Note on a family of surfaces with $p_g = q = 2$ and $K^2 = 7$

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To Professor F. Catanese on the occasion of his 70th birthday

Abstract

We study a family of surfaces of general type with $p_g = q = 2$ and $K^2 = 7$, originally constructed by C. Rito in [Rit18]. We provide an alternative construction of these surfaces, that allows us to describe their Albanese map and the corresponding locus $\mathcal{M}$ in the moduli space of the surfaces of general type. In particular we prove that $\mathcal{M}$ is an irreducible component, two dimensional and generically smooth.

1 Introduction

In the last two decades, several authors worked intensively on the classification of irregular algebraic surfaces (i.e., surfaces S with $q(S) > 0$) and produced a considerable amount of results, see for example the survey papers [BaCaPi06, MP12, Pen13] for a detailed bibliography on the subject.

In particular, irregular surfaces of general type with $\chi(\mathcal{O}_S) = 1$, that is, $p_g(S) = q(S) \geq 1$ were investigated. By, nowadays classical, Debarre inequality [Deb81, Théorème 6.1] we have $p_g \leq 4$. Surfaces with $p_g = q = 4$ and $p_g = q = 3$ are completely classified, see [Bea82, CaCiML98, HP02, Pir02]. On the other hand, for the the case $p_g = q = 2$, which presents a very rich and subtle geometry, we have so far only a partial understanding of the situation; we refer the reader to [Cat00, Cat11, Cat15, Pen09, PePol13a, PePol13b, PePol14, PiPol17, PRR20, Zuc03] for an account on this topic and recent results.

As the title suggest, in this paper we consider a family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$. The existence of these surfaces was originally established by Rito in [Rit18]; the present work provides an alternative construction of them, that allows us to describe their Albanese map and their moduli space.

Our results can be summarized as follows.

Theorem 1.1. *There exists a 2-dimensional family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ such that, for all elements $S \in \mathcal{M}$, the Albanese map $\alpha\colon S \to A$ is a generically finite double cover onto a $(1,2)$-polarized non simple abelian surface A. The family $\mathcal{M}$ provides an irreducible, generically smooth component of the Gieseker moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,7}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$.*

It is worth to notice here, that there is only another known family of surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ found by Cancian and Frapporti in [CanFr15] and described in details in [PiPol17]. This last family is characterized by the fact that its elements have a

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different Albanese map than the elements in $\mathcal{M}$. Namely, the Albanese map is a generically finite *triple* cover of a principally polarized abelian surface. Hence, being the degree of the Albanese map a topological invariant (see Proposition 5.1), the family $\mathcal{M}$ provides a substantially new piece in the fine classification of minimal surfaces of general type with $p_g = q = 2$ in the spirit of [Cat84, Cat89, Cat90].

The paper is organized as follows.

In Section 2 we explain our construction in details, pointing out the similarities and the differences with [Rit18], and computing the invariants of the resulting surfaces (Proposition 2.2). We study their Albanese map $\alpha\colon S \to A$, giving a precise description of its image, isogenous to a product of two curves of genus 1, and of its branch curve.

In Section 3 we use our description to study the modular image of Rito's family, showing that it is irreducible of dimension 2. We notice here that we are not able to prove the irreducibility of the family by the original construction, essentially because when constructing the surface as double cover of A one has to choose a line bundle on A in a set of 16 choices, each choice giving a different surface, and we do not see any obvious way to show that they belong to the same component. Our alternative description gives instead a *parametrization* of $\mathcal{M}$ with an irreducible variety, the product of two families of elliptic curves "decorated" by some torsion points.

The last two sections contain results of deformation theory headed to compute $h^1(S, T_S) = 2$ (Proposition 5.9) from which it follows that the family $\mathcal{M}$ is open in the moduli space.

Section 4 is devoted to a general result, Theorem 4.2, about the deformations of the blow up in a point, that was crucial for the proof and that we find of independent interest. The situation is the following: consider a point p in a smooth surface B, a curve D in B smooth at p and a vector $v \in T_pB$. A standard exact sequence associate to v an infinitesimal deformation $\mathcal{B}$ of the blow-up of B in p. Then Theorem 4.2 says that $\mathcal{B}$ contains an infinitesimal deformation of the strict transform of D if and only if the class of v in the normal vector space T_pB/T_pD extends to a global section of the normal bundle of D in B.

Finally Section 5 is devoted to the study of the first-order deformations of the surfaces in $\mathcal{M}$. To show $h^1(S, T_S) = 2$, we show in fact that the map $H^1(S, T_S) \to H^1(A, T_A)$ is injective, and its image is given by the infinitesimal deformations of A that are still isogenous to a product.

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Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers. By *surface* we mean a projective, non-singular surface S, and for such a surface K_S denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic.*

2 The construction

In this section we give an alternative, but equivalent, construction to the surface S of general type with $p_g = q = 2$ and $K^2 = 7$ constructed by Rito in [Rit18].

Let us fix the following points on $\mathbb{P}^2$:

$$p_0 = (1 : 0 : 0),\ p_1 = (0 : 1 : 0),\ p_2 = (0 : 0 : 1),\ p_3 = (1 : 1 : 1),\ p_4 = (1 : a : b).$$

Moreover, let us denote by r_i with $i = 1, \dots, 4$ the four lines joining p_0 with each p_i resectively, i.e.,

$$r_1 = (x_2),\ r_2 = (x_1),\ r_3 = (x_1 - x_2),\ r_4 = (bx_1 - ax_2),$$

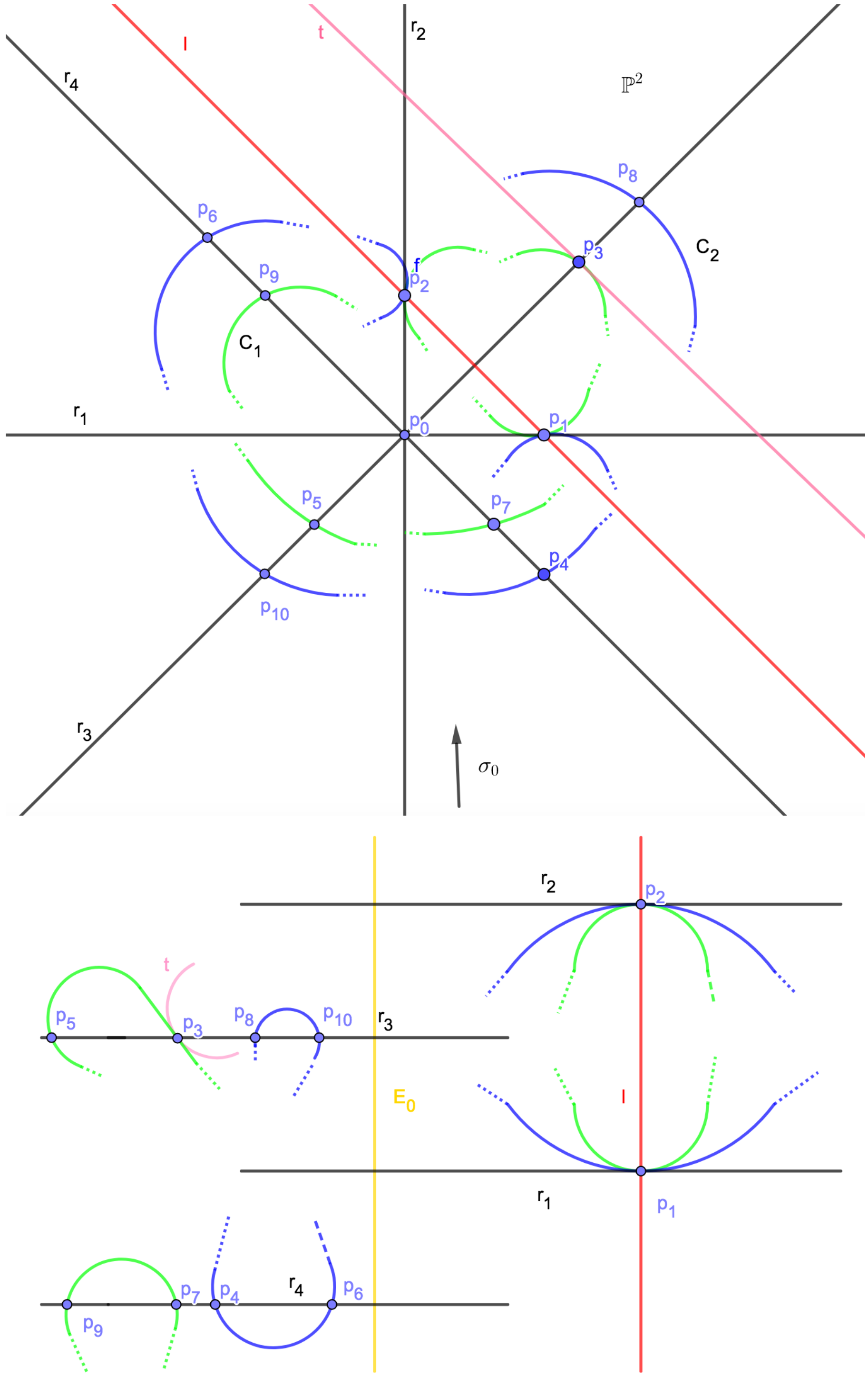


Figure 1: $\sigma_0 \colon Bl_{p_0}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

and the two conics:

$$C_1 = (x_0^2 - x_1x_2), \quad C_2 = (abx_0^2 - x_1x_2).$$

Note that both conics are tangent to r_1 and r_2 respectively in p_1 and p_2. Finally, $p_3 \in C_1$ and $p_4 \in C_2$.

Fix a square root c of ab and consider the following points on the curves we have just defined

$$p_5 = (1 : -1 : -1),\ p_6 = (1 : -a : -b),\ p_7, p_9 = (\pm c : a : b),\ p_8, p_{10} = (\pm 1 : c : c).$$

Finally, let $\ell = (x_0)$ be the line through p_1 and p_2 and $t = (2x_0 - x_1 - x_2)$ be the tangent line to C_1 through p_3, see Figure 1 to have a visual representation of the situation.

Up to now, we followed Rito in [Rit18], changing the notation only for the curve t (R in Rito's notation). Now, we proceed a bit differently. Let us apply the following birational transformations of $\mathbb{P}^2$:

1. We blow up the point p_0 and we get $\sigma_0 \colon Bl_{p_0}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ with exceptional divisor E_0, (see Figure 1 again).

   Considering the pencil of lines through p_0 on $Bl_{p_0}(\mathbb{P}^2)$ we have a rational pencil of curves with self-intersection 0, which include the strict transforms of the four lines r_i, $i = 1, \ldots, 4$. We notice that on $Bl_{p_0}(\mathbb{P}^2)$ we can lift the natural involution on $\mathbb{P}^2$

   $$j \colon (x_0 : x_1 : x_2) \mapsto (-x_0 : x_1 : x_2)$$

   which has as fixed divisor $E_0 + \sigma_0^*(\ell)$.

2. The quotient by this involution $Bl_{p_0}(\mathbb{P}^2)/j$ is the Segre–Hirzebruch surface $\mathbb{F}_2$.

   The images of the four lines r_i are fibres of the fibration on $\mathbb{F}_2$. Moreover, the only negative section of this fibration coincide with the image of E_0.

3. We blow up on $\mathbb{F}_2$ the images of the points p_1 and p_2, introducing two exceptional divisors E_1 and E_2.

   We recall that the images of the lines r_1 and r_2 and of the conics C_1 and C_2 pass all through these points. Performing this operation the images of r_1 and r_2 became -1-curves (see Figure 2).

4. We contract the images of the curves r_1 and r_2. The resulting surface is exactly $\mathbb{P}^1 \times \mathbb{P}^1$.

Summarizing we have obtained a rational map of degree 2

$$\sigma \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

We denote with the same letters the strict transform on $\mathbb{P}^1 \times \mathbb{P}^1$ of all the curves considered on $\mathbb{P}^2$, since no confusion arises (see Figure 3). The bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with ramification divisors

$$D_1 = 0,\ D_2 = E_1 + E_2 + r_3 + r_4,\ D_3 = C_1 + C_2 + E_0 + l,$$

is obviously the product $T_1 \times T_2$ of two double covers $\phi_j \colon T_j \to \mathbb{P}^1$ branched at 4 points, two curves of genus 1 (see Figure 3).

The fibre product of the bidouble cover $T_1 \times T_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ with σ_0 gives the bidouble cover of $\mathbb{P}^2$ studied in [Rit18][Section 3, Step 1], where it is shown that it is birational to an abelian surface that we denote by A (it was V' in [Rit18]). We can summarize this construction with the following diagram.

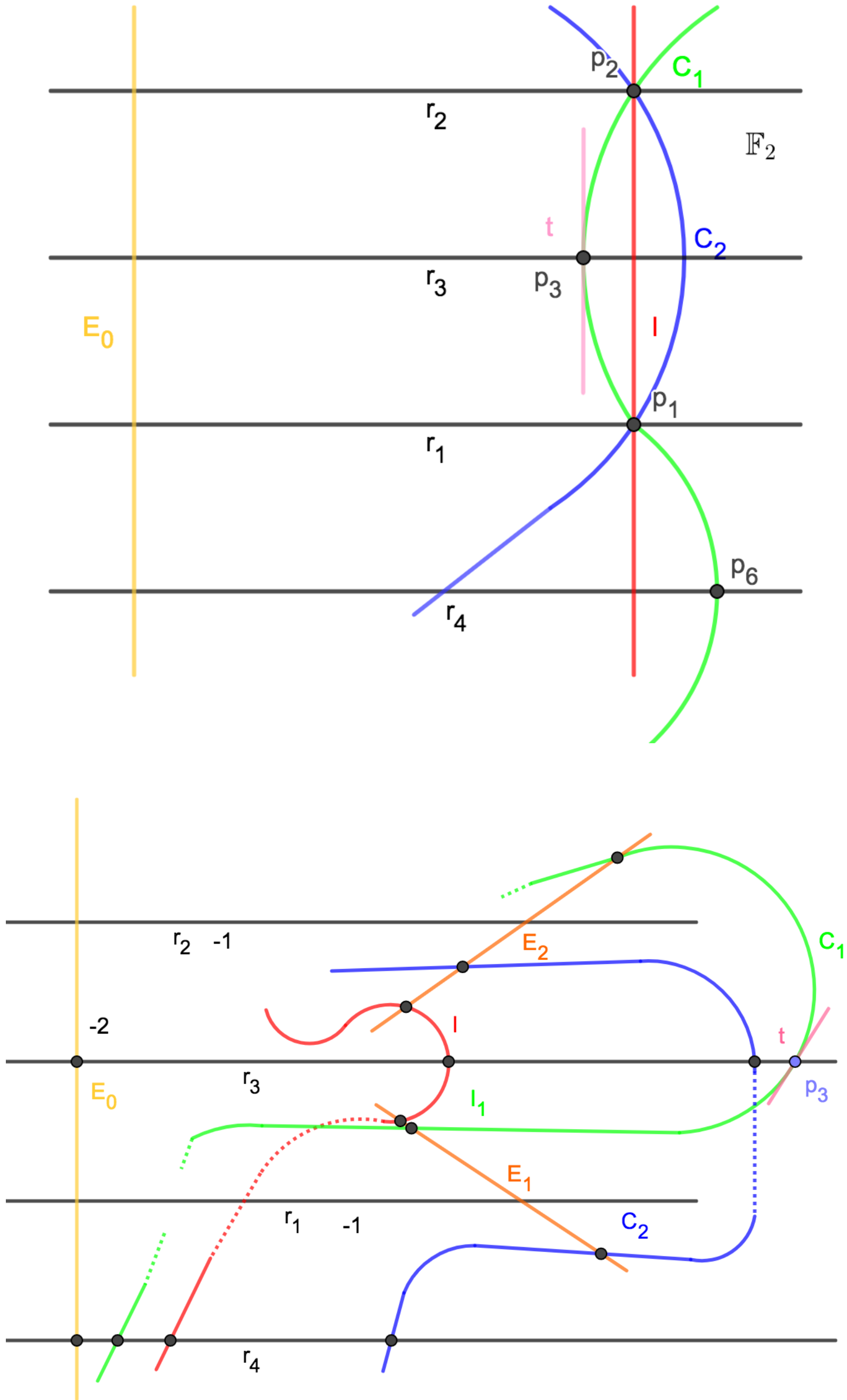


Figure 2: The blow up on $\mathbb{F}_2$ of the images of the points p_1 and p_2

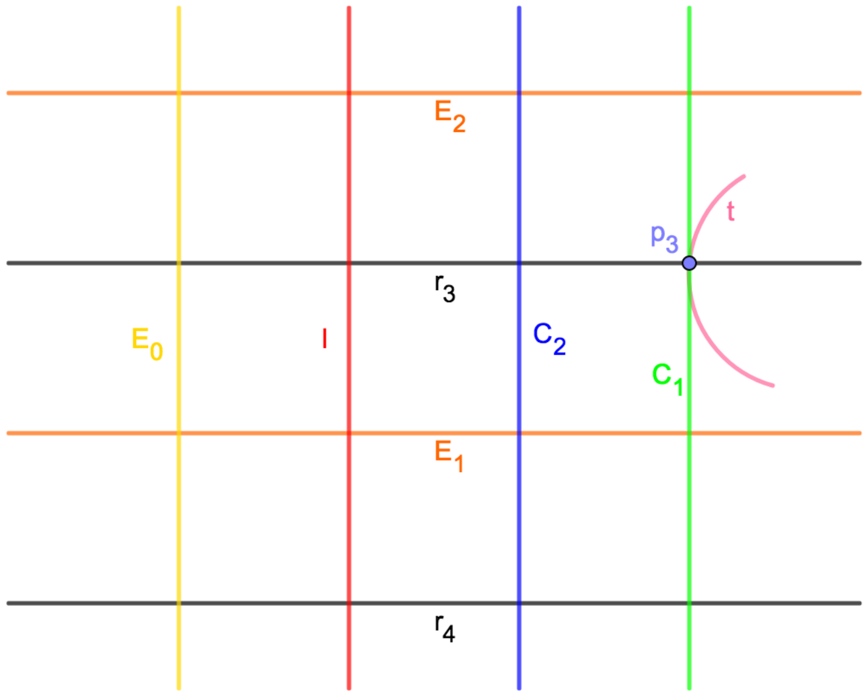


Figure 3: Contracting r_1 and r_2 we find $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[d, dotted, "2:1"'] & A \arrow[l, dotted, "\pi", "2^2:1"'] \arrow[d, "\iota"', "2:1"] \\
\mathbb{F}_2 \arrow[d, dotted] & T_1 \times T_2 \arrow[dl, "2^2:1"] \\
\mathbb{P}^1 \times \mathbb{P}^1 &
\end{tikzcd}$$

Note that the map $\iota\colon A \to T_1 \times T_2$ is an isogeny of degree 2.

We see (compare [Rit18][Section 3, Step 2]) that the strict transform of the curve C_1 is tangent to the curve t on A at a point p. This point is a tacnode (singularity of type $(2,2)$) for the strict transform of the curve t. So the divisor $t + C_1$ is reduced and has a singularity of type $(3,3)$.

Remark 2.1. We see that we recover the construction due to Rito [Rit18] of an abelian surface with a $(1,2)$-polarization, Please notice that in [Rit18] the abelian surface A was labelled by V' and the curves C_1 and t by $\hat{C}_1$ and $\hat{R}$.

In [Rit18] it is shown that the divisor $t + C_1$ is even, i.e. there is a divisor L such that

$$t + C_1 \equiv 2L,$$

and that

$$(t + C_1)^2 = 16.$$

So L is a polarization of type $(1,2)$. This is exactly the situation described by the second author and F. Polizzi in [PePol13a, Remark 2.2]. There the authors suggest how to construct a surface with $p_g = q = 2$ and $K_S^2 = 7$ as a generically finite double cover of A branched along a divisor as $t+C_1$. We follow the suggestion slavishly and we summarize the situation with the following special case of [Rit18, Proposition 1]

Proposition 2.2. *Let A be an Abelian surface. Assume that A contains a reduced curve $t+C_1$ and a divisor L such that $t+C_1 \equiv 2L$, $(t+C_1)^2 = 16$ and $t+C_1$ contains a $(3,3)$-point and no other singularity. Let S be the smooth minimal model of the double cover of A with branch locus $t+C_1$. Then $p_g(S) = q(S) = 2$ and $K_S^2 = 7$.*

Let us now construct S step by step starting from A.

1. First, we resolve the singularity in p. To do that, we need to blow up A twice, first in p and then in a point infinitely close to p. Let us denote these two blow ups by

$$B' \xrightarrow{\sigma_4} B \xrightarrow{\sigma_3} A.$$

   On B', let us denote by F the exceptional divisor relative to σ_4, by E' the strict transform of the exceptional divisor E relative to σ_3, by C_1 the strict transform of C_1 and, finally, by R the strict transform of t (see Figure 4).

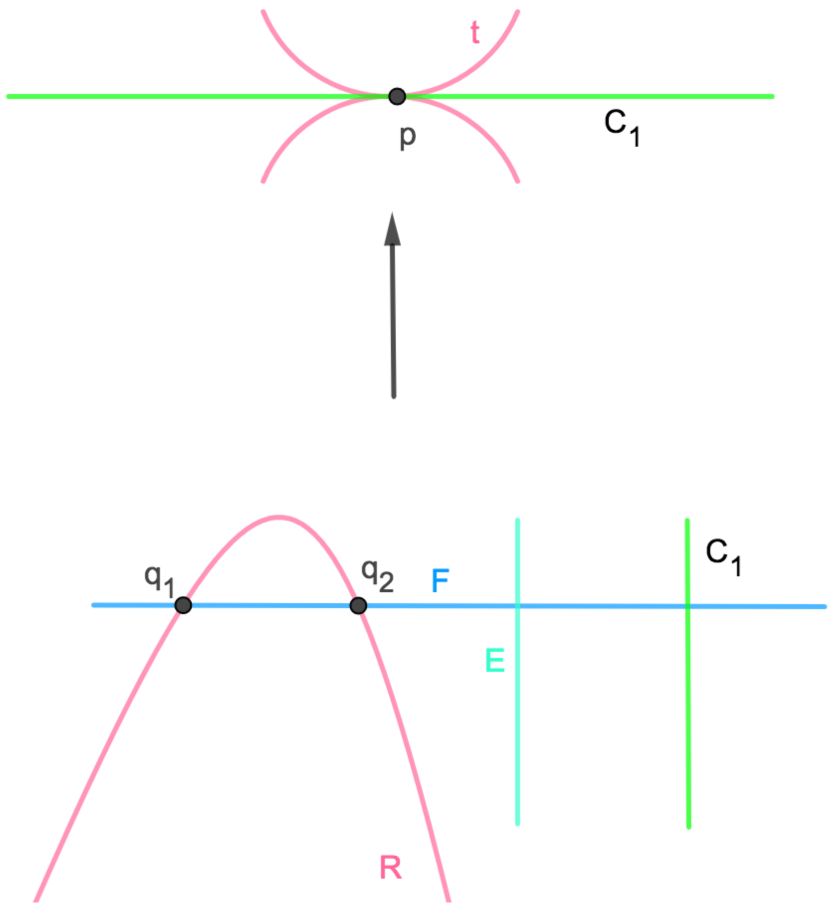


Figure 4: The birational map $\sigma_3 \circ \sigma_4 \colon B' \to A$

In addition, one gathers the following information: $E' \cong \mathbb{P}^1$ and $(E')^2 = -2$, $F \cong \mathbb{P}^1$ and $F^2 = -1$, $g(C_1) = 1$ and $C_1^2 = -2$.

2. Second, we consider a double cover of $\beta\colon S' \longrightarrow B'$ ramified over $R + C_1 + E'$ (that's even since $t + C_1$ is even on A). The surface S' is a surface of general type, not minimal. Indeed, it contains a -1-curve, which is $\hat{E} = \beta^{-1}(E')$. The ramification divisor is denoted $\hat{R} + \hat{C_1} + \hat{E}$. Notice that $\hat{C_1}$ has genuis 1 and $\hat{C_1}^2 = -1$.

3. Finally, to get S we contract the -1-curve $\hat{E}$.

We can summarize the construction of S with the following diagram.

$$\begin{tikzcd}
S' \arrow[r, "\beta"] \arrow[d] & B' \arrow[d, "\sigma_4"] \\
S \arrow[dr, "\alpha"'] & B \arrow[d, "\sigma_3"] \\
 & A
\end{tikzcd}$$

We note that since α is the Albanese morphism of S, we obtained in particular that the Albanese variety of these surfaces is isogenous to a product of elliptic curves:

Proposition 2.3. *The Albanese variety A of the surface S is isogenous to a product via an isogeny $\iota\colon A \to T_1 \times T_2$ of degree 2.*

3 Rito's family is irreducible of moduli dimension 2

The surfaces S are constructed by a configuration of plane curves determined by two parameters (as noticed already in [Rit18, Section 3, Step 4]), that we denoted by a, b, and a choice of a linear system $|L|$ such that $|2L|$ contains the divisor $|C_1 + t|$. So there are 2^4 possible choice for L, since we can always add to L a $2-$torsion line bundle. In this section we prove that the family is connected, irreducible of moduli dimension 2.

Definition 3.1. Denote by $\mathcal{M}$ the locus of the surfaces S above in the Gieseker moduli space of the surfaces of general type.

The isogeny ι induces two natural fibrations $f_i\colon A \longrightarrow T_i$ with fibres Λ_i, $i = 1, 2$ of genus 1. The fibres of each fibration are isomorphic to the base of the other fibration: $i \neq j \Rightarrow \Lambda_i \cong T_j$.

Both fibrations have been considered in [Rit18, Section 3, Step 3] on a birational model of A denoted by V. Composing f_i with the double cover ϕ_i we obtain two pencils, that are induced by two pencils on $\mathbb{P}^2$, the lines through the point p_0 and the conics tangent to the lines r_i in the points p_i, $i = 1, 2$.

Without loss of generality we can assume that the first pencil $A \xrightarrow{f_1} T_1 \xrightarrow{\phi_1} \mathbb{P}^1$ is the one given by the lines in $\mathbb{P}^2$ through the point p_0. The branching points of ϕ_1 correspond to the lines r_1, r_2, r_3 and r_4, that, in the natural coordinates, give the 4 points $(1:0), (0:1), (1:1)$ and $(a:b)$, with cross-ratio $\frac{a}{b}$.

Similarly, the branching points of ϕ_2 correspond, in the pencil of the conics tangent to the lines r_i in the points p_i, $i = 1, 2$, to the conics C_1, C_2, $2l$ and $r_1 + r_2$. Writing the pencil as $\langle x_0^2, x_1 x_2 \rangle$ we get a parametrization of $\mathbb{P}^1$ such that the branching points of ϕ_2 have coordinates $(1:1), (1:ab), (1:0)$ and $(0:1)$, with cross-ratio ab.

We deduce the following

Proposition 3.2. *Every connected component of $\mathcal{M}$ has dimension* 2.

Proof. The base of the family of the surfaces S has a finite proper map on an open subset of $\mathbb{C}^2$ given by the parameters (a, b). So, if $\mathcal{C}$ is any irreducible component of it, $\dim \mathcal{C} = 2$.

The relative Albanese morphism maps $\mathcal{C}$ to the moduli space of the Abelian surfaces with a polarization of type $(1, 2)$. By Proposition 2.3 the image of $\mathcal{C}$ is contained in the $2-$dimensional subvariety $\mathcal{I}$ of those isogenous to a product of curves. Since these curves are double covers of $\mathbb{P}^1$ branched at 4 points with cross-ratio respectively $\frac{a}{b}$ and ab the general pair of curves of genus 1 appears in the image of $\mathcal{C}$: the map $\mathcal{C} \to \mathcal{I}$ is generically finite and therefore dominant.

Since isomorphic manifolds have isomorphic Albanese varieties, $\mathcal{C} \to \mathcal{I}$ factors through the moduli space of the surfaces of general type, and then the moduli dimension of $\mathcal{C}$ is 2. $\square$

Remark 3.3. The four genus 1 curves $\hat{r}_1$, $\hat{r}_2$, $\hat{r}_3$, $\hat{r}_4$, are mapped by f_1 to the ramification points of $\phi_1 \colon T_1 \to \mathbb{P}^1$. We denote by a_j the image of $\hat{r}_j$. Similarly, we denote by b_1, b_2, b_3, b_4 the ramification points of ϕ_2 as follows: b_1 corresponding to the conic C_1, b_2 corresponding to C_2, b_3 corresponding to $2l$, b_4 corresponding to $r_1 + r_2$.

Recall that for $\{i, j, h, k\} = \{1, 2, 3, 4\}$, $\mathcal{O}_{T_1}(a_i - a_j) \cong \mathcal{O}_{T_1}(a_k - a_l)$ and $\mathcal{O}_{T_2}(b_i - b_j) \cong \mathcal{O}_{T_2}(b_k - b_l)$ are the $2-$torsion line bundles on the curves T_1, T_2.

We can now determine the isogeny. Recall that an étale double cover of a variety is determined up to isomorphism by a $2-$torsion line bundle on it, the antiinvariant part of the direct image of the structure sheaf of the cover.

Lemma 3.4. *The antiinvariant part of $\iota_* \mathcal{O}_A$ is $\mathcal{O}_{T_1}(a_1 - a_2) \boxtimes \mathcal{O}_{T_2}(b_1 - b_2)$.*

Proof. By Remark 3.3 we can write every line bundle of torsion 2 on $T_1 \times T_2$ as $\mathcal{O}_{T_1}(a_i - a_j) \boxtimes \mathcal{O}_{T_2}(b_k - b_l)$. We compute separately each factor by restricting to a fibre of type Λ_1 resp. Λ_2. In fact, restricting the isogeny to a fibre of type Λ_1 (respectively Λ_2) we obtain an étale double cover of T_2 (respectively T_1) given by the restriction of the above bundle $\mathcal{O}_{T_2}(b_k - b_l)$ (respectively $\mathcal{O}_{T_1}(a_i - a_j)$).

We did the computation by using the fibres corresponding respectively to r_3 and C_1. We write here only the first case, leaving the fully analogous other computation to the reader.

Then consider the curve $\hat{r}_3 = f_1^{-1}(a_3) \subset A$, we need to show that the antiinvariant part of $(\iota_{|\hat{r}_3})_* \mathcal{O}_{\hat{r}_3}$ is $\mathcal{O}_{T_2}(b_1 - b_2)$.

It is invariant by the $(\mathbb{Z}/2\mathbb{Z})^2$ action on A given by the bidouble cover π, and in fact $\hat{r}_3$ lies in the locus of the fixed points of one of the three involutions. Thus π induces a nontrivial involution on it, whose quotient is the double cover $\hat{r}_3 \to r_3$ branched on $p_3 + p_5 + p_8 + p_{10}$ (see Figure 1). The involution j acts on r_3 permuting those points as $p_3 \leftrightarrow p_5$, $p_8 \leftrightarrow p_{10}$ lifting to an involution on $\hat{r}_3$ without fixed points. Taking the quotient we get a commutative diagram

$$\begin{tikzcd} \hat{r}_3 \arrow[r, "\iota"] \arrow[d, "\pi"] & T_2 \arrow[d] \\ r_3 \arrow[r, "\sigma"] & \mathbb{P}^1. \end{tikzcd}$$

Let us call q_j the ramification point in A of $\pi_{|\hat{r}_3}$ mapping to p_j. Then $\iota(q_3) = \iota(q_5) = b_1$, $\iota(q_8) = \iota(q_{10}) = b_2$. So $\iota^* \mathcal{O}_{T_2}(b_1 - b_2) = \mathcal{O}_{\hat{r}_3}(q_3 + q_5 - q_8 - q_{10}) \cong \mathcal{O}_{\hat{r}_3}$: this implies the claim. $\square$

We write explicitly the 16 linear systems $|L|$.

Proposition 3.5.
$$L = \pi^*\mathcal{O}_{\mathbb{P}^2}(1) + f_1^*(\mathcal{T}_1(a_1)) + f_2^*(\mathcal{T}_2),$$
where the $\mathcal{T}_j$ vary on the line bundles on T_j with $\mathcal{T}_1^2 = \mathcal{O}_{T_1}(a_1 - a_2)$, $\mathcal{T}_2^2 = \mathcal{O}_{T_2}(b_1 - b_3)$.

Note that the $\mathcal{T}_j$ are 4−torsion line bundles on T_j.

Proof. Set $\hat{C}_3$ for the fiber of f_2 on b_3, so corresponding to the conic $2l$. We compute

$$\begin{cases} \pi^*t = t \\ \pi^*C_1 = 2C_1 + 2\hat{r}_1 + 2\hat{r}_2 + 2E \\ \pi^*l = \hat{C}_3 + \hat{r}_1 + \hat{r}_2 + 2E \end{cases} \Rightarrow 2C_1 + \hat{C}_3 + t + 3(\hat{r}_1 + \hat{r}_2) + 4E \in |\pi^*\mathcal{O}_{\mathbb{P}^2}(4)|.$$

it follows
$$C_1 + t \in |2\big(\pi^*\mathcal{O}_{\mathbb{P}^2}(2) - \hat{r}_1 - \hat{r}_2 - 2E\big) - (\hat{r}_1 + \hat{r}_2) - (C_1 + \hat{C}_3)|.$$
Hence half of the branch divisor is of the form

$$\big(\pi^*\mathcal{O}_{\mathbb{P}^2}(2) - \hat{r}_1 - \hat{r}_2 - 2E\big) - \Lambda_1 - \Lambda_2 = \pi^*\mathcal{O}_{\mathbb{P}^2}(1) + \hat{C}_3 - \Lambda_1 - \Lambda_2$$

where Λ_i are fibres of f_i, such that $2\Lambda_1 = \hat{r}_1 + \hat{r}_2$ and $2\Lambda_2 = C_1 + \hat{C}_3$. □

We can now "parametrize" $\mathcal{M}$.

Lemma 3.6. *Keeping the notation above, the surface S is given by the choice of*

- *one elliptic curve T_1 marked with a point a_3 and a 4−torsion line bundle $\mathcal{T}_1$;*
- *one elliptic curve T_2 marked with a point b_1, a 4−torsion line bundle $\mathcal{T}_2$ and a 2−torsion line bundle $\mathcal{O}_{T_2}(b_1 - b_2) \ncong \mathcal{T}_2^2$.*

Here by *elliptic curve marked with a point* we mean that we have fixed a group structure on the curve for which that point is the neutral element.

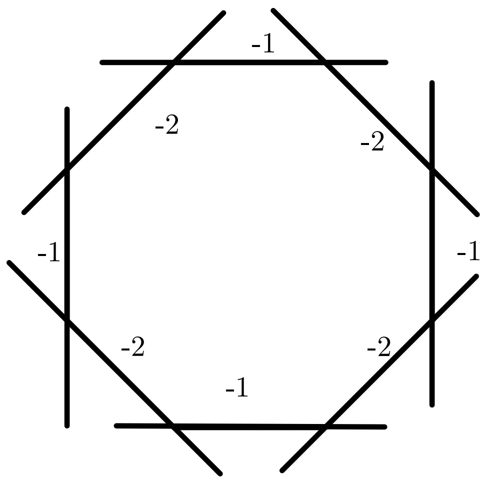
Proof. By Lemma 3.4 and Proposition 3.5 the isogeny $A \to T_1 \times T_2$ is the one given by the 2-torsion bundle
$$\mathcal{O}_{T_1}(a_1 - a_2) \boxtimes \mathcal{O}_{T_2}(b_1 - b_2) \cong \mathcal{T}_1^2 \boxtimes \mathcal{O}_{T_2}(b_1 - b_2)$$
We consider on each elliptic curve T_j the group structure such that the marking point is the neutral element. Writing $T_i = \mathbb{C}/\Lambda_i$ we obtain $A = \mathbb{C}^2/\Lambda$ where Λ is a sublattice of $\Lambda_1 \times \Lambda_2$ of index 2. The action of the Klein group $K \cong (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{C}^2$ given by $(z_1, z_2) \mapsto (\pm z_1, \pm z_2)$ preserves both Λ and $\Lambda_1 \times \Lambda_2$ thus giving a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "/K"] & T_1 \times T_2 \arrow[d, "/K"] \\ D \arrow[r] & \mathbb{P}^1 \times \mathbb{P}^1. \end{tikzcd}$$

The bidouble cover $T_1 \times T_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ is ramified at the union of 8 elliptic curves, corresponding to the four 2−torsion points on each factor: say a_1, a_2, a_3, a_4 on T_1 and b_1, b_2, b_3, b_4 on T_2. Setting $\mathcal{T}_1^2 = \mathcal{O}_{T_1}(a_1 - a_2)$, $\mathcal{T}_2^2 = \mathcal{O}_{T_2}(b_1 - b_3)$ as in Proposition 3.5 we have determined, among these points, who are $a_3, a_4, b_1, b_2, b_3, b_4$. Note that we have no way to distinguish a_1 and a_2.

A direct computation shows that the branching locus of the double cover $D \to \mathbb{P}^1 \times \mathbb{P}^1$ has four connected components, and precisely two components of each ruling, corresponding to a_1, a_2, b_3, b_4 (in Figure 3 respectively E_1, E_2, l, E_0). Therefore D is a Del Pezzo surface of degree

4 with 4 nodes. Solving the 4 nodes we obtain a weak Del Pezzo surface with a configuration of 8 rational curves whose incidence graph is an octagon with alternating self intersections -1 and -2: the strict transforms of the ramification lines have self intersection -1 whereas the exceptional curves have self intersection -2.



Now we consider, among the -1 curves in the octagon, the one labeled b_3: contract first the other three -1 curves and then the two exceptional curves now of self intersection -1: the resulting surface is $\mathbb{P}^2$ and the remaining three sides of the octagon map to three lines, let's call them l (the one coming from the -1-curve "'b_3"), r_1 and r_2. The preimages of the lines of $\mathbb{P}^1 \times \mathbb{P}^1$ labeled a_3, a_4, b_1, b_2 are respectively two lines r_3 and r_4 and two conics C_1, C_2 forming the configuration of curves in Figure 1.

We choose one of the points in $C_1 \cap r_3$ and draw the tangent t to C_1 in that point. Pulling-back $t + C_1$ we obtain a curve in A as in Proposition 2.2 and define S as the double cover of A branched in it associated, as suggested by Proposition 3.5, to the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^2}(1) + f_1^*(\mathcal{T}_1(a_1)) + f_2^*(\mathcal{T}_2)$. The resulting surface does not depend on the choice of the point in $C_1 \cap r_3$ since the two curves we obtain in A differs just by a translation of order 2. $\square$

Now, we shall deal with the problem of irreducibility of $\mathcal{M}$.

Let us recall some well known fact about modular curves, see e.g. [DS05, Section 1.5]. The principal congruence subgroup of level N is

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup of level N if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{Z}^+$. The most important congruence subgroups are

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

The modular curve $\mathcal{Y}(\Gamma)$ for Γ is defined as

$$\mathcal{Y}(\Gamma) = \Gamma \setminus \mathcal{H} = \{\Gamma \cdot z | z \in \mathcal{H}\}.$$

and the special cases of modular curves for $\Gamma_1(N)$ denoted by $\mathcal{Y}_1[N] = \mathcal{H}/\Gamma_1(N)$.

Theorem 3.7. *Points of $\mathcal{Y}_1[N]$ correspond to pairs (E, P), where E is an elliptic curve and $P \in E$ is a point of exact order N. Two such pairs (E, P) and (E_0, P_0) are identified when there is an isomorphism of E onto E_0 taking P to P_0.*

We are interested in the case when $N = 4$ and in the special modular curve $\mathcal{Y}_1[4]$ which parametrizes elliptic curves with 4-torsion points.

Now, let $\mathcal{Y}_1[2, 4]$ the space parametrizing elliptic curves with a 2-line bundle point $\mathcal{Q}$ and a 4-torsion line bundle $\mathcal{T}$ such that $\mathcal{T}^2 \neq \mathcal{Q}$, than we have the following proposition.

Proposition 3.8. *$\mathcal{Y}_1[2, 4]$ is irreducible and generically smooth of dimension* 1.

Proof. We proceed as explained in the Appendix A of [PePol13a]. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve (and $\widehat{E}$ its dual abelian variety), $E[n]$ the subgroup of order n torsion points on E and $\widehat{E}[n] \subset \widehat{E}$ the subgroup of n torsion line bundles. Moreover let $G = \mathrm{SL}_2(\mathbb{Z})$ be the modular group. Then G is the orbifold fundamental group of $\mathcal{H}/G$ and there is an induced monodromy action of G on both $E[n]$ and $\widehat{E}[n]$, see [Har79].

By the Appell-Humbert theorem, the elements of $\widehat{E}[2]$ can be canonically identified with the 4 characters $\Lambda \to \mathbb{C}^*$ with values in $\{\pm 1\}$ (see [BL04, Chapter 2]) which are

$$\chi_0 := (1,\,1), \quad \chi_1 := (1,\,-1), \quad \chi_2 := (-1,\,1), \quad \chi_3 := (-1,\,-1).$$

Let $\{\omega_1, \omega_2\}$ be a suitable basis of Λ by [BL04, proof of Proposition 8.1.3], the monodromy action of

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$$

induced over a character χ is as follows:

$$\begin{aligned} (M \cdot \chi)(\omega_1) &= \chi(\omega_1)^{\alpha} \chi(\omega_2)^{\beta} \\ (M \cdot \chi)(\omega_2) &= \chi(\omega_1)^{\gamma} \chi(\omega_2)^{\delta}. \end{aligned} \tag{1}$$

Therefore we have

$$M \cdot \chi_1 = ((-1)^{\beta},\,(-1)^{\delta}), \quad M \cdot \chi_2 = ((-1)^{\alpha},\,(-1)^{\gamma}), \quad M \cdot \chi_2 = ((-1)^{\alpha+\beta},\,(-1)^{\gamma+\delta}),$$

Whereas the 16 elements of $\widehat{E}[4]$ correspond to he 16 characters $\Lambda \to \mathbb{C}^*$ with values in $\{\pm i\}$:

$$\begin{array}{llll} \psi_1 := (1,\,1), & \psi_2 := (1,\,-1), & \psi_3 := (-1,\,1), & \psi_4 := (-1,\,-1,\,) \\ \psi_5 := (1,\,i), & \psi_6 := (-1,\,i), & \psi_7 := (1, -i), & \psi_8 := (-1,\,-i), \\ \psi_9 := (i,\,1), & \psi_{10} := (i,\,-1), & \psi_{11} := (-i,\,1), & \psi_{12} := (-i,\,-1). \\ \psi_{13} := (i,\,i), & \psi_{14} := (-i,\,i), & \psi_{15} := (i,\,-i), & \psi_{16} := (-i,\,-i). \end{array}$$

And by equations (1) one can compute the induced action of M over a character ψ.

Thus, to prove the first part of the proposition it is sufficient to check that the monodromy action of G is transitive on the set

$$\{(\mathcal{Q},\,\mathcal{T}) \in \big(\widehat{E}[2] \setminus \mathcal{O}_E\big) \times \big(\widehat{E}[4] \setminus \widehat{E}[2]\big) \,|\, \mathcal{T}^2 \neq \mathcal{Q}\}.$$

This is a straightforward computation which can be carried out as the one in the proof of [PePol13a, Proposition A1] and it is left to the reader.

Therefore we can consider the set of triples

$$(z,\,\chi, \psi), \quad z \in \mathcal{H},\; \chi \in \{\chi_1, \chi_2, \chi_3\} \subset \widehat{E_z}[2],\; \psi \in \{\psi_5, \ldots, \psi_{16}\} \subset \widehat{E_z}[4].$$

The group G acts on the set of triple (z, χ, ψ), with the natural action of the modular group on $\mathcal{H}$ and by the induced monodromy action on the second two ones. The corresponding quotient $\mathcal{Y}_1[2,4]$ is a quasi-projective variety. Moreover

$$\pi\colon \mathcal{Y}_1[2,4] \longrightarrow \mathcal{H}/G$$

given by the forgetful map, is an étale covers on a smooth Zariski open set $\mathcal{Y}_1^0 \subset \mathcal{H}/G$; then it is generically smooth. Finally, by construction $\mathcal{Y}_1[2,4]$ is a normal varieties, because it only has quotient singularities. Then, since it is connected, it must be also irreducible.

□

Proposition 3.9. *The moduli space $\mathcal{M}$ is irreducible, generically smooth of dimension* 2.

Proof. By the Lemma 3.6 the construction of a surface S depends on the following data:

- one elliptic curve T_1 *marked* with a point a_3 and a $4-$torsion line bundle $\mathcal{T}_1$;
- one elliptic curve T_2 *marked* with a point b_1, a $4-$torsion line bundle $\mathcal{T}_2$ and a $2-$torsion line bundle $\mathcal{O}_{T_2}(b_1 - b_2) \ncong \mathcal{T}_2^2$.

In other words there is a dominant morphism

$$\mathcal{Y}_1[4] \times \mathcal{Y}_1[2,\, 4] \to \mathcal{M}.$$

We observe that $\mathcal{Y}_1[4]$ is a generically smooth quasi-projective variety, connected, and irreducible of dimension 1, [DS05, Chapter 2]. By Proposition 3.8 $\mathcal{Y}_1[2,\, 4]$ is irreducible and generically smooth of dimension 1. This concludes the proof since $\dim \mathcal{M} = 2$ by Proposition 3.2. □

4 Some remarks on the deformations of a blown up surface

In this section we shall present some classicall results on deformation of a pairs. The main result is Theorem 4.2, possibly known to the experts, although we could not find it in the literature. This section will be employed systematically in the Moduli Space Section 5 and Theorem 4.2 mainly for the Remark 5.7.

Let us first recall some basic definition.

Let B an algebraic nonsingular variety over an algebraically closed field k. The *first order deformation* of B is a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "\pi"] \\ Spec(k) \arrow[r] & Spec(k[\epsilon]) \end{tikzcd}$$

where π is a flat morphism, $Spec(k[\epsilon]) = Spec(k[t]/t^2)$ and such that the induced morphism

$$B \to Spec(k) \times_{Spec(k[\epsilon])} \mathcal{B}$$

is an isomorphism. There is a natural notion of isomorphism between first order deformations, see [Ser06, Section 1.2]. The set of first order deformations, up to isomorphisms, is usually denoted by $T^1(B)$ and it has a natural structure of complex vector space (see [Sch68]). If B has a semiuniversal deformation $\tilde{B} \to Def(B)$ then every first order deformation is induced by a unique map $Spec(k[\epsilon]) \to Def(B)$ and then there exists an isomorphisms of vector spaces

$$T_0 Def_B \cong T^1(B) \cong H^1(B, T_B),$$

for the last isomorphism see e.g. [Ser06, Proposition 1.2.9].

Now, we look at deformations of subvarieties in a given variety. Given a closed embedding $D \subset B$, the *first order deformation* of D in B is a cartesian diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[r, hook] \arrow[d, "\pi"] & B \times Spec(k[\epsilon]) \arrow[d] \\ Spec(k) \arrow[r, hook] & Spec(k[\epsilon]) \arrow[r, equal] & Spec(k[\epsilon]) \end{tikzcd}$$

where π is flat and it is induced by the projection from $B \times Spec(k[\epsilon])$. Again we can give a cohomological interpretation to these deformations, indeed there is a natural identification between the first order deformations of D in B and $H^0(D, \mathcal{N}_{D/B})$, where $\mathcal{N}_{D/B}$ is the normal sheaf of D in B, see e.g. [Ser06, Proposition 3.2.1].

Before introducing the last two situations we are interested in, let us recall the following definition.

Definition 4.1. Let $D_1, \ldots, D_k$ be divisors in a smooth manifold X and $x_1, \ldots, x_k$ equations for them. Define $\Omega^1_S(\log D_1, \ldots, \log D_k)$ to be the subsheaf (as $\mathcal{O}_X$-module) of $\Omega^1_X(D_1 + \ldots + D_k)$ generated by Ω^1_X and by $\frac{dxj}{x_j}$ for $j = 1, \ldots k$.

The next situation we want to look at is the case of deformation of a pair (B, D) where $j \colon D \hookrightarrow B$ is a closed embedding. The deformation theory of morphisms is more subtle if we want to allow both the domain and the target to deform nontrivially. A first order deformation of the pair (D, B) is a commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "J"] \arrow[dr, "\pi_D"'] & & \mathcal{B} \arrow[dl, "\pi_B"] \\ & Spec(k[\epsilon]) & \end{tikzcd}$$

where π_D and π_B come from first deformations of D and B respectively and J is a closed embedding. There is a natural notion of isomorphism between first order deformations of pairs see e.g. [Ser06, Section 3.4]. And, we denote by Def'_j the set if isomorphism classes of first order deformations of the pair (B, D), which are locally trivial. Also in this case we have a cohomological interpretation, by [Ser06, Proposition 3.4.17], Def'_j has a formal semiuniversal deformation and its tangent space is isomorphic to $H^1(T_{B'}(-\log D'))$, where $T_{B'}(-\log D')$ is the sheaf of germs of tangent vectors to B' which are tangent to D'.

Finally, let us consider the following situation. Let B be a compact complex smooth surface, $p \in B$ and $\sigma \colon B' \to B$ the blow up of B in p with exceptional divisor E. Let D be an effective divisor on B which has multiplicity c in p. Moreover, let us denote by $D' = \sigma^*(D) - cE$ the strict transform of D in B' and assume that D' is a smooth normal crossing divisor. We want to describe the relations between the deformations of the pair (B', D') with those of D in B.

We know that the first order deformations of the pair (B', D') are parameterized by the vector space $H^1(T_{B'}(-\log D'))$. The natural map

$$\vartheta \colon H^1(T_{B'}(-\log D')) \to H^1(T_{B'})$$

corresponds to the forgetful map, which forget the deformation of D'. By [Har10, Exercise 10.5] we have an exact sequence

$$0 \to \sigma_* T_{B'} \to T_B \to T_p B \to 0$$

where $T_pB \cong \mathbb{C}^2$ is the tangent space of B in p seen as skyscreaper sheaf concentrated in p. Then we consider the long exact sequence in cohomology and in particular the connecting homomorphism

$$\psi\colon T_pB \to H^1(\sigma_* T_{B'}) \cong H^1(T_{B'}).$$

The next result give us a better understanding of the intersection between the images of the maps ϑ and ψ in $H^1(T_{B'})$.

Theorem 4.2. *Keeping the same notation as before, assume that D is smooth at p, so $c = 1$, and choose an element $v \in T_pB$.*

Then $\psi(v)$ is contained in $Im(\vartheta)$ if and only if the class of v in the normal vector space T_pB/T_pD extends to a global section of the normal bundle $v_D \in H^0(D, \mathcal{N}_{D|B})$.

In particular v is tangent to D if and only if v_D vanishes in p.

Proof. We start constructing a family of first order deformations of B'.

Let U be an affine chart of B centered in p with local coordinates x, y such that $D = \{x = 0\}$. We consider a section $s_{a,b}$ of the trivial family $B \times Spec(\mathbb{C}[\epsilon]) \to Spec(\mathbb{C}[\epsilon])$ whose image is contained in $U \times Spec(\mathbb{C}[\epsilon])$

$$\begin{tikzcd} B \times Spec(\mathbb{C}[\epsilon]) \supset & U \times Spec(\mathbb{C}[\epsilon]) \arrow[r] & Spec(\mathbb{C}[\epsilon]) \arrow[l, bend left, "s_{a,b}"] \end{tikzcd}$$

obtained by mapping (x, y, ϵ) to $(a\epsilon, b\epsilon, \epsilon)$, so that the image is locally the complete intersection

$$x - a\epsilon = y - b\epsilon = 0.$$

Blowing up this section we obtain the following families over $Spec(\mathbb{C}[\epsilon])$

$$\begin{tikzcd} \mathcal{B}'_{a,b} \arrow[r] \arrow[rr, bend left, "\Phi_{a,b}"] & B \times Spec(\mathbb{C}[\epsilon]) \arrow[r] & Spec(\mathbb{C}[\epsilon]) \arrow[d, equal] \\ \mathcal{U}'_{a,b} \arrow[r] \arrow[u, hook] & U \times Spec(\mathbb{C}[\epsilon]) \arrow[r] \arrow[u, hook] & Spec(\mathbb{C}[\epsilon]) \end{tikzcd}$$

where $\Phi_{a,b}$ is a first-order deformation of B'. The Kodaira-Spencer correspondence associates to $\Phi_{a,b}$ a class in $\kappa(\Phi_{a,b}) \in H^1(B', T_{B'})$, its Kodaira-Spencer class. This can be explicitly computed: following *e.g.* the proof of [Ser06, Proposition 1.2.9] we find

$$\kappa(\Phi_{a,b}) = \psi\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right).$$

The blown up chart $\mathcal{U}'_{a,b}$ is the subscheme of $U \times \mathbb{P}^1 \times B \times Spec(\mathbb{C}[\epsilon])$ defined by

$$Y(x - a\epsilon) = X(y - b\epsilon),$$

where (X, Y) are homogeneous coordinates on the factor $\mathbb{P}^1$. It is the union of two affine charts, given respectively by imposing $X \neq 0$ and $Y \neq 0$.

Let us work locally and restrict to the affine chart of $\mathcal{U}'_{a,b}$ given by $Y \neq 0$, and let us introduce the new coordinate $z = \frac{X}{Y}$. Then, we can eliminate x by

$$x = zy + (a - bz)\epsilon$$

and the exceptional divisor $\mathcal{E}$ of the blow-up is $\{y - b\epsilon = 0\}$ in the coordinates y, z.

Since $D = \{x = 0\}$, the strict transform of D on B' is, in the coordinates y, z, the divisor $D' = \{z = 0\}$. Now, $\kappa\,(\Phi_{a,b})$ is in the image of ϑ if and only if D' can be extended to a divisor $\mathcal{D}'_{a,b}$ in $\mathcal{B}'_{a,b}$. The image of $\mathcal{D}'_{a,b}$ in $B \times Spec(\mathbb{C}[\epsilon])$ is

$$\mathcal{D}_{a,b} = \{x + \delta(x, y)\epsilon = 0\},$$

an infinitesimal deformation of D in B over $Spec\,(\mathbb{C}[\epsilon])$ so that $\delta(x, y)$ is the affine trace of a global section of the normal bundle $\mathcal{N}_{D|B}$, an element $\delta \in H^0(D, \mathcal{N}_{D|B})$ ([Ser06, Proposition 3.2.1]), locally given by the class of a vector field $\delta(x, y)\frac{\partial}{\partial x}$.

The pullback of $\mathcal{D}_{a,b}$ on $\mathcal{B}'_{a,b}$ contains the exceptional divisor $\mathcal{E}$, thus

$$y - b\epsilon \text{ divides } zy + (a - bz - \delta(zy, y))\epsilon = z(y - b\epsilon) + (a - \delta(zy, y))\epsilon,$$

that implies $\delta(0, 0) = a$. Conversely, if $\delta(0, 0) = a$ the pull-back of $\mathcal{D}_{a,b}$ contains $\mathcal{E}$ and then its strict transform gives an extension $\mathcal{D}'_{a,b}$ of D' in $\mathcal{B}'_{a,b}$.

Since the class of $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ in T_pB/T_pD equals the class of $a\frac{\partial}{\partial x}$, then $\psi\,(v)$ is in the image of θ if and only if there is some $\delta \in H^0(B, \mathcal{N}_{D|B})$ whose value at p is the class of v. □

The situation is even simpler if D is a rigid divisor.

Corollary 4.3. *Let D be a divisor which is smooth in p and $H^0(D, \mathcal{O}_D(D)) = 0$. Let $v \in T_pB$ such that $\psi(v) \in Im(\vartheta)$. Then v is tangent to D.*

Keeping the same notation as above, the application we have in mind for the next can be summarized in

Proposition 4.4. *Let $B' \to B$ the blow up of B in p and D' the strict transform of D a divisor passing through p. Let us further suppose that $D \geq D_1 + D_2$ with D_1 and D_2 smooth and transversal in p. Moreover, let us assume that $H^0(D_i, \mathcal{O}_{D_i}(D_i)) = 0$ for $i = 1, 2$. Then*

$$\vartheta(H^1(B', T_{B'}(-\log D'))) \cap \psi(T_pB) = \{0\}. \tag{2}$$

Proof. We have that ϑ factors through the analogous map for D_j, $j = 1, 2$:

$$H^1(T_{B'}(-\log(D'))) \to H^1(T_{B'}(-\log(D_j))) \to H^1(B', T_{B'}) \qquad \text{for } j = 1, 2.$$

Hence, the image of ϑ is contained in the image of both $H^1(T_{B'}(-\log(D_j)))$ for $j = 1, 2$. Than we apply the Corollary 4.3 and we obtain a vector v which is tangent to both D_1 and D_2. Finally, observe that if a vector is tangent to two transversal curves must vanish. □

Corollary 4.5. *Let D be as in Proposition 4.4 and suppose moreover $H^0(D', \mathcal{O}_{D'}(D')) = 0$. Then the composition*

$$H^1(B', T_{B'}(-\log D')) \to H^1(B, T_B)$$

is injective.

Proof. The proof follows directly from the Proposition 4.4 and the following diagram with exact row and column.

$$\begin{tikzcd}
 & H^0(\mathcal{O}_{D'}(D')) = 0 \arrow[d] & \\
 & H^1(T_{B'}(-\log D')) \arrow[d, "\vartheta"] \arrow[dr] & \\
T_pB \arrow[r, "\psi"] & H^1(T_{B'}) \arrow[r] & H^1(T_B).
\end{tikzcd}$$

□

5 The moduli space

The following result can be found in [Cat11, Section 5].

Proposition 5.1. *Let S be a minimal surface of general type with $q(S) \geq 2$ and Albanese map $\alpha\colon S \to A$, and assume that $\alpha(S)$ is a surface. Then this is a topological property. If in addition $q(S) = 2$, then the degree of α is a topological invariant.*

Proof. By [Cat91] the Albanese map α induces a homomorphism of cohomology algebras

$$\alpha^*\colon H^*(\mathrm{Alb}(S),\, \mathbb{Z}) \longrightarrow H^*(S,\, \mathbb{Z})$$

and $H^*(\mathrm{Alb}(S), \mathbb{Z})$ is isomorphic to the full exterior algebra

$$\bigwedge^{*} H^1(\mathrm{Alb}(S),\, \mathbb{Z})) \cong \bigwedge^{*} H^1(S,\, \mathbb{Z}).$$

In particular, if $q = 2$ the degree of the Albanese map equals the index of the image of $\bigwedge^4 H^1(S,\, \mathbb{Z})$ inside $H^4(S,\, \mathbb{Z})$ and it is therefore a topological invariant. $\square$

Let S be a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 7$ and Albanese map of degree 2; assume that K_S is ample. By Proposition 5.1 it follows that one may study the deformations of S by relating them to those of the flat double cover $\beta\colon S' \to B'$. By [Ser06, p. 162] we have an exact sequence

$$0 \longrightarrow T_{S'} \longrightarrow \beta^* T_{B'} \longrightarrow \mathcal{N}_\beta \longrightarrow 0, \tag{3}$$

where $\mathcal{N}_\beta$ is a coherent sheaf supported on the ramification divisor $\hat{R}+\hat{C}_1+\hat{E}$ called the *normal sheaf of* β.

Lemma 5.2. *Keeping the notation above it holds*

$$H^i(S', \mathcal{N}_\beta) = H^i(\mathcal{O}_{\hat{R}}(2\hat{R})) \oplus H^i(\mathcal{O}_{\hat{C}_1}(2\hat{C}_1)) \oplus H^i(\mathcal{O}_{\hat{E}}(2\hat{E})),\ i = 0, 1. \tag{4}$$

Moreover we have:

$$h^0\big(\mathcal{O}_{\hat{R}}(2\hat{R})\big) = 0, \quad h^0\big(\mathcal{O}_{\hat{C}_1}(2\hat{C}_1)\big) = 0, \quad h^0\big(\mathcal{O}_{\hat{E}}(2\hat{E})\big) = 0,$$

$$h^1\big(\mathcal{O}_{\hat{R}}(2\hat{R})\big) = 2, \quad h^1\big(\mathcal{O}_{\hat{C}_1}(2\hat{C}_1)\big) = 2, \quad h^1\big(\mathcal{O}_{\hat{E}}(2\hat{E})\big) = 1.$$

Proof. The ramification divisor of the double cover $\beta\colon S' \longrightarrow B$ is the disjoint union of the divisors $\hat{E}$, $\hat{R}$ and $\hat{C}_1$, this is enough for (4).

Since $\hat{C}$ is an elliptic curve with $\hat{C}^2 = -1$, we have that $2\hat{C}$ is not effective on $\hat{C}$ and by Riemann–Roch we conclude that $h^1\big(\mathcal{O}_{\hat{C}_1}(2\hat{C}_1)\big) = 2$.

The computations for $\hat{E} \cong \mathbb{P}^1$ are straightforward.

Finally we work on $\hat{R}$. Recall that $g(\hat{R}) = 3$ and $\hat{R}^2 = 0$. Thus, by Riemann–Roch we have $\chi\big(\mathcal{O}_{\hat{R}}(2\hat{R})\big) = -2$. Therefore, it is sufficient to prove that $h^0\big(\mathcal{O}_{\hat{R}}(2\hat{R})\big) = 0$.

We notice that $H^0(\mathcal{O}_{\hat{R}}(2\hat{R})) = H^0(\mathcal{O}_R(R)) = H^0(\mathcal{N}_{R|B})$. Recall that by adjunction the normal bundle of a curve in an abelian surface equals its canonical bundle, so $\mathcal{N}_{t|A} = \omega_t$. The map $\nu = (\sigma_4 \circ \sigma_3)|_R\colon R \longrightarrow t$ is the normalization of t. Let $q_1, q_2 \in R$ such that $\nu(q_i) = p$ with $i = 1, 2$ and recall that p is the tacnode of t. We have

$$\omega_R = \nu^*\omega_t \otimes \mathcal{O}_R(-2q_1 - 2q_2), \quad \mathcal{N}_{R|B} = \nu^*\mathcal{N}_{t|A} \otimes \mathcal{O}_R(-4q_1 - 4q_2),$$

this yields

$$\mathcal{N}_{R|B} = \omega_R \otimes \mathcal{O}_R(-2q_1 - 2q_2). \tag{5}$$

By construction R is a smooth irreducible curve of genus 3 with a $(\mathbb{Z}/2\mathbb{Z})^2$-action, by [BO20, Lemma 2.15] R is not hyperelliptic. Thus, R is a plane quartic curve invariant under the action

$$(x_0 : x_1 : x_2) \mapsto (\pm x_0, \pm x_1, \pm x_2),$$

the equation defining it is biquadratic, and the divisor $q_1 + q_2$ is invariant. This means that q_1 and q_2 have a stabilizer of order 2 and lie on a coordinate line x_j.

By (5)

$$H^0(\mathcal{O}_{\hat{R}}(2\hat{R})) = 0 \Leftrightarrow (x_j) \text{ is not a bitangent}$$

Since the quartic equation defining R is biquadratic, this would imply that R is singular in q_1 and q_2, but this is absurd. $\square$

Recall that S' is a surfaces of general type, hence $h^0(T_{S'}) = 0$ and using the bit of information of the previous lemma, the sequence (3) induces the following long sequence in cohomology.

$$0 \longrightarrow H^1(T_{S'}) \longrightarrow H^1(\beta^* T_{B'}) \longrightarrow H^1(\mathcal{N}_\beta) \longrightarrow H^2(T_{S'}) \longrightarrow H^2(\beta^* T_B) \longrightarrow 0.$$

Proposition 5.3. *Keeping the notation as above, then the sheaf $\beta^* T_B$ satisfies*

$$\begin{aligned} h^0(S, \beta^* T_{B'}) =& 0, \\ h^1(S, \beta^* T_{B'}) =& 6 + h^1(B', T_{B'} \otimes \mathcal{L}_{B'}^{-1}), \\ h^2(S, \beta^* T_{B'}) =& 2 + h^2(B', T_{B'} \otimes \mathcal{L}_{B'}^{-1}). \end{aligned}$$

Proof. Since $\beta\colon S' \to B'$ is a finite map, by using projection formula and the Leray spectral sequence we deduce

$$h^i(S', \mathcal{O}_{S'}) = h^i(B', \mathcal{O}_{B'}) + h^i(B', \mathcal{L}_{B'}^{-1}), \quad i = 0, 1, 2.$$

Recall that $p_g(S') = q(S') = 2$ and B' is an abelian surface blown up twice, then we have

$$h^0(B', \mathcal{L}_{B'}^{-1}) = 0, \quad h^1(B, \mathcal{L}_{B'}^{-1}) = 0, \quad h^2(B', \mathcal{L}_{B'}^{-1}) = 1, \tag{6}$$

By the same argument above we have

$$h^i(S', \beta^* T_{B'}) = h^i(B', \beta_* \beta^* T_{B'}) = h^i(B', T_{B'}) + h^i(B', T_{B'} \otimes \mathcal{L}_{B'}^{-1}), \quad i = 0, 1, 2.$$

We look first at σ_3. There is a short exact sequence

$$0 \longrightarrow T_B \to \sigma_3^* T_A \to \mathcal{O}_E(-E) \to 0, \tag{7}$$

see [Ser06, p. 73] for the general setting of a blow up. Then a direct computation shows

$$h^0(B, T_B) = 0, \quad h^1(B, T_B) = 4, \quad h^2(B, T_B) = 2.$$

The analogous computation for σ_4, for the exact sequence

$$0 \longrightarrow T_{B'} \to \sigma_4^* T_B \to \mathcal{O}_F(-F) \to 0. \tag{8}$$

yields

$$h^0(B', T_{B'}) = 0, \quad h^1(B', T_{B'}) = 6, \quad h^2(B', T_{B'}) = 2.$$

Therefore the claim follows.

$\square$

Let us consider the exact sequence

$$0 \longrightarrow T_{B'} \to (\sigma_4 \circ \sigma_3)^* T_A \to \mathcal{N}_{\sigma_4 \circ \sigma_3} \to 0, \tag{9}$$

where the last sheaf is supported on E and F. We tensor (9) by $\mathcal{L}_{B'}^{-1}$ and we obtain the sequence

$$0 \longrightarrow T_{B'} \otimes \mathcal{L}_{B'}^{-1} \to (\mathcal{L}_{B'}^{-1})^{\oplus 2} \to \mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1} \to 0.$$

Considering the induced long exact sequence in cohomology, by (6)

$$h^1\big(T_{B'} \otimes \mathcal{L}_{B'}^{-1}\big) = h^0\big(\mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1}\big) \tag{10}$$

and

$$h^2\big(T_{B'} \otimes \mathcal{L}_{B'}^{-1}\big) = h^1\big(\mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1}\big) + 2 \tag{11}$$

Lemma 5.4. *It holds*

$$h^0\big(\mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1}\big) = 2.$$

Proof. Recall that we set $E' = \sigma_4^* E$. Let us consider the exact sequence (7), it lifts on B' as

$$0 \to \sigma_4^* T_B \to \mathcal{O}_{B'}^{\oplus 2} \to \sigma_4^* \mathcal{O}_E(-E) \to 0.$$

We put this last exact sequence together with (8) as respectively the middle horizontal sequence and the first vertical sequence in a diagram. Chasing the diagram we obtain the following

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T_B \arrow[r, equal] \arrow[d] & T_B \arrow[d] & & \\
0 \arrow[r] & \sigma_4^*(T_{B_0}) \arrow[r] \arrow[d] & \mathcal{O}_B^{\oplus 2} \arrow[r] \arrow[d] & \sigma_4^*(\mathcal{O}_E(-E)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_F(-F) \arrow[r] \arrow[d] & \mathcal{N}_{\sigma_4 \circ \sigma_3} \arrow[r] \arrow[d] & \sigma_4^*(\mathcal{O}_E(-E)) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{12}$$

which is a diagram with exact rows and columns. Let us look at the last horizontal sequence. Recall that $F \cong \mathbb{P}^1 \cong E'$, thus $\mathcal{O}_F(-F) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{E'}(-E') \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Moreover, we the sheaf $\sigma_4^*(\mathcal{O}_E(-E))$ is locally free and it is supported on $F \cup E'$. Its restriction to the irreducible components are

$$\sigma_4^*(\mathcal{O}_E(-E))|_{E'} \cong \mathcal{O}_{\mathbb{P}^1}(1), \qquad \qquad \sigma_4^*(\mathcal{O}_E(-E))|_F \cong \mathcal{O}_{\mathbb{P}^1}.$$

We tensor the short exact sequence (12) by $\mathcal{L}_B^{-1} \cong \mathcal{O}_{B'}(R + E' + C_1)$ and we get

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1} \to \sigma_4^*(\mathcal{O}_E(-E)) \otimes \mathcal{L}_{B'}^{-1} \to 0.$$

The long exact sequence in cohomology yields

$$H^i(\mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1}) \cong H^i(\sigma_4^*(\mathcal{O}_E(-E)) \otimes \mathcal{L}_{B'}^{-1}), \qquad \forall\, i.$$

By the intersection computation

$$(R+E'+C_1)E' = -2 \text{ and } (R+E'+C_1)F = 4.$$

he sheaf $\sigma_4^*(\mathcal{O}_E(-E)) \otimes \mathcal{L}_{B'}^{-1}$ is a locally free sheaf on $E' \cup F$ which has degree -1 on F and degree 2 on E'. Hence its global sections coincide with the sections of $H^0(\mathcal{O}_E(2))$ which vanish on $E' \cup F$. This last intersection is just a point and thus

$$H^0\big(\mathcal{N}_{\sigma_4 \circ \sigma_3} \otimes \mathcal{L}_{B'}^{-1}\big) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2.$$

□

Corollary 5.5. *It holds*

$$h^1\big(\beta^* T_{B'}\big) = 8, \quad h^2\big(\beta^* T_{B'}\big) = 6.$$

Proof. This follows from Proposition 5.3, Lemma 5.4 and equations (10), (11). □

Remark 5.6. Let $q \in S$ be the point blown-up by $S' \to S$. The short exact sequence obtained pushing forward (3) produces a cohomology exact sequence

$$0 \to T_q S \to H^1(S', T_{S'}) \to H^1(S, T_S) \to 0.$$

Recall that if $\beta \colon S' \longrightarrow B'$ is a finite two to one cover, then $H^1(S', T_{S'}) = H^1(B', \beta_* T_{S'})$ splits as invariant and anti-invariant part. Since q is an isolated fixed point of the involution, the image of $T_q S$ is contained in $H^1(S', T_{S'})^-$. By (*e.g.* Pardini [Par91, Lemma 4.2]) we have

$$(\beta_* T_{S'})^+ \cong T_{B'}(-\log(R+E'+C_1)) \qquad\qquad (\beta_* T_{S'})^- \cong T_{B'} \otimes \mathcal{L}_{B'}^{-1}$$

By the Lemma 5.4 and (10) then $h^1(\beta_* T_{S'})^- = 2$, and so $T_q S$ maps isomorphically onto $H^1(S', T_{S'})^-$.

In particular the map

$$H^1(B', T_{B'}(-\log(R+E'+C_1))) \to H^1(S, T_S),$$

is an isomorphism.

Remark 5.7. Corollary 4.5 apply to the blow-up $\sigma_4 \colon B' \to B$ with $D' = R + E' + C_1$ since all required rigidites have been proved in Lemma 5.2.

So the natural map

$$H^1\big(T_{B'}(-\log(R+E'+C_1))\big) \hookrightarrow H^1\big(T_B\big).$$

is injective. Since the map $H^1\big(T_B\big) \to H^1\big(T_A\big)$ is an isomorphism,

$$H^1\big(T_{B'}(-\log(R+E'+C_1))\big) \hookrightarrow H^1\big(T_A\big).$$

is injective as well.

Hence we have a commutative diagram

$$\begin{tikzcd}
H^1(T_{B'}(-\log((R+E'+C_1))) \arrow[r] \arrow[d, "\cong"] & H^1(T_{B'}) \arrow[d] \\
H^1(T_S) \arrow[r, hook] & H^1(T_A).
\end{tikzcd}$$

The left vertical map is an isomorphism by Remark 5.6. The composition of the top horizontal arrow and the right vertical arrow is the map in Remark 5.7, so injective, and therefore the lower horizontal map is injective.

We prove the following general result.

Lemma 5.8. *Let A be an abelian surface isogenous to a product of elliptic curves $T_1 \times T_2$. Let $H \subset H^1(T_A)$ be the hyperplane corresponding to the projective deformations of A. Finally, let $H_j \subset H^1(T_A)$ be the hyperplanes corresponding to the deformations preserving the fibration $A \longrightarrow T_j$ for $j = 1, 2$. Then*

$$H_j \nsubseteq H.$$

Proof. The isogeny maps $H^1(T_A)$ isomorphically to $H^1(T_{T_1 \times T_2})$ by a map preserving H, H_1 and H_2. Therefore we may assume without loss of generality $A = T_1 \times T_2$.

For a product of curves the period matrix assumes the form

$$\Lambda = \Omega\,\mathbb{Z}^4, \quad \Omega : \mathbb{Z}^4 \longrightarrow \mathbb{C}^2, \qquad x \longmapsto \Omega\, x = \left(\begin{array}{c|c} \Delta_1 & \tau \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \delta \end{array} \right)$$

It is well known that one can identify the deformation space $H^1(T_A)$ of a polarized abelian surface $A = V/\Lambda$ with the space of the square matrices τ (see [HKW93, Chapter 1]). For $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we obtain the deformation given by

$$\left(\begin{array}{c|c} \Delta_1 & \tau \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 0 & \alpha + a\epsilon & b\epsilon \\ 0 & 1 & c\epsilon & \delta + d\epsilon \end{array} \right).$$

The Riemann–Conditions for an abelian surface with a principal polarization yields the existence of an integral basis $\{\lambda_i\}_i$ for Λ and a complex basis $\{e_i\}$ for V such that the period matrix can be normalized so that the matrix τ is symmetric with positive imaginary part (see [GH94] p.306), so

$$H = \{b = c\}.$$

The subspaces H_j are respectively

$$b = 0 \qquad\qquad\qquad c = 0$$

and this concludes the lemma. $\square$

Proposition 5.9. *It holds*

$$h^1(T_S) = 2.$$

Proof. The image of the map $H^1(T_S) \to H^1(A)$ is contained in the hyperplane H of Lemma 5.8, since the Albanese variety of every surface of general type is an abelian variety.

We proved that $H^1(T_{B'}(-\log((R+E'+C_1))) \cong H^1(T_S)$ and the induced map $\varphi\colon H^1(T_{B'}(-\log(R+E'+C_1))) \to H^1(T_A)$ is injective. So it is enough to prove $\dim Im(\varphi) = 2$

The function φ factorizes as in the following commutative diagram.

$$\begin{tikzcd}
H^1(T_{B'}(-\log(R+E'+C_1)) \arrow[r] \arrow[d] & H^1(T_A) \\
H^1(T_{B'}(-\log(C_1)) \arrow[r] & H^1(T_A(-\log C_1)) \arrow[u, "\varepsilon"']
\end{tikzcd}$$

where C_1 is the elliptic curve in Figure 4. We recall that A is isogenous to the product of two elliptic curves $T_1 \times T_2$ and C_1 is a fibre of the induced elliptic fibration f_2 on T_2.
So the image of ϵ is contained in H_2. Then by Lemma 5.8 the dimension of $Im(\varphi)$ is at most 2. On the other hand it is at least 2 by Proposition 3.2. and therefore it equals 2. $\square$

Proposition 5.10. *The following holds: $\mathcal{M}$ is an irreducible, generically smooth component of the moduli space of the surfaces of general type of dimension 2.*

Proof. We have shown that $\mathcal{M}$ is irreducible of dimension 2 in Proposition 3.9. Then by Proposition 5.9 $Def(S)$ is smooth of dimension 2 at each point. It follows that $\mathcal{M}$ is an irreducible component, and that this component is generically smooth. $\square$

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