n}{2}\right)$, $s_1 = \left(\frac{n}{2}, \frac{n}{2}\right)$.

For all $p \in \{0, 1, \infty\}$ there are n^2 points of $C \times C$ lying over $(p, p) \in \mathbb{P}^1 \times \mathbb{P}^1$. Since $\langle g_p \rangle \cap \langle h_p \rangle$ has order 2, they split in $\frac{n^2}{\frac{n^2}{2}} = 2$ orbits, so producing each 2 nodes on the quotients $X_n := (C \times C)/G$. Hence X_n has exactly $3 \cdot 2 = 6$ nodes.

By [BCGP12] S_n is regular and

$$K_{S_n}^2 = \frac{8(g_1 - 1)(g_2 - 1)}{|G|} = \frac{8\left(\frac{n(n-3)}{2}\right)^2}{n^2} = 2(n-3)^2,$$

$$\chi(\mathcal{O}_{S_n}) = \frac{K_{S_n}^2 + 6}{8} = \frac{2n^2 - 12n + 24}{8} = \frac{n^2 - 6n + 12}{4},$$

$$p_g(S_n) = \frac{n^2 - 6n + 8}{4} = \frac{(n-4)(n-2)}{4} = \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right).$$

$\square$

Remark 3.8. X_n are (singular models of) so-called *product-quotient* surfaces, introduced in [BP12], [BCGP12].

4. THE DEFORMATIONS OF S_n AND X_n

This section is dedicated to the proof of our main result.

Theorem 4.1. *Let $n \in \mathbb{N}$ be an even number ≥ 8, not divisible by 3. Then S_n is rigid and $h^1(\Theta_{S_n}) = 6$.*

In particular, S_n is an infinite series of minimal regular surfaces of general type with unbounded invariants which are rigid, but not infinitesimally rigid.

Proof. Set $Z := C \times C$ and let $\pi\colon Z \to X_n = Z/G$ be the quotient map.

Since C is a triangle curve, $h^1(\Theta_C)^G = 0$ and then $h^1(\Theta_Z)^G = 2h^1(\Theta_C)^G = 0$.

Since π is unramified in codimension 1, the result follows by Corollary 1.4 if condition 2 in Theorem 1.3 holds for X_n.

This will be proven in Proposition 4.3. $\square$

First observe that by [Cat89, Proposition 1.6] we know that
$$H^0(\Omega_{S_n}^1 \otimes \Omega_{S_n}^2) \cong H^0(\Omega_{C \times C}^1 \otimes \Omega_{C \times C}^2)^G.$$

By the Künneth formula (cf. [Kau67]) we have:

(4.1) $\quad H^0(\Omega_{C \times C}^1 \otimes \Omega_{C \times C}^2) = \left(\left(H^0(\omega_C^{\otimes 2}) \otimes H^0(\omega_C)\right) \oplus \left(H^0(\omega_C) \otimes H^0(\omega_C^{\otimes 2})\right)\right).$

The group G acts on both sides of equation (4.1) producing a Künneth decomposition of each eigenspace $H^0(\Omega_{C \times C}^1 \otimes \Omega_{C \times C}^2)^{(\chi)}$. The result for the G-invariant part is, by Remark 3.6,

(4.2) $\quad H^0(\Omega_{C \times C}^1 \otimes \Omega_{C \times C}^2)^G =$

$$= \bigoplus_{\chi \in G^*} \left(\left(H^0(\omega_C^{\otimes 2})^{(\chi)} \otimes H^0(\omega_C)^{(-\chi')}\right) \oplus \left(H^0(\omega_C)^{(\chi)} \otimes H^0(\omega_C^{\otimes 2})^{(-\chi')}\right)\right)$$

where, if $\chi = (\alpha, \beta)$, then $\chi' = (\alpha', \beta')$ with

(4.3) $$\qquad {}^t A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \quad \mod n.$$

Proposition 4.2. *Set $k_0 = k_1 = (1,0)$, $k_\infty = (0,1) \in G$.*

Assume that there is a set of six characters $\mathcal{C} := \{\chi_0, \chi_0', \chi_1, \chi_1', \chi_\infty, \chi_\infty'\} \subset G^*$, such that

(1) $\chi_0 \equiv \chi'_0 \equiv (0,1)$, $\chi_1 \equiv \chi'_1 \equiv (1,1)$, $\chi_\infty \equiv \chi'_\infty \equiv (1,0)$ *mod 2;*
(2) $\forall p \in \{0,1,\infty\}$, $\chi_p(k_p) \neq \chi'_p(k_p)$;
(3) *if* $\chi \in \mathcal{C}$, *then* $H^0(\omega_C)^{(\chi)} \neq \{0\}$,
(4) *if* $\chi \in \mathcal{C}$, *then* $H^0(\omega_C^{\otimes 2})^{(-\chi')} \neq \{0\}$, *where* χ *and* χ' *are related as in (4.3).*

Then condition 2 in Theorem 1.3 holds for X_n.

Proof. For each $p \in \{0,1,\infty\}$ choose a point $q_p \in C \times C$ whose image in $\mathbb{P}^1 \times \mathbb{P}^1$ is (p,p). Then the image of $\{q_p : p \in \{0,1,\infty\}\}$ in X_n, say $\{\nu_p : p \in \{0,1,\infty\}\}$, consists of three nodes.

For every $\chi = (\alpha,\beta) \in \mathcal{C}$ the summand

$$H^0(\omega_C)^{(\chi)} \otimes H^0(\omega_C^{\otimes 2})^{(-\chi')}$$

in (4.2) has by assumption (3) and (4) a nontrivial section ξ_χ.

The stabilizer of q_p is, in the notation of Remark 3.5, $\langle g_p \rangle \cap \langle h_p \rangle \cong \mathbb{Z}/2\mathbb{Z}$, generated by s_p, where $s_0 = \left(\frac{n}{2},0\right)$, $s_\infty = \left(0,\frac{n}{2}\right)$, $s_1 = \left(\frac{n}{2},\frac{n}{2}\right)$.

We choose local coordinates (x_{q_p}, y_{q_p}) at q_p such that x_{q_p} is a local parameter for the first factor C, y_{q_p} is a local parameter for the second copy of C,

$$s_p^* x_{q_p} = -x_{q_p}, \quad s_p^* y_{q_p} = -y_{q_p},$$

and

$$\xi_\chi = x_{q_p}^{\lambda_p} dx_{q_p} \otimes y_{q_p}^{\mu_p} (dy_{q_p})^2$$

Since a complete linear system on $\mathbb{P}^1$ has no base points, $\lambda_p, \mu_p \in \{0,1\}$.

Moreover $s_p^* x_{q_p}^{\lambda_p} dx_{q_p} = (-1)^{\lambda_p+1} x_{q_p}^{\lambda_p} dx_{q_p} = \chi(s_p) x_{q_p}^{\lambda_p} dx_{q_p}$ implies

$$s_p \in \ker \chi \Leftrightarrow \lambda_p = 1.$$

Similarly $s_p^* y_{q_p}^{\mu_p} (dy_{q_p})^2 = (-1)^{\mu_p} y_{q_p}^{\mu_p} (dy_{q_p})^2 = \chi(s_p) y_{q_p}^{\mu_p} (dy_{q_p})^2$ implies

$$s_p \in \ker \chi \Leftrightarrow \mu_p = 0.$$

Next we look for three more points $q'_p \in C \times C$, which lie over the remaining three nodes $\{\nu'_p : p \in \{0,1,\infty\}\}$, such that ν'_p maps to $(p,p) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Note that $G \times G$ acts on $C \times C$ and that one half of the points in the orbit of q_p under this action maps to ν_p and the other half to ν'_p. More precisely, if $(g,g') = ((a,b)(a',b'))$, then $(g,g')q_0$ maps to the node ν_0 if and only if $a + a'$ is even. Similarly $(g,g')q_1$ resp. $(g,g')q_\infty$ maps on the node ν_1 resp. ν_∞ if and only if $b + b'$ resp $a + a' + b + b'$ is even.

Choosing q'_p such that $(k_p,(0,0))q'_p = q_p$, we get by our choice of k_p that q'_p maps to ν'_p for $p \in \{0,1,\infty\}$.

We pull back the local coordinates around q_p to local coordinates around q'_p:

$$x_{q'_p} = k_p^* x_{q_p}, \quad y_{q'_p} = y_{q_p}.$$

Then

$$x_{q'_p}^{\lambda_p} dx_{q'_p} \otimes y_{q'_p}^{\mu_p} (dy_{q'_p})^2 = (k_p,(0,0))^* \xi_\chi = \chi(k_p)\xi_\chi,$$

hence

$$\xi_\chi = \chi(k_p)^{-1} x_{q'_p}^{\lambda_p} dx_{q'_p} \otimes y_{q'_p}^{\mu_p} (dy'_{q_p})^2.$$

By (1.1), if ξ_χ has local expression $f(x, y)dx \otimes (dy)^2$ in the chosen local coordinates near one of our points q_p (resp. q'_p), then $\alpha_{\nu_p}(\xi_\chi)$ (resp. $\alpha_{\nu'_p}(\xi_\chi)$) equals $\frac{\partial f}{\partial x}(0, 0)$.

We observe now that $\forall p \neq p' \in \{0, 1, \infty\}$, $s_p \in \ker \chi_p, \ker \chi'_p$ and $s_{p'} \notin \ker \chi_p, \ker \chi'_p$. So the local expression of $\xi_{\chi_p}, \xi_{\chi'_p}$ at q_p has $f = x$ whereas its local expression at $q_{p'}$ has $f = y$. Then

$$(\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty})(\xi_{\chi_0}) = (1, \chi_0(h_0)^{-1}, 0, 0, 0, 0)$$
$$(\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty})(\xi_{\chi'_0}) = (1, \chi'_0(h_0)^{-1}, 0, 0, 0, 0)$$
$$(\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty})(\xi_{\chi_1}) = (0, 0, 1, \chi_1(h_1)^{-1}, 0, 0)$$
$$(\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty})(\xi_{\chi'_1}) = (0, 0, 1, \chi'_1(h_1)^{-1}, 0, 0)$$
$$(\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty})(\xi_{\chi_\infty}) = (0, 0, 0, 0, 1, \chi_\infty(h_\infty)^{-1})$$
$$(\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty})(\xi_{\chi'_\infty}) = (0, 0, 0, 0, 1, \chi'_\infty(h_\infty)^{-1})$$

Since by condition 2 the six vectors on the right are linearly independent, then $\alpha_{\nu_0}, \alpha_{\nu'_0}, \alpha_{\nu_1}, \alpha_{\nu'_1}, \alpha_{\nu_\infty}, \alpha_{\nu'_\infty}$ are linearly independent too. $\square$

Finally we can prove

Proposition 4.3. *For all $n \geq 8$ even and not divisible by 3 condition 2 in Theorem 1.3 holds for X_n.*

Proof. It suffices to find a set $\mathcal{C}$ of characters of G as in Proposition 4.2.

We take the characters

$$\chi_0 = (2, 1) \qquad\qquad \chi_1 = (1, 3) \qquad\qquad \chi_\infty = (1, 2)$$
$$\chi'_0 = (4, 1) \qquad\qquad \chi'_1 = (3, 1) \qquad\qquad \chi'_\infty = (1, 4)$$

Conditions (1) and (2) are obvious. Condition (3) follows by (3.1).

Finally, condition (4) follows by Proposition 3.4 if $\forall \chi = (\alpha, \beta) \in \mathcal{C}$ and $\forall (\alpha', \beta') \in \{0, 1\}^2 \cup \{(0, n-1), (n-1, 0)\} \cup \{(1, n-1), (n-1, 1)\} \cup \{(1, n-2), (n-2, 1)\}$

$${}^t A^{-1} \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \not\equiv \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \mod n$$

We need then to distinguish the two cases $n = 3m \pm 1$. Note that in both cases $n \geq 8$ gives $m \geq 3$.

If $n = 3m - 1$, ${}^t A^{-1} = \begin{pmatrix} -m & -2m \\ 2m & m \end{pmatrix}$, hence

$${}^t A^{-1} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} m+1 \\ m-2 \end{pmatrix}, \quad {}^t A^{-1} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \equiv \begin{pmatrix} m+2 \\ m-2 \end{pmatrix}, \quad {}^t A^{-1} \begin{pmatrix} -1 \\ -2 \end{pmatrix} \equiv \begin{pmatrix} 2m+1 \\ 2m-2 \end{pmatrix},$$

$${}^t A^{-1} \begin{pmatrix} -4 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ n-3 \end{pmatrix}, \quad {}^t A^{-1} \begin{pmatrix} -3 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} 2m+1 \\ 2m-3 \end{pmatrix}, \quad {}^t A^{-1} \begin{pmatrix} -1 \\ -4 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ n-2 \end{pmatrix}.$$

If $n = 3m + 1$, $^tA^{-1} = \begin{pmatrix} m & 2m \\ -2m & -m \end{pmatrix}$ and

$$^tA^{-1}\begin{pmatrix} -2 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} 2m+2 \\ 2m-1 \end{pmatrix}, \quad ^tA^{-1}\begin{pmatrix} -1 \\ -3 \end{pmatrix} \equiv \begin{pmatrix} 2m+3 \\ 2m-1 \end{pmatrix}, \quad ^tA^{-1}\begin{pmatrix} -1 \\ -2 \end{pmatrix} \equiv \begin{pmatrix} m+2 \\ m-1 \end{pmatrix},$$

$$^tA^{-1}\begin{pmatrix} -4 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ n-3 \end{pmatrix}, \quad ^tA^{-1}\begin{pmatrix} -3 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} m+2 \\ m-2 \end{pmatrix}, \quad ^tA^{-1}\begin{pmatrix} -1 \\ -4 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ n-2 \end{pmatrix}.$$

$\square$

5. Higher dimensional examples

The aim of this section is to give examples of rigid compact complex manifolds which are not infinitesimally rigid in all dimensions $d \geq 3$.

The main result is the following

Theorem 5.1. *Let $n \geq 8$ be an even integer such that $3 \nmid n$, and let X be a compact complex rigid manifold.*

Then $S_n \times X$ is rigid, but not infinitesimally rigid.

In particular there are rigid, but not infinitesimally rigid, manifolds of dimension d and Kodaira dimension κ for all possible pairs (d, κ) with $d \geq 5$ and $\kappa \neq 0, 1, 3$ and for $(d, \kappa) = (3, -\infty), (4, -\infty), (4, 4)$.

Let X and Y be compact complex manifolds. Then by the Künneth formula (cf. [Kau67]) we have:

$$(5.1) \quad H^1(\Theta_{X \times Y}) = H^1(\Theta_X) \oplus (H^0(\Theta_X) \otimes H^1(\mathcal{O}_Y)) \oplus$$
$$\oplus (H^1(\mathcal{O}_X) \otimes H^0(\Theta_Y)) \oplus H^1(\Theta_Y).$$

Before proving the Theorem we need the following result, which is probably well-known. For lack of a suitable reference we will give a sketch of proof.

Lemma 5.2. *Let X, Y be compact complex manifolds, such that*
$$H^1(\Theta_{X \times Y}) = H^1(\Theta_X) \oplus H^1(\Theta_Y).$$
Then $\mathrm{Def}(X \times Y) = \mathrm{Def}(X) \times \mathrm{Def}(Y)$.

Here $\mathrm{Def}(X)$ denotes as usual the base of the Kuranishi family of the compact complex manifold X.

Proof. We choose a Hermitian metric on X respectively on Y (i.e. on the respective holomorphic tangent bundles $T_X^{1,0}$ resp. $T_Y^{1,0}$). This induces (identifying $T_{X \times Y}^{1,0}$ with the direct sum of the pullbacks of $T_X^{1,0}$ resp. $T_Y^{1,0}$) a Hermitian metric on $X \times Y$.

With respect to these Hermitian metrics we have the adjoint operators on differentiable $(0, p)$ forms on each of these spaces, i.e.,

$$\bar{\partial}_\bullet^* \colon \mathcal{E}^{(0,p+1)}(T_\bullet^{1,0}) \to \mathcal{E}^{(0,p)}(T_\bullet^{1,0}),$$

such that for $(\eta_X, \eta_Y) \in \mathcal{E}^{(0,p+1)}(T_X^{1,0}) \oplus \mathcal{E}^{(0,p+1)}(T_Y^{1,0})$ we have

$$\bar{\partial}_{X \times Y}^*(\eta_X + \eta_Y) = \bar{\partial}_X^*(\eta_X) + \bar{\partial}_Y^*(\eta_Y).$$

Then we may (by a slight abuse of the notation) consider $\bar{\partial}_X^*$, resp. $\bar{\partial}_Y^*$ as the restrictions of $\bar{\partial}^* := \bar{\partial}_{X \times Y}^*$. The same holds then for the induced Laplace operator, Green operator G and the harmonic projector H.

Going through the construction of the Kuranishi family (for the details we refer to [Cat88]), identifying $H^1(\Theta_{X \times Y}) = H^1(\Theta_X) \oplus H^1(\Theta_Y)$ with $\mathbb{C}^m \times \mathbb{C}^l$ with coordinates $t = (t_1, \ldots, t_m)$, and $s = (s_1, \ldots, s_l)$ (corresponding to two bases $\eta_1, \ldots, \eta_m$ and $\eta_1', \ldots, \eta_l'$ of the harmonic 1-forms with values in $T_{\bullet}^{1,0}$), and using that then

$$\mathrm{Def}(X \times Y) = \{(t, s) \in \mathbb{C}^m \times \mathbb{C}^l : H[\varphi(t,s), \varphi(t,s)] = 0\},$$

where $\varphi(t,s)$ is the unique solution of the equation

$$\varphi(t,s) = \sum t_i \eta_i + \sum s_j \eta_j' + \frac{1}{2}\bar{\partial}^* G[\varphi(t,s), \varphi(t,s)].$$

It is then easy to see that $\varphi(t,s) = \varphi_1(t) + \varphi_2(s)$, where φ_i are the solutions of the respective equations for each of the factors. This concludes the proof. $\qquad \square$

Proof of Theorem 5.1. Observe that $H^1(\mathcal{O}_{S_n}) = H^0(\Theta_{S_n}) = 0$. Therefore applying the Künneth formula we obtain:

$$H^1(\Theta_{S_n \times X}) = H^1(\Theta_{S_n}) \oplus H^1(\Theta_X) \neq \{0\}.$$

Then $S_n \times X$ is not infinitesimally rigid. Moreover, by the above Lemma, we have that

$$\mathrm{Def}(S_n \times X) = \mathrm{Def}(S_n) \times \mathrm{Def}(X).$$

Therefore, since S_n and X are rigid, also $S_n \times X$ is rigid.

Choosing $X = (\mathbb{P}^1)^{d-2}$ we get examples for all dimension $d \geq 3$ with $\kappa = -\infty$.

Choosing $X = S_m$ ($m \geq 8$ even with $3 \nmid m$) we get examples with $(d, \kappa) = (4, 4)$.

Choosing a rigid manifold X of Kodaira dimension $\kappa \in \{0, 2, \ldots, \dim X\}$ (cf. [BC18, Theorems 3.4, 3.5]), we get rigid and not infinitesimally rigid examples for all dimensions $d \geq 5$ and all possible Kodaira dimensions except $0, 1, 3$. $\qquad \square$

Remark 5.3. Observe that, since rigid manifolds of general type are globally rigid, we found globally rigid manifolds which are not infinitesimally rigid of every dimension $d \geq 2$, $d \neq 3$.

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