Computer Aided Algebraic Geometry: Constructing Surfaces of Genus Zero.

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Abstract Everybody knows that mathematics has a key role in the development of the modern technology. It is less known that the modern technology gives something back to mathematics. In this note we give an account on how the combination of classical results as the Riemann Existence Theorem with the use of computers and computational algebra programs answered interesting old-standing problems in classical algebraic geometry, namely regarding the construction and the classification of new surfaces of general type. We also give a full list of the surfaces constructed with this method up to now, and present the next challenges on the subject.

1 Introduction

Most people divide the mathematicians in two subsets, pure mathematicians and applied mathematicians. They imagine the mathematicians in the first class as old, thinking about very abstract problems, and working only with pen and paper, whence the mathematicians in the second class are expected to be younger, interested only in very concrete questions and spending most of their time in front of the screen of a computer or some other electronic device.

I know indeed many mathematicians who fit perfectly in one of the two models above. But one can do interesting research without corresponding to any of them. In particular, one can study classical problems, problems considered already many decades ago, sometimes more than a century ago, and use the new technologies to answer that questions.

I am mainly interested in the study of smooth compact complex algebraic surfaces, which I will for short denote just by surfaces from now on. This theory essentially started in the XIX$^{\text{th}}$ century, thanks to the impulse of the *Italian school* leaded by the genius of G. Castelnuovo and F. Enriques. One of the most interesting questions at that time was the one posed by Max Noether which we can express in modern terms as *Is a surface of genus zero birational to a projective plane?* The (geometric) genus of a surface is the dimension of the vector space of the holomorphic two forms. Max Noether knew that the analogous question in dimension one has positive answer, but could not settle the two-dimensional case.

Enriques answered negatively to this question few years later before the end of the XIXth century, by his beautiful construction of a sextic surface double along the edges of a tetrahedron ([15]).

Many similar questions have been answered in the next century. *Is a surfaces homeomorphic to a projective plane isomorphic to it?* Yes ([24]). *Is a surface homeomorphic to the product of two projective lines isomorphic to it?* No ([18]). *Is a surface with the same homology groups of the projective plane isomorphic to it?* No ([20]). *Can we list all surfaces with the same homology groups of the projective plane?* Yes ([22, 11]).

The answer to the last question is a clear example of what I mentioned at the beginning. By a theorem of Yau ([25]) each surface with the same Betti numbers of the projective plane is rigid and a quotient of the open ball by an automorphism group acting properly discontinuously. It follows that there are only finitely many surfaces with those Betti numbers, but to get a full list of them a computer was necessary, together with four clever mathematicians, and now we can say

Theorem ([11])
There are exactly 101 pairwise not isomorphic surfaces with the Betti numbers of the projective planes.

All these surfaces have genus zero. More precisely the 100 surfaces different from the projective planes are surfaces of general type, which means that the canonical bundle, the bundle of the two forms, is as *positive* as it can be: tensoring it with itself k times, and leaving k go to infinity, one gets bundles with many holomorphic global sections, and more precisely forming a vector space whose dimension grows as the square of k. Despite this *positivity*, the genus (which is the dimension for k=1) is as small as possible, zero!

The surfaces of general type of genus zero enter indeed in all the questions above, and in most of the open analogous questions. Hirzebruch ([18]) produced countably many surfaces not pairwise isomorphic which are homeomorphic to the product of two lines, now known as *Hirzebruch surfaces*. The natural question follows: *Is every surface homeomorphich to the product of two projective lines isomorphic to one of the Hirzebruch surfaces?* It is known that if there is a surface homeomorphic to the product of two lines not in Hirzebruch's list, then it is a surface of general type and genus zero.

There are many more motivations to study this class of surfaces coming from different subjects as surface theory, differential geometry, and cycle theory: see the survey [5]. We just add that recently also their derived category has shown to be very special, and have been used to show the existence of certain special objects, the phantoms, which were conjectured not to exist, see [10].

A full classification of these surfaces is still out of reach. [5] describes the state of the art of this research one year ago. Some more constructions appeared last year: [1, 2, 8, 14, 21, 23].

A strategy for constructing these surfaces origins from an idea of Beauville ([7]). Beauville considers the quotient of a product of two curves by a finite group G of automorphisms acting freely. These surfaces are now known as *surfaces isogenous to a product*. If the order of G is $(g_1\text{-}1)$ $(g_2\text{-}1)$ then the genus p_g equals the irregularity q, which is the dimension of the vector space of the holomorphic 1-forms, which is easy to compute and vanishes in most cases. This gives minimal surfaces with ample canonical class K of self-intersection 8.

The self-intersection of the canonical class of a minimal surface of general type is an integer which may vary among 1 and 9, and is a topological invariant, so the surfaces isogenous to a product can touch only few topological types of surfaces of genus zero, and it is natural to ask if we can generalize the construction to be able to touch also other values of K^2.

The generalization we suggest is to consider quasi-étale quotients, which are quotients of a product of two curves by a finite group G acting freely out of finitely many points. The quotient is singular and we need to consider a resolution of its singularities. This gives many new examples of surfaces of genus zero as we will see in the following.

In the next sections we give an account on the results about surfaces of genus zero which have been found up to now with this method. In the following section we explain the details of the method, and in section 4 the algorithm we implemented in MAGMA to find all these surfaces. Finally, in the last section we list the most interesting open problem on the subject.

2 Quasi-Étale Surfaces

We start by giving the main definitions.

Definition A surface X is *isogenous to a product* if X is the quotient of a product $C_1 \times C_2$ of two algebraic curves by the action of a finite group of automorphisms G acting freely.

We are mainly interested in the following generalization.

Definition A surface X is a *a quasi-étale quotient* if X is the quotient of the product $C_1 \times C_2$ by the action of a finite group of automorphisms G acting freely out of finite subset of $C_1 \times C_2$.

We use the word quasi-étale because the natural projection $\pi: C_1 \times C_2 \rightarrow X$ is quasi-étale in the sense of [13]. A quasi-étale quotient which is not isogenous to a product is never smooth, but it is singular at finitely many points, the images of the *small* orbits of the G-action on $C_1 \times C_2$, the orbits of cardinality smaller than the order of G. Since we are interested in the classification of smooth surfaces of general type, we introduce the following

Definition A surface S is a *a quasi-étale surface* if it is the minimal resolution of the singularities of a quasi-étale quotient of X.

Catanese ([12]) has noticed that

1) For every quasi-étale quotient X there are infinitely many triples (C_1, C_2, G) such that $X = (C_1 \times C_2)/G$, but the triple producing X with G of minimal order is unique up to isomorphism, and we call it *a minimal realization* of X; in this case we say that the action of G on $C_1 \times C_2$ is *minimal.*
2) A minimal finite group action on $C_1 \times C_2$ is either *unmixed*, which means that it is the action *diagonally* induced by two faithful actions of G respectively on C_1 and C_2 or *mixed*, in which case $C := C_1 = C_2$, half of the elements of G exchange the two factors, and the other half form a subgroup G^0 of index 2 acting minimally unmixed on $C \times C$.

Working on Catanese's remarks, Frapporti proved that the quasi-étale assumption is algebraically very simple.

Theorem ([16])

Assume that G acts minimally on $C_1 \times C_2$.

Then $X = (C_1 \times C_2)/G$ is not a quasi-étale quotient (equivalently, the natural map π: $C_1 \times C_2 \rightarrow X$ is not quasi-étale) if and only if the action is mixed and the exact sequence

$$1 \rightarrow G^0 \rightarrow G \rightarrow Z_{/2Z} \rightarrow 1$$

does not split (equivalently there are no elements of $G - G^0$ of order 2).

This allows to check very simply if an action is quasi-étale. Having an explicit quasi-étale action, the singular points of X are finitely many. A careful analysis of them allows to compute all the invariants of the quasi-étale surfaces S.

We are mainly interested in the opposite argument: fixed the invariants, can we produce surfaces with those invariants? And, more ambitiously, can we give a full list of all quasi-étale surfaces with those invariants?

The answer to both questions are positive, but the latter only in a weak sense. Indeed we have written a computer program in the MAGMA ([9]) language producing in principle (assuming to have enough time and RAM) all quasi-étale surfaces of general type with given genus, irregularity and K^2. Still, this is not fully satisfactory for us. We wish to have a full list of surfaces of general type of genus zero, and these have automatically irregularity zero, but there is in principle no lower bound for K^2.

As mentioned in the introduction the self-intersection of the canonical class of the minimal model of a surface of general type of genus zero is a number among 1 and 9, but since blowing up a point the selfintersection of the canonical class drops by one, the K^2 of a surfaces of general type with genus zero is not bounded from below.

Anyway, quasi-étale surfaces are not far from being minimal. Indeed, it is easy to see that the canonical class of a surface of general type isogenous to a product is ample, and therefore the surface is minimal, so in this case we just need to run the program for all values of K^2 from 1 to 9. The result can be shortly summarized in the following.

Theorem ([3, 16])

There are exactly 18 irreducible families of surfaces of general type of genus zero isogenous to a product, forming 18 irreducible components of the moduli space of the surfaces of general type.

Unfortunately this is not true for quasi-étale surfaces: the first example of a not minimal quasi-étale surface was found by Mistretta and Polizzi ([19]). Running our programs for K^2 from 1 to 9 we proved

Theorem ([4, 6, 17])

There are exactly 88 irreducible families of minimal quasi-étale surfaces

and

Theorem ([6, 17]))

There is exactly one quasi-étale surface S of general type with positive K^2 not minimal; it is a surface with $K^2=1$ whose minimal model has $K^2=3$

We have organised the 89 families in the table 1 below

Table 1 All families of minimal quasi-étale surfaces of general type of genus zero.

K^2	m/u	g_1	g_2	o(G)	Id(G)	Sing(X)	Signature(s)	$\|\pi_1\|$	N
8	U	21	4	60	5	-	$2,5^2;3^4$	∞	1
8	U	16	5	60	5	-	$3^2,5;2^5$	∞	1
8	U	13	6	60	5	-	$2^3,3;5^3$	∞	1
8	U	25	3	48	48	-	$2,4,6;2^6$	∞	1
8	U	9	5	32	27	-	$2^3,4;2^2,4^2$	∞	1
8	U	6	6	25	2	-	$5^3;5^3$	∞	2
8	U	13	3	24	12	-	$3,4^2;2^6$	∞	1
8	U	9	3	16	11	-	$2^3,4;2^6$	∞	1

8	U	5	5	16	3	-	$2^2,4^2;2^2,4^2$	∞	1
8	U	5	5	16	14	-	$2^5;2^5$	∞	1
8	U	4	4	9	2	-	$3^4;3^4$	∞	1
8	U	5	3	8	5	-	$2^6;2^5$	∞	1
8	M	17		256	3678	-	4^3	∞	3
8	M	17		256	3679	-	4^3	∞	1
8	M	9		64	92	-	2^5	∞	1
6	U	3	7	16	11	2½	$2^3,4;2^4,4$	∞	1
6	U	19	3	48	48	2½	$2^3,4;2^4,4$	∞	1
6	U	19	8	168	42	2½	$2^3,4;2^4,4$	84	2
6	U	19	16	360	118	2½	$2^3,4;2^4,4$	60	2
6	U	19	11	240	189	2½	$2^3,4;2^4,4$	120	1
6	U	4	16	60	5	2½	$2^3,4;2^4,4$	∞	1
5	U	3	17	48	48	1/3+2/3	$2,4,6;2^4,3$	∞	1
5	U	9	9	96	227	1/3+2/3	$2^3,3;3,4^2$	32	1
5	U	9	3	24	12	1/3+2/3	$2^4,3;3,4^2$	∞	1
5	U	9	11	120	34	1/3+2/3	$2,5,6;3,4^2$	40	1
5	U	9	6	60	5	1/3+2/3	$3,5^2;2^3,3$	40	1
5	U	9	6	60	5	1/3+2/3	$3,5^2;2^3,3$	20	1
5	U	9	5	48	48	1/3+2/3	$4^2,6;2^3,3$	∞	1
4	U	13	3	48	48	4½	$2^2,4^2;2,4,6$	∞	1
4	U	3	3	8	5	4½	$2^5;2^5$	∞	1
4	U	13	3	48	48	4½	$2^5;2,4,6$	∞	1
4	U	4	4	18	4	4½	$2^2,3^2;2^2,3^2$	27	1
4	U	5	5	32	27	4½	$2^3,4;2^3,4$	32	1
4	U	3	3	8	2	4½	$2^2,4^2;2^2,4^2$	∞	1
4	U	7	3	24	12	4½	$2^5;3,4^2$	∞	1
4	U	4	4	18	3	4½	$3,6^2;2^2,3^2$	∞	1
4	U	21	4	120	34	4½	$3,6^2;2,4,5$	∞	1
4	U	4	3	16	11	4½	$2^5;2^3,4$	∞	1
4	U	4	11	60	5	4½	$2,5^2;2^2,3^2$	15	1
4	U	10	25	360	118	2/5+3/5	$2,4,5;3^2,5$	6	1
4	U	5	25	160	234	2/5+3/5	$2,4,5;4^2,5$	8	3
4	U	10	5	60	5	2/5+3/5	$2^3,5;3^2,5$	12	1
4	M	5		32	7	4½	2^5	32	1
4	M	5		32	22	4½	2^5	∞	1
4	M	9		128	836	4½	4^3	32	1
3	U	10	25	360	118	1/5+4/5	$2,4,5;3^2,5$	6	1
3	U	5	25	160	234	1/5+4/5	$2,4,5;4^2,5$	8	3
3	U	10	5	60	5	1/5+4/5	$2^3,5;3^2,5$	12	1

3	U	11	3	48	48	2½+1/3+2/3	2^2,3,4;2,4,6	8	1
2	U	7	3	48	48	6½	2^3,4;2,4,6	4	1
2	U	4	4	36	10	6½	2,6^2;2^3,3	3	1
2	U	4	6	60	5	6½	2,5^2;2^3,3	5	1
2	U	3	3	16	2	6½	4^3;4^3	8	1
2	U	3	3	16	11	6½	2^3,4;2^3,4	8	1
2	U	3	22	168	42	6½	2,3,7;4^3	4	2
2	U	11	3	120	34	6½	2,6^2;2,4,5	3	1
2	U	10	7	168	42	2½+¼+¾	2,4,7;3^2,4	3	2
2	U	10	16	360	118	2½+¼+¾	2,4,5;3^2,4	3	2
2	U	16	4	120	34	2½+¼+¾	3,4,6;2,4,5	3	2
2	U	5	3	24	12	2(1/3+2/3)	2^2,3^2;3,4^2	8	1
2	U	3	5	24	13	2(1/3+2/3)	2,6^2;2^2,3^2	8	1
2	U	6	6	75	2	2(1/3+2/3)	3^2,5;3^2,5	5	2
2	U	5	6	60	5	2(1/3+2/3)	3^2,5;2^3,3	4	1
2	M	3		16	3	6½	2^5	8	1
2	M	5		64	82	6½	4^3	8	1
2	M	5		64	32	½+2¾	2^3,4	4	1
2	M	7		36	9	½+2¾	2^2,3^2	3	1
2	M	9		128	1535	3½+2¼	2^3,4	8	1
2	M	17		768	1083540	2(1/3+2/3)	3^2,4	4	1
2	M	17		768	1083541	2(1/3+2/3)	3^2,4	4	1
1	U	3	3	24	12	4½+1/3+2/3	2^3,3;3,4^2	4	1
1	U	3	15	168	42	4½+1/3+2/3	2,3,7;3,4^2	2	1
1	U	3	5	48	48	4½+1/3+2/3	2,4,6;2^3,3	2	1
1	M	2		16	11	2½+2¾	2^3,4	4	1

[a] Legenda: every rows correspond to an irreducible family of quasi-étale surfaces. In the first column we put K^2 of the surface. In the second column we put U or M according if the action is unmixed or mixed. In the third column we put the genus of the first curve, C in the mixed case, C_1 in the unmixed case. In the fourth column we put the genus of C_1 in the unmixed case, nothing in the mixed case. In the fifth column we put the order of the group G. In the sixth column we put the identifier of G among the groups of its order in the MAGMA database. In the seventh column we put the basket of singularities; since all the singular points are cyclic quotient singularities we represent them as fractions so that, for example, 3½+2¼ means that there 5 singular points, 3 quotient singularities of type ½ and 2 of type ¼. In the eighth column we put the signatures (only one in the unmixed case) in exponential notation, so that 2^3,4 correspond to {2,2,2,4}. In the ninth column we put the order of the fundamental group of the resulting surfaces. In the last column we put the number of families up to isomorphisms.

3 Constructing Curves with a Group Action

By Riemann Existence Theorem, to give an action of a group G^0 onto a curve C is equivalent to give

- The quotient curve C/G^0;
- Finitely many points $p_1,\dots,p_r$, in C/G^0, the branching points of the projection map $C \rightarrow C/G^0$;
- A suitable set of generators of the fundamental group of the complement of these points in C/G^0, including loops $\gamma_1,\dots,\gamma_r$ such that each γ_i is a small circle around p_i;
- A suitable set of generators of G^0; these are the images of the above generators of the fundamental group of the complement of the p_i by the standard monodromy map.

The signature of this system of generators is the unordered list of the orders of the elements of this last set of generators which are the images of the γ_i by the monodromy map.

So, to construct an unmixed quotient surface, we have to choose two sets of points $p_1,\dots,p_r$, in C_1/G^0 and $q_1,\dots,q_r$, in C_2/G^0, corresponding systems of generators as above, and that's all. To give a mixed quasi-étale quotient it is enough to give ([16]) a set of points, a set of generators of both the fundamental group and the group G^0, thus determining the action of G^0 on the first factor of $C\times C$, and then the unsplit extension $1 \rightarrow G^0 \rightarrow G \rightarrow Z_{/2Z} \rightarrow 1$.

The formulas for the invariants of the resulting surface are simple to compute. For the irregularity

$$q=g(C_1)+g(C_2)$$

in the unmixed case and

$$q=g(C)$$

in the mixed case.

The formulas for the other invariants are slightly more complicated. For each singular point x in X we can compute its analytic type, which only depends on the stabilizers of the corresponding points in $C_1\times C_2$. There are two positive rational numbers k_x and b_x ([6, 17]) depending only on the analytic type of the singularity such that

$$K^2=8(g(C_1)-1)(g(C_2)-1)/|G|+\Sigma_x k_x$$

$$p_g=K^2/8-1+q+\Sigma_x b_x$$

We can now construct all quasi-étale surfaces and compute their numerical invariants.

4 Contructing Surfaces

Constructing random surfaces, it is not very likely to give anything interesting. From the discussion in the introduction is clear that we are interested in a systematic procedure for constructing surfaces with prescribed invariants, for example for genus zero.

The formulas in the last section can be “reversed” to an algorithm which does exactly that, as follows. Note that the unmixed and the mixed case have to be treated separately (and indeed we have two distinct programs) but the idea of the algorithm is the same.

- Choose the invariants p_g, q and K^2 (these integers are the input).
- In the unmixed case, consider all possible pairs of genera of the quotient curves with $q=g(C_1/G)+g(C_2/G)$; in the mixed case $q=g(C/G^0)$.
- Determine all possible “baskets” of singularities x such that $\Sigma_x b_x=K^2/8-1+q-p_g$; it is not difficult to show, from the explicit formula for b_x, that there are only finitely many possible baskets like that, and there is a simple algorithm to compute this list.
- There are some inequalities which in both cases, once fixed the invariants and the basket of singularities, restricts the possible signatures to a finite set: our algorithm computes, for each basket, the corresponding list.
- Hurwitz formula computes the order of G^0 (which coincides with G in the unmixed case) from the invariants and the signatures; then (for each pairs of signatures in the unmixed case, for each signature in the mixed case) we run a search among all groups of the right order for (pairs of) systems of generators of the prescribed signature(s). Of course this can be done only if the order of the group is small enough;

up to now we could exclude all other cases by group theoretical arguments.

- In the unmixed case, for each such system of generators we compute all unsplit extensions of degree 2 of G^0.
- Each of the output we have gives a surface, whose singularities we can compute. If the basket of singularities is exactly the one prescribed by the basket we are considering, then we have a surface with the invariants we have fixed at the beginning. Else we just throw it in the waste.

Both algorithms (in the mixed and unmixed case) are now implemented in MAGMA, and available upon request to the author, although the algorithm for the unmixed case works at the moment only in the regular case (q=0). These programs produce, for $p_g=0$ and $K^2>0$, the 90 families in section 2.

5 Problems

Problem 1: *Bound K^2 from below by a function of p_g and q.*

An upper bound is provided by Bogomolov-Miyaoka.Yau inequality, while there is no lower bound in general, but we have some evidence that there should be some lower bound for surfaces constructed in this way. We have some conjectures and only partial results in this direction. This would allow us to obtain the full list of all quasi-étale surfaces of genus zero.

Problem 2: *If q=1, the Albanese map is a morphism onto an elliptic curve. Compute the genus of general fibre.*

This is simple in the unmixed case, where we can always assume $g(C_1)=1$ and then the Albanese map is an isotrivial fibration with fibre C_2. But it is absolutely non trivial in the mixed case. Once solved this problem we could use our programs to test some interesting existence conjecture in the irregular case.

Problem 3: *Simplify the algorithm!*

The program complexity is very big; if we drop K^2 in the input by 1 the time necessary to complete the computation is multiplied approximately by 30. We run the program up to $K^2=-2$, but with this program we cannot go much further. Moreover every tentative to use the program for $p_g>q$ ended, up to now, with a computer running out of memory.

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