ON WAHL'S PROOF OF $\mu(6) = 65$

Introduction

In this note we present a short proof of the following theorem of D. Jaffe and D. Ruberman:

Theorem [Ja-Ru]**.** *A sextic hypersurface in $\mathbb{P}^3$ has at most 65 nodes.*

The bound is sharp by Barth's construction [Ba] of a sextic with 65 nodes.

Following Beauville [Be], to a set of n nodes on a surface is associated a linear subspace of $\mathbb{F}^n$ (where $\mathbb{F}$ is the field with two elements) whose elements corresponds to the so-called *even subsets* of the set of the nodes. Studying this code Beauville proved that the maximal number of nodes of a quintic surface is 31.

The same idea was used by Jaffe and Ruberman, but their proof is not so short as the one of Beauville, partly because at that time a complete understanding of the possible cardinalities of an even set of nodes was missing.

Almost at the same time, J. Wahl [Wa] proposed a much shorter proof of the same result. He proved indeed the following (see the beginning of the next section for the missing definitions)

Theorem [Wa]**.** *Let $V \subset \mathbb{F}^{66}$ be a code, with weights in $\{24, 32, 40\}$. Then $\dim(V) \leq 12$.*

He claimed that Jaffe-Ruberman's theorem follows as a corollary since the code associated to a nodal sextic has dimension at least $n - 53$ (see section 1 of [Ca-To] for this computation). In fact, he used an incorrect result stated by Casnati and Catanese in [Ca-Ca], asserting that the possible cardinalities of an even set of nodes on a sextic are only $24, 32$ and 40. Recently Catanese and Tonoli showed indeed

Theorem [Ca-To]**.** *On a sextic nodal surface in $\mathbb{P}^3$, an even set of nodes has cardinality in $\{24, 32, 40, 56\}$.*

Note however that [Ca-To] used a result by Jaffe and Ruberman, namely that there is no even set of nodes of cardinality 48.

By the above theorem the proof of the theorem of Jaffe and Ruberman reduces to the following

Theorem A. *Let $V \subset \mathbb{F}^{66}$ be a code with weights in $\{24, 32, 40, 56\}$. Then $\dim(V) \leq 12$.*

This statement is in fact theorem 8.1 of [Ja-Ru]. Anyway, its proof is much more complicated than Wahl's one and moreover requires computers computations. In this short note we give an elementary proof, using and integrating Wahl's ideas.

1. Notation and general results from coding theory

A **code** is (in this note) a vector subspace $V \subset \mathbb{F}^n$, where $\mathbb{F}$ is the field with two elements. A **word** is a vector $v = (v_1, \ldots, v_n) \in \mathbb{F}^n$. Its **support** $\mathrm{Supp}(v)$ is the set $\{i \mid v_i \neq 0\}$ of coordinates that do not vanish in v, its **weight** $|v|$ is the cardinality of its support. The **length** of a code is the cardinality of the union of the supports of all its elements. A code $V \subset \mathbb{F}^n$ is said to be **spanning** if it has length n.

A code is **even** if all its words have even weight, **doubly even** if all its weights are divisible by 4. The number of words of weight i in the code V is denoted by $a_i(V)$ or simply a_i when no confusion arises. The **weight enumerator** of the code V is the homogeneous polynomial

$$W_V(x, y) = \sum a_i x^{n-i} y^i.$$

The standard scalar product in $\mathbb{F}^n$ associates to each code its dual code , *i.e.*, its annihilator $V^* \subset \mathbb{F}^n$, which has complementary dimension. We set $a_i^* := a_i(V^*)$.

cazzatine **Remark 1.1.** *1) $V \subset \mathbb{F}^n$ is spanning if and only if $a_1^* = 0$.*

2) If $v^* \in V^*$ has weight 2, the subset of V given by all words v with $\mathrm{Supp}(v) \cap \mathrm{Supp}(v^*) = \emptyset$ *is a subcode of codimension at most* 1 *(and length at most $n - 2$).*

3) A doubly even code is automatically isotropic, i.e., $V \subset V^*$.

The **MacWilliams identity** (cf. [McW-Sl]) states that the weight enumerator $W_{V^*}(x, y)$ of the dual code V^* equals $W_V(x + y, x - y)/2^d$, *i.e.*,

MW (1.1)
$$\sum a_i^* x^{n-i} y^i = \frac{1}{2^d} \left(\sum a_i (x + y)^{n-i} (x - y)^i \right).$$

As explained in [Wa], comparing the coefficients of $x^{n-i}y^i$ for $i \leq 3$ in both sides of (1.1) gives (since $a_0 = a_0^* = 1$):

lem2.4 **Lemma 1.2.** [Wa, Lemma 2.4] *Let $V \subset \mathbb{F}^n$ be a spanning code of dimension d. Then:*

1 (1.2a) $$\sum_{i>0} a_i = 2^d - 1$$

2 (1.2b) $$\sum i a_i = 2^{d-1} n$$

3 (1.2c) $$\sum i^2 a_i = 2^{d-1}(a_2^* + n(n + 1)/2)$$

4 (1.2d) $$\sum i^3 a_i = 2^{d-2} \left(3(a_2^* n - a_3^*) + n^2(n + 3)/2 \right)$$

The following proposition gives dimension and weights of a projected linear code.

prop2.8 **Proposition 1.3.** [Wa, Prop. 2.8] *Let $V \subset \mathbb{F}^n$ be a code of dimension d. Fix a word $w \in V$ and consider the projection $\pi \colon \mathbb{F}^n \to \mathbb{F}^{n-|w|}$ onto the complement of the support of w. Then*

(1) *If w is not a sum of two disjoint words in V, then $V' := \pi(V)$ is a code of dimension $d' = d - 1$.*
(2) $|\pi(v)| = \frac{1}{2}(|v| + |v + w| - |w|)$.

Proof. If $\ker \pi_{|V}$ contains, besides w, another word v, one can write a disjoint sum $w = v + (w - v)$. Thus, in the hypothesis of (1), $\dim \ker \pi_{|V} = 1$ and therefore $d' = d - 1$.

For (2), let r be the cardinality of the intersection of the two supports of v and w. Then $|v| = r + |\pi(v)|$ and $|v| + |w| = |v + w| + 2r$. □

2. The proof

lem2.6 **Lemma 2.1.** [Wa, Lemma 2.6] *The dimension of a code with weights in* $\{24, 32\}$ *is at most* 9.

Proof. Let n be the length of the code and d its dimension. Solving the linear system given by (1.2a) and (1.2b), $a_{24} = 2^{d-4}(64-n) - 4$, $a_{32} = 2^{d-4}(n-48) + 3$. Substituting in (1.2c)

$$2^8\left(2^{d-6}\cdot 9\cdot(2^6-n) + 2^{d-2}\cdot(n-48) + 3\right) = 2^{d-1}(a_2^* + n(n+1)/2)$$

If $d > 9$, then 2^{d-1} divides the R.H.S. but not the L.H.S., a contradiction. □

rem1 **Remark 2.2.** *A code* $V \subset \mathbb{F}^{67}$ *with weights* ≥ 24 *has necessarily* $a_{56} \leq 1$.

Proof. Indeed, if there are two different words of weight 56, their sum has weight at least 24 and then the cardinality of the intersection of their supports is at least $1/2(56+56-24) = 44$. Therefore their span has length $\geq 44 + 2\cdot(56-44) = 68$. □

lem2 **Lemma 2.3.** *The dimension of a code* $V \subset \mathbb{F}^{67}$ *with weights in* $\{24, 32, 56\}$ *is at most* 10.

Proof. If $a_{56} = 0$ the result follows by Lemma 2.1.

Otherwise, by Remark 2.2, $a_{56} = 1$. The intersection of V with any hyperplane not containing its unique word of weight 56 is a code V' of dimension $\dim(V) - 1$ with weights in $\{24, 32\}$ and the result follows again by Lemma 2.1. □

Proof of Theorem A. Suppose that there exists a code $V \subset \mathbb{F}^{66}$ with weights in $\{24, 32, 40, 56\}$ of dimension 13. Let n be its length and consider V as a spanning code in $\mathbb{F}^n$.

By Lemma 2.3 we have $a_{40} > 0$. For each word $w \in V$ with weight 40 we consider the projection π_w onto the complement of the support of w. By Proposition 1.3, $V' := \pi_w(V) \subset \mathbb{F}^{n-40}$ is a doubly even code of dimension 12. So V' is an isotropic subspace, $n - 40 \geq 24$ and we obtain $n \geq 64$: more precisely $n \in \{64, 65, 66\}$.

Suppose $n = 64$. For each word $w \in V$ of weight 40, $\pi_w(V)$ is isotropic of dimension 12 in $\mathbb{F}^{24}$, so $\pi_w(V) = (\pi_w(V))^*$. Let $\mathbb{I} \in \mathbb{F}^{24}$ be the vector with all coordinates 1: $\mathbb{I} \in (\pi_w(V))^*$ (since $\pi_w(V)$ is even) and therefore $\mathbb{I} \in \pi_w(V)$.

If $v \in V$ is a word such that both the weights $|v|, |v+w|$ are ≤ 40, then by Proposition 1.3 $|\pi_w(v)| \leq 20$; therefore by remark 2.2 $a_{56}(V) = 1$ and $\mathbb{I} = \pi_w(\overline{v})$ for the unique word $\overline{v} \in V$ with $|\overline{v}| = 56$.

Fix one coordinate not in the support of $\overline{v}$ and let $V'' \subset V$ be the subcode defined by the vanishing of the given coordinate. Since $\mathbb{I} = \pi_w(\overline{v})$, the support of w contains the complementary of the support of $\overline{v}$: then $w \notin V''$. Since this holds for each $w \in V$ with $|w| = 40$, then V'' has no word of weight 40; it is a code of dimension 12 with weights in $\{24, 32, 56\}$, contradicting lemma 2.3.

Suppose $n = 65$. Solving the equations (1.2a)-(1.2d), we obtain $a_{56} = \frac{1}{2}(a_2^* - a_3^* - 5)$ and thus $a_2^* > 0$. Let then $z \in V^*$ be a word of length 2.

For each word $w \in V$ of weight 40, $a_2^*(\pi_w(V)) = 0$: in fact, for any word $z' \in (\pi_w(V))^*$ of weight 2, $\mathrm{Span}(V', z')$ is an isotropic subspace of dimension 13 in $\mathbb{F}^{25}$, absurd. Therefore every word w of weight 40 satisfies $\mathrm{Supp}(w) \supset \mathrm{Supp}(z)$.

By remark 1.1 the subset of V given by all words v with $\mathrm{Supp}(v) \cap \mathrm{Supp}(z) = \emptyset$ is a subcode of dimension at least 12 with weights in $\{24, 32, 56\}$, contradicting Lemma 2.3.

Then $n = 66$. Solving the equations (1.2a)-(1.2d), we obtain $a_{56} = a_2^* - \frac{1}{2}(a_3^* + 13)$ and thus $a_2^* \geq 7$. We choose two words $z_1 \neq z_2$ in V^* of weight 2.

If we show that for each word $w \in V$ of weight 40, $a_2^*(\pi_w(V)) \leq 1$, then $\mathrm{Supp}(w)$ intersects $Z = \mathrm{Supp}(z_1) \cup \mathrm{Supp}(z_2)$. Therefore, by remark 1.1, the subset of V

given by all words v with $\mathrm{Supp}(v) \cap Z = \emptyset$ is a code of dimension at least 11 and weights among $\{24, 32, 56\}$, contradicting again Lemma 2.3.

So it remains to show only that for each word $w \in V$ of weight 40, $a_2^*(\pi_w(V)) \leq 1$.

If $z' \in (\pi_w(V))^*$ is a word of weight 2, then $V'' := \mathrm{Span}(\pi_w(V), z') \subset \mathbb{F}^{26}$ is an isotropic subspace of dimension 13, and thus $\mathbb{I} \in V'' = (V'')^*$. Being $\pi_w(V)$ doubly even, $\mathbb{I}, z' \in V'' \setminus \pi_w(V)$, and therefore $\mathbb{I} + z'$ is a word in $\pi_w(V)$ of weight 24. Thus $a_2^*(\pi_w(V)) \leq a_{24}(\pi_w(V))$.

If $v \in V$ is a word such that both the weights $|v|, |v+w|$ are ≤ 40, then by Proposition 1.3 $|\pi_w(v)| \leq 20$; therefore $a_{24}(\pi_w(V)) \leq a_{56}(V) \leq 1$ (the last inequality by remark 2.2).

$\square$

References

Ba [Ba] W. Barth, Two projective surfaces with many nodes, admitting the symmetries of the icosahedron. *J. Alg. Geom.*, 5 (1996), 173–186 .

Bea [Be] A. Beauville, Sur le nombre maximum de point doubles d'une surface dans $\mathbb{P}^3$ ($\mu(5) = 31$). In *Algebraic Geometry, Angers, 1979, A. Beauville ed., Sijthoff & Noordhoff*, (1980), 207–215.

CaCa [Ca-Ca] G. Casnati and F. Catanese, Even sets of nodes are bundle symmetric. *J. Diff. Geom.*, 47 (1997), 237–256. Erratum. *J. Diff. Geom.* 50 (1998), no. 2, 415.

CaTo [Ca-To] F. Catanese, F. Tonoli, Even sets of nodes on sextic surfaces To appear in *J.E.M.S*, 2006.

JR [Ja-Ru] D. B. Jaffe and D. Ruberman, A sextic surface cannot have 66 nodes. *J. Alg. Geom.*, 6 (1997), 151–168 .

mw [McW-Sl] F. J. MacWilliams, N. J. A. Sloane, The theory of error-correcting codes I, II. *North-Holland Mathematical Library*, Vol. 16. North-Holland Publishing Co., Amsterdam-New York-Oxford, (1977), i–xv + 1–369, resp. i–ix + 370–762.

wahl [Wa] J. Wahl, Nodes on sextic hypersurfaces in P^3. *J. Differential Geom.* 48 (1998), no. 3, 439–444.