CHISINI'S CONJECTURE FOR CURVES WITH SINGULARITIES OF TYPE $x^n = y^m$

SANDRO MANFREDINI, ROBERTO PIGNATELLI

Abstract. This paper is devoted to a very classical problem that can be summarized as follows: let S be a non singular compact complex surface, $\pi : S \to \mathbb{P}^2$ a finite morphism having simple branching, B the branch curve: then (cf. [Fu2]) "to what extent does B determine $\pi : S \to \mathbb{P}^2$"?
The problem was first studied by Chisini ([Ch]) who proved that B determines S and π, assuming B to have only nodes and cusps as singularities, the degree d of π to be greater than 5, and a very strong hypothesis on the possible degenerations of B, and posed the question if the first or the third hypothesis could be weakened.
Recently Kulikov ([Ku]) and Nemirovski ([Ne]) proved the result for $d \geq 12$, and B having only nodes and cusps as singularities.
In this paper we weaken the hypothesis about the singularities of B: we generalize the theorem of Kulikov and Nemirovski for B having only singularities of type $\{x^n = y^m\}$, in the additional hypothesis of smoothness for the ramification divisor (automatic in the "nodes and cusps" case). Moreover we exhibit a family of counterexamples showing that our additional hypothesis is necessary.

1. Introduction.

As introduced in the abstract, in this paper we study finite morphisms having simple branching over curves with singularities of type $\{x^n = y^m\}$. In order to state the problem and our results, we need to introduce a little bit of notations.

Definition 1.1 A *normal generic cover* is a finite holomorphic map $\pi : S \longrightarrow \mathbb{C}^2$, which is an analytic cover branched over a curve B such that S is a connected normal surface and the fiber over a smooth point of B is supported on $\deg\pi - 1$ distinct points.

Two normal generic covers (S_1, π_1), (S_2, π_2) with the same branch locus B are called (analytically) *equivalent* if there exists an isomorphism $\phi : S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$.

The main interest for generic covers comes from the well known fact that,

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by the Weierstrass preparation theorem, given an analytic surface $S \subset \mathbb{C}^n$, a generic projection $S \stackrel{\pi}{\to} \mathbb{C}^2$ is (at least locally, in order to insure $\deg \pi < \infty$) a normal generic cover branched over a curve (see [GuRo]).
A standard way to study generic covers, is the following: given a generic cover $\pi : S \to \mathbb{C}^2$ with branch curve B, one defines the *monodromy homomorphism* $\rho : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_{\deg \pi}$ as the action of this fundamental group on the fiber of a regular value.
The pair (B, ρ) gives the "building data" of the cover: one can reconstruct the cover from (B, ρ) (cf. [GrRe]).
Despite of the explicit construction, to understand the singularity of the cover from the building data is a very difficult problem (except in specific cases). It is, for example, still an open problem to classify all the possible "building data" coming from smooth surfaces.
In [MP] we give a complete classification of the normal generic covers branched over irreducible curves of type $\{x^n = y^m\}$ in terms of what we called there "monodromy graphs": we will recall briefly the definition of monodromy graphs and the above result in section 1. Let us point out that, according to the Puiseux classification, this class of singularities is a natural first step for a complete classification.
Our first result (to which is devoted section 1), is a "more friendly" classification theorem, that will be crucial in the following sections.
Let h, k, a, b be positive integers with $(h, k) = 1$, and consider the surface $S_{h,k,a,b}$ in $\mathbb{C}^4$ defined by the equations $hz^k + kw^h - (h+k)x^a = zw - y^b = 0$. Let $F : S_{h,k,a,b} \to \mathbb{C}^2$ be the projection on the (x, y)-plane.

Theorem 1.2. *The map $F : S_{h,k,a,b} \to \mathbb{C}^2$ is a generic cover branched over $x^{a(h+k)} = y^{bhk}$ of degree $h + k$.*
Conversely, up to exchanging x and y, every generic cover $\pi : S \to \mathbb{C}^2$ of degree $d \geq 3$ branched over $\{x^n = y^m\}$, with (n, m)=1, is equivalent to one of the previous maps.

In section 2 we consider the "global" case of projective generic covers.
Definition 1.3 A *projective generic cover* is a finite morphism $\pi : S \longrightarrow \mathbb{P}^2$, branched over an irreducible curve B such that S is an irreducible projective surface and the fiber over a smooth point of B has cardinality $\deg\pi - 1$.
This is the same as requiring that $\pi^*(B) = 2R + C$, with R irreducible and C reduced, and that $\pi_{|_R} : R \to B$ is 1:1 over smooth points of B.
As in the previous case, for each irreducible projective surface S, a generic projection $\pi : S \to \mathbb{P}^2$ is a projective generic cover branched over a (projective plane) curve B.
We say that a projective generic cover is *smooth* if the surface S and the ramification divisor R are non-singular.
Actually, when S is non-singular, a "general" generic projection has ramification divisor R non-singular. Let us point out that, if B has only nodes and cusps as singularities, R is automatically smooth.
Again, we will consider projective generic covers up to analytic equivalence:

(S_1, π_1), (S_2, π_2) with the same branch locus B are *equivalent* if there exists an isomorphism $\phi : S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$.
Chisini's conjecture asserts the following (see [Ch])

Conjecture 1.4 (Chisini)**.** *Let B be the branch locus of a smooth projective cover $\pi : S \to \mathbb{P}^2$ of degree $deg\pi \geq 5$. Then π is unique up to equivalence.*

In other words, if S is smooth and the degree high enough, the curve B determines the cover.
In fact, Chisini proved the result in the above mentioned additional hypothesis that the branch curve B has only nodes and cusps as singularities, and that B has some particular degeneration. In the same paper, he posed the question if this two last hypothesis could be weakened.
The bound for the degree of π is needed according to a counterexample, due to Chisini and Catanese (see [Ca]) of a sextic curve with 9 cusps which is the branch curve of 4 non equivalent smooth projective covers, three of them are of degree 4 and one is of degree 3.
Recently, V.S. Kulikov (see [Ku]) developed a new approach proving Chisini's conjecture for curves with only nodes and cusps as singularities, and the additional hypothesis that the degree of π is greater than a certain function of the degree, genus and number of cusps of the branch locus. After that, S. Nemirovski (see [Ne]), using the Bogomolov-Miyaoka-Yau inequalities, found a uniform bound, 12, for Kulikov function.
Putting the two results together we have the following theorem:

Theorem 1.5 ([Ku], [Ne])**.** *Let B be the branch locus of a smooth projective cover $\pi : S \to \mathbb{P}^2$ of degree $deg\pi \geq 12$, with only nodes and cusps as singularities. Then π is unique up to equivalence.*

In section 3, we use theorem 1.2 in order to improve the previous results as follows:

Theorem 1.6. *Let B be the branch locus of a smooth projective generic cover $\pi : S \to \mathbb{P}^2$ having only singularities of type $x^{n_i} = y^{m_i}$. Then, if*

$$\deg\pi > \frac{4(3d+g-1)}{2(3d+g-1) - \sum_{i=1}^{r}\left(\frac{\min(m_i,n_i)}{\gcd(m_i,n_i)} - 1\right)}$$

where $2d = degB$ and $g = g(B)$ is its genus, then π is unique.

Theorem 1.7. *In the above hypothesis, if $deg\pi \geq 12$ then π is unique.*

Finally, in section 4, we will construct a family of projective generic covers and we will show that the hypothesis of smoothness for R is necessary, finding pairs of non equivalent projective generic covers of arbitrarily large degree having the same branch curve. More precisely we prove (we defer the definitions of $\bar{f}_i, \bar{g}_j$ to section 4)

Proposition 1.8. *Let $t \in \mathbb{N}$, $t \geq 1$, B be the projective plane curve given by the equation*

$$\bar{g}_{4t+1}(x,w)^{2t(2t+1)} = \bar{f}_{2t(2t+1)}(y,w)^{4t+1}.$$

Then there are two generic covers $S' \xrightarrow{\pi'} \mathbb{P}^2$, $S'' \xrightarrow{\pi''} \mathbb{P}^2$, with S', S'' smooth, degrees respectively $4t+1$ and $4t+2$.
The ramification divisor is singular except in the case $t = 1$ and the degree of the cover is 6.

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2. Equations.

Consider the following surface $S_{h,k}$ in $\mathbb{C}^4$ ($S_{h,k,1,1}$ in the introduction)

$$\begin{cases} hz^k + kw^h = (h+k)x \\ zw = y \end{cases} \tag{2.1}$$

where $1 \leq h < k$ are coprime integers.
The jacobian matrix is

$$\begin{pmatrix} h+k & 0 & hkz^{k-1} & hkw^{h-1} \\ 0 & 1 & w & z \end{pmatrix}$$

from which we see that $S_{h,k}$ is smooth and we can choose z, w as local coordinates near $(0,0,0,0)$ for $S_{h,k}$.
Consider the map $F_{h,k} : S_{h,k} \longrightarrow \mathbb{C}^2$ which is the restriction to $S_{h,k}$ of the projection of $\mathbb{C}^4$ on the (x,y)-plane.

Proposition 2.2. *$F_{h,k}$ is a normal generic cover of degree $h+k$ branched over the curve $x^{h+k} = y^{hk}$*

Proof. We have that $F^{-1}_{h,k}(0,0) = (0,0,0,0)$ and one can easily check that the degree of $F_{h,k}$ is $h+k$.
The equations of the ramification divisor R in the local coordinates (z,w) of $S_{h,k}$ are given by the vanishing of the determinant of the submatrix of the jacobian matrix

$$\begin{pmatrix} hkz^{k-1} & hkw^{h-1} \\ w & z \end{pmatrix}$$

that is $z^k = w^h$.
Substituting into the equations of $S_{h,k}$ in $\mathbb{C}^4$, we get that the locus defined by the equation $y^{hk} = (z^{hk}w^{hk} =)x^{h+k}$ in the (x,y)-plane contains the branch curve B. But this locus is irreducible since $(h,k)=1$, so we found the equation of the branch curve.
We are left with the "genericness" check. Of course (by irreducibility), it is enough to check it over a smooth point of B, and we take the point $(1,1)$.

$F_{h,k}^{-1}(1,1)$ is the set of points of the form $(1,1,z,w)$ described by the equations $\{\frac{hz^k+kw^h}{h+k} = zw = 1\}$.
Then $z \neq 0$, $w = \frac{1}{z}$ and, (multiplying by z^h), we have to compute the solutions of

$$\left(\frac{hz+k}{h+k}\right)^{h+k} = z^h,$$

i.e. the roots of the polynomial $P(z) = (hz+k)^{h+k} - (h+k)^{h+k}z^h$.
We have to show that P has exactly $h+k-1$ distinct roots; its first and second derivatives are

$$P'(z) = h(h+k)[(hz+k)^{h+k-1} - (h+k)^{h+k-1}z^{h-1}]$$

$$P''(z) = h(h+k)[h(h+k-1)(hz+k)^{h+k-2} - (h-1)(h+k)^{h+k-1}z^{h-2}].$$

But $P(z) = P'(z) = 0$ implies

$$(hz+k)(h+k)^{h+k-1}z^{h-1} = (h+k)^{h+k}z^h$$

and since 0 is not a root of P, $hz+k = (h+k)z$, i.e. $z=1$.
Since $P(1) = P'(1) = 0$ but $P''(1) \neq 0$, we conclude that $z=1$ is a double root of P and all the others are simple roots. ❑

From the proof of previous proposition we get also

Remark 2.3. *The ramification divisor R is cut (on $S_{h,k}$) by the hypersurface $z^k = w^h$, while the preimage of the branch locus B is $2R + C$ where C is the union of the curves cut by the hypersurfaces $z^k = \alpha w^h$ for $\alpha \neq 1$ a root of $P(t) = (ht+k)^{h+k} - (h+k)^{h+k}t^h$.*

Now we introduce the complete class of covers we need for our classification theorem.

Consider the pullback of $F_{h,k}$ under the base change given by the map

$$f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2, \quad f_{a,b}(x,y) = (x^a, y^b).$$

We obtain the surface $S_{h,k,a,b}$ of equations

$$\begin{cases} hz^k + kw^h = (h+k)x^a \\ zw = y^b \end{cases} \tag{2.4}$$

and the map $F_{h,k,a,b} : S_{h,k,a,b} \to \mathbb{C}^2$ given by the two coordinates (x,y).

Now we can introduce the main result of this section.

Theorem 2.5. *The maps $F_{h,k,a,b}$ are generic covers of degree $h+k$, branched over $x^{a(h+k)} = y^{bhk}$*

Conversely, up to exchanging x and y, every generic cover $\pi : S \to \mathbb{C}^2$ of degree $d \geq 3$ branched over $\{x^n = y^m\}$, with $(n,m)=1$, is equivalent to one of the previous maps.

The first part of the statement is the following lemma.

Lemma 2.6. *The maps $F_{h,k,a,b}$ are normal generic covers of degree $(h+k)$ branched over the curve $x^{a(h+k)} = y^{bhk}$.*

Proof. The statement comes from previous proposition using the base change map $f_{a,b}$. The normality of $S_{h,k}$ implies the normality of $S_{h,k,a,b}$ by theorem 2.2 of [MP]. ❑

In order to prove the second part, we use the well known fact ([GrRe]) already mentioned in the introduction that the pair (branch curve B, monodromy homomorphism) determines the cover. We will introduce now precisely the monodromy homomorphisms and the monodromy graphs that represent them, in term of which we gave in [MP] a classification theorem for generic covers branched over irreducible curves of type $\{x^n = y^m\}$, result that we will finally briefly recall.
Let (S, π) be a normal generic cover of degree $\deg\pi = d$.
Every element in the fundamental group $\pi_1(\mathbb{C}^2 \setminus B)$ of the set of regular values of π, induces a permutation of the $d=\deg\pi$ points of the fiber over the base point, thus a homomorphism $\varphi : \pi_1(\mathbb{C}^2 \setminus B) \longrightarrow \mathcal{S}_d$, called the *monodromy* of the cover. The "generic" condition means that for each geometric loop (i.e. a simple loop around a smooth point of the curve) its monodromy is a transposition. The homomorphisms with this property are called *generic* monodromies.
So, in order to classify generic covers $\pi : S \to \mathbb{C}^2$ of degree d, with S a normal surface, branched over some curve B, one need to classify generic monodromies $\varphi : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$.
We did it (for curves B of type $\{x^n = y^m\}$) in [MP], representing the monodromy of a normal generic cover of degree d branched on the curve $\{x^n = y^m\}$ by a labeled graph Γ, called *monodromy graph*. We will denote by $Gr_{d,n}$ the set of all (isomorphism classes) of graphs with d vertices and n labeled edges.
Note that the monodromies of equivalent generic covers differ by an inner automorphism of $\mathcal{S}_d$, so we will say that two monodromies $\varphi_1, \varphi_2 : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$ are *equivalent* if there exists $\sigma \in \mathcal{S}_d$ such that

$$\varphi_1(\gamma) = \sigma\varphi_2(\gamma)\sigma^{-1}$$

for all $\gamma \in \pi_1(\mathbb{C}^2 \setminus B)$.
The representation is done as follows: let $\varphi : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$ be a generic monodromy; if $\gamma_1, \dots, \gamma_n$ is a set of geometric loops that generates $\pi_1(\mathbb{C}^2 \setminus B) \cap \{y = 1\}$ (in particular they generate $\pi_1(\mathbb{C}^2 \setminus B)$, see, e.g., [O], [MP] for more a detailed description of this fundamental group), we write d vertices labeled $\{1, \dots, d\}$. Now $\mathcal{S}_d$ acts naturally on the set of our vertices, and then, $\forall i \in \{1, \dots, n\}$, we draw the edge labeled i between the two points exchanged by $\varphi(\gamma_i)$. Finally we have to delete the labeling of the vertices (this corresponds to consider φ up to the equivalence relation introduced above).
Remark that the monodromy graph does not carry all the informations needed to reconstruct the cover: Γ has n edges, but we lost m.
For a fixed $\Gamma \in Gr_{d,n}$, we say that m is *compatible* with Γ if Γ defines a

normal generic cover branched over $x^n = y^m$.
Then, a pair $(\Gamma \in Gr_{d,n}, m)$ with m compatible with Γ, determines the cover.
Finally, one notes that this construction is not symmetric in the two variables x, y. So, simply exchanging m and n, one gets a natural involution that sends compatible pairs $(\Gamma \in Gr_{d,n}, m)$ in compatible pairs $(\Gamma' \in Gr_{d,m}, n)$: we call this operation "duality".
We need a last definition:
Definition 2.7 A polygon with d vertices, valence a and increment j, with j and d relatively prime, is a graph with $n{=}ad$, d vertices, such that $\forall s, t$ the edges labeled s and t have

two vertices in common if and only if $s{-}t{=}\lambda d$
one vertex in common if and only if $s{-}t{=}\lambda d{+}j$ or $s{-}t{=}\lambda d{-}j$
no vertices in common otherwise.

This complicated definition is in fact probably better explained by the following example.

A polygon with 5 vertices, valence 3 and increment 2.

Now we are able to introduce the main result of [MP].

Theorem 2.8. *The monodromy graphs for generic covers $\pi : S \to \mathbb{C}^2$ of degree $d \geq 3$ branched over the curve $\{x^n = y^m\}$, with $(n, m){=}1$, are the following:*

1. *"Polygons" with d vertices, valence $\frac{n}{d}$ (or $\frac{m}{d}$) and increment j, with $(j, d){=}1$, $j < \frac{d}{2}$, $j(d{-}j)|m$ (resp. $j(d{-}j)|n$) . Moreover, d must divide n (resp. m).*
2. *"Double stars" of type $(j, d{-}j)$ and valence $\frac{n}{j(d-j)}$ (or $\frac{m}{j(d-j)}$), with $(j, d){=}1$, $j < \frac{d}{2}$, $j(d{-}j)|n$ (resp. $j(d{-}j)|m$). Moreover, d must divide m (resp. n).*

Duality takes graphs of type 1 in graphs of type 2, and vice-versa.

We skip here the definition of the double stars (cf. [MP]), that we do not need.

Shortly, in theorem 2.8, we have shown that generic covers branched over an irreducible curve of type $\{x^n = y^m\}$ are classified by pairs (polygon in $Gr_{d,n}$, m multiple of $j(d-j)$), up to exchanging x and y.
Let us recall that the above pairs describe generic covers also when the hypothesis $(n,m) = 1$ fails, but in this case we have examples of covers that can't be described in this way (with monodromy graphs of different type).
In view of theorem 2.8, in order to prove the remaining part of theorem 2.5, we only need the following

Proposition 2.9. *The normal generic cover branched over $x^{an} = y^{bm}$ associated to the polygon with n edges, increment h and valence a is the cover $F_{h,n-h,a,b}$.*

Proof. We have to compute the monodromy graphs of the covers $F_{h,k,a,b}$.
Let us start by considering the case $a = b = 1$, i.e. the covers $F_{h,k}$.
$F_{h,k}$ is a normal generic cover branched over $B = \{x^n = y^m\}$ with $n = h+k$ and $m = hk$. Notice that the assumption $(h,k) = 1$ implies $(n,m) = 1$.
By theorem 2.8 the monodromy graph Γ is, up to exchanging x and y, a polygon. In fact we do not need to exchange x and y: otherwise we would have $d|m$, while $\deg F_{h,k} = n$, and $(n,m) = 1$.
So Γ has to be a polygon of valence 1 $(d = n)$, and some increment h'. Set $k' = n - h'$.
By [MP], corollary 4.2, the smoothness of $S_{h,k}$ forces $m = h'k'$ (the minimal compatible integer for Γ).
But now $h' + k' = h + k$ and $h'k' = hk$, then $\{h', k'\} = \{h, k\}$.
Summing up we proved that the monodromy graph of $F_{h,k}$ is a polygon with valence 1 and increment h (or k). Of course, the corresponding m is hk.
Now remark that, $\forall a, b$, $F_{h,k,a,b}$ can be obtained by fiber product from $F_{h,k}$ and the map $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $f_{a,b}(x,y) = (x^a, y^b)$. As shown in [MP], this fiber product acts on the "building data" of the cover multiplying the valence by a, and the compatible m by b. So the corresponding monodromy graph is a polygon with $d = h + k$ vertices, valence a, increment h (or k).
Conversely, the cover associated to a pair ("polygon with n edges, valence a, and increment h", m) is $F_{h,n-h,a,\frac{m}{h(n-h)}}$, as stated. ❑

This concludes the proof of theorem 2.5.
One immediately gets the following corollary, whose first statement completes corollary 4.2 in [MP].

Corollary 2.10. *The cover $F_{h,k,a,b}$ is smooth $\iff$ $a = b = 1$ or $h = b = 1$.*
The cover and the ramification divisor are both smooth $\iff$ $h = a = b = 1$.

Proof. The first statement comes by the equations 2.4, whence the second can be easily checked in local coordinates as in remark 2.3. ❑

In the following section we will use the following consequence:

Corollary 2.11. *Let n and m be coprime integers.*
There exists a non-singular normal generic cover $\pi : S \longrightarrow \mathbb{C}^2$ branched over $x^n = y^m$ for which the ramification divisor is non-singular if and only if $|m-n| = 1$, or $d = 2$, $n = 1$.
In the first case the cover is unique of degree $d = max(m, n)$ and its monodromy graph is the polygon with d edges, increment 1 and valence 1.
In the second case the cover is given by the projection on the x, y-plane of the surface $z^2 = x - y^m$.

Proof. For $d \geq 3$, by previous corollary, we have only the covers having as monodromy graph the polygon with d edges, increment 1 and valence 1. For $d = 2$, the remark that for every curve $\{f(x, y) = 0\}$ there is exactly one double cover given by projection on the x, y-plane of the surface $z^2 = f$, gives immediately the result. ❑

We conclude this section with a direct computation of the monodromy graph associated to $\pi = F_{h,k,a,b}$, although we don't need it in the rest of the paper: the uninterested reader can skip directly to the next section.
We refer to [MP] for the notations used in what follows.
In order to see how the minimal standard generators act on the preimage of $(1 - \varepsilon, 1)$, we examine the inverse image of the path $(\lambda\beta, 1)$ for $0 \leq \lambda \leq 1$ in the (x, y)-plane, where $\beta^{h+k} = 1$.
Since $zw = 1$, we can substitute for w in the first equation 2.1 obtaining

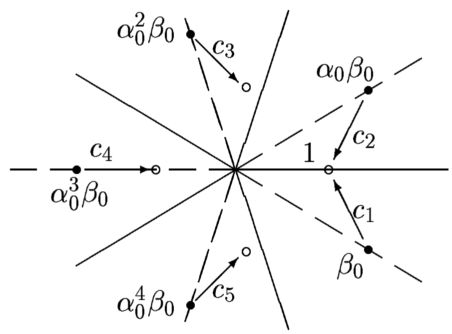
$$hz^{h+k} - (h + k)z^h \lambda\beta + k = 0 \tag{2.12}$$

We claim that if $\lambda \neq 0, 1$, then z^{h+k} is real if and only if $h+k$ is odd and $z = s\beta^{-1}$ with s negative.
Indeed, if z^{h+k} is real, then $z = s\beta^{-1}$ for some real s and s is a zero of the real polynomial function

$$f(s) = hs^{h+k} - (h + k)\lambda s^h + k.$$

Since $f'(s) = h(h + k)s^{h-1}(s^k - \lambda)$, f will have $s=0$ as critical point if $h>1$ and one other critical point $s=\sqrt[k]{\lambda}$ if k is odd, or two other critical points $s = \pm\sqrt[k]{\lambda}$ if k is even.
If $s^k=\lambda$, then $f(s) = k(1-\lambda s^h)$ which is strictly positive because either $s<0$ or $0<\lambda, s<1$. Thus f has only strictly positive critical values, hence, it has at most one zero s_0 and if f does have a zero, then $h + k$ is odd and $s_0<0$.
If $\lambda=1$, the same argument shows that z^{h+k} is real if and only if $s=1$ or $h+k$ is odd and $s<0$.
Note that, since $(h, k) = (h, h+k) = (k, h+k) = 1$, the equations $z^k = \beta$, $z^k = \beta^{-1}$ for $\beta \neq 1$, $\beta^{h+k}=1$, have a unique common solution: namely $z=\beta^s$ where $sk \equiv -sh \equiv 1 \bmod h+k$.
Now, if z_0 is a root of 2.12 with $\beta=1$, then $z=\frac{z_0}{\beta^s}$ is a root of 2.12, so we may restrict to the case $\beta=1$.
Note that if $\lambda=0$ then $z^{h+k} = -\frac{k}{h}$, while if $\lambda=\beta=1$ 2.12 has $z=1$ as double root, a real negative root if $h+k$ is odd and no other real roots.

Set $\beta_0 = \sqrt[h+k]{h+k}e^{-i\frac{\pi}{h+k}}$ and $\alpha_0 = e^{i\frac{2\pi}{h+k}}$. Then, each component of $F^{-1}_{h,k}(\lambda,1)$ will start from one of the points $\alpha_0^r\beta_0$ (each component from a different point) for $r=0,\dots,h+k-1$. Call c_r the component of $F^{-1}_{h,k}(\lambda,1)$ which starts from $\alpha_0^{r-1}\beta_0$.
Then, c_1 is contained in the region $-\pi < (h+k)arg(z) < 0$, c_2 is contained in the region $0 < (h+k)arg(z) < \pi$ and they both have $z=1$ as ending point. Also, $c_{h+k+3-r} = \overline{c_r}$ for $3 \le r < \frac{h+k+3}{2}$ are complex conjugated paths and c_r must be contained in one of the two regions $(2r-4)\pi < (h+k)arg(z) < (2r-3)\pi$ or $(2r-3)\pi < (h+k)arg(z) < (2r-4)\pi$ (see picture). Note that if $h+k$ is odd $c_{[\frac{h+k}{2}]+2}$ is contained in the negative real half-line.



Configuration of the paths $F^{-1}_{h,k}(\lambda,1)$ in case $h+k=5$.

Number the points $z_1,\dots,z_{h+k}$in $F^{-1}_{h,k}(1-\varepsilon,1)$ by the path c_r they belong to. It is clear that z_1 and z_2 are near $z=1$ and that the action of γ_1 exchanges z_1 and z_2.
Now, to see which is the action of γ_{h+1}, follow the motion of the points over the path $((1-\varepsilon)(1-t),1)$ and the path $(t(1-\varepsilon)\alpha_0^h,1)$ for $0\le t\le 1$. Recall that the paths over $(t(1-\varepsilon)\alpha_0^h,1)$ are obtained from the paths c_r by multiplying by $\alpha_0^{-sh}=\alpha_0$, thus, the action of γ_{1+h} exchanges z_2 and z_3.
By the same argument the action of γ_{1+rh} will exchange z_{r+1} and z_{r+2}, where indices are taken to be cyclical (mod$h+k$), i.e. the monodromy graph associated to $F_{h,k}$ is the polygon with $h+k$ edges increment h and valence 1 (see 2.7).

3. Chisini's Conjecture

In this section we will obtain similar results as those in [Ku] and [Ne] for curves with singularities of type $x^n = y^m$.
Let $B \subset \mathbb{P}^2$ be an irreducible curve with only singularities of type $\{x^n = y^m\}$. In the whole section, for every such a curve, we write

$$Sing(B) = \{p_1,\dots,p_r\}$$

where locally, near p_i, $\forall\, i=1,\dots,r$, B is equivalent to $x^{s_in_i} = y^{s_im_i}$ with $(n_i,m_i)=1$, and we set $n_i<m_i$ (unless $n_i = m_i = 1$).

Proposition 3.1. *Suppose B is the branch curve of a smooth projective generic cover (cf. 1.3) $\pi : S \longrightarrow \mathbb{P}^2$, and let R be the ramification locus of π.*
Then, restricted to the preimage of a small neighborhood of p_i, π is given by deg $\pi - n_i s_i$ connected components, $U_1, \dots, U_{s_i}$, $V_1, \dots, V_{deg\pi-(n_i+1)s_i}$ such that, restricted to one of the U_j, π gives a generic cover of degree n_i+1 branched over one of the s_i local irreducible components of B (different components for different j), while restricted to each V_k, π is an isomorphism. Moreover, if $n_i \geq 2$, then $m_i = n_i + 1$, and (locally) π restricted to U_j is equivalent to the cover $F_{1,n_i,1,1}$ for each $j = 1, \dots, s_i$.

Proof. Since we assumed R non-singular, it is locally irreducible: then for each $p \in R$ there exists a neighborhood $U \ni p$ such that $\pi(R \cap U)$ is irreducible and hence $\pi_{|_U}$ is a smooth normal generic cover branched over an irreducible curve. Since the image of an irreducible curve is still an irreducible curve, the cover splits locally as disjoint union of covers each branched over one of the (local) irreducible components of B.
In order to prove the first part of the statement, we still have to compute the degrees of the cover restricted to the "relevant" components, that come directly by corollary 2.11.
In case $n_1 \geq 2$, by the assumption of smoothness of the surface and of the ramification divisor R, corollary 2.11 forces $m_i = n_i + 1$ =the (local) degree of the cover: the local equation for these covers comes from proposition 2.9. ∎

Remark 3.2. *By the degrees computed in the previous proposition we have that deg $\pi \geq \max\{s_i(n_i + 1)\}$.*

We introduce some notations: let $\pi : S \to \mathbb{P}^2$ be a smooth projective generic cover, B the branch curve, B^* the dual curve to B, R the ramification locus, $C := \pi^*(B) - 2R$.
We set $E := \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ (so that $K_S = -3E + R$), $N := \deg\pi$, $d := \frac{\deg B}{2}$, $\delta := \deg B^*$, $g := g(B) = g(B^*) = g(R)$. With a standard abuse of notation, we will not distinguish a divisor from the associated line bundle.
In order to prove the main theorem of this section, we follow now the arguments of Kulikov in our more general case. Although some of the proof of Kulikov works without correction, we decided, for the convenience of the reader, to repeat also those proof, with the exception of proposition 3.8.
We start with some numerical relations.

Lemma 3.3.

(1) $d \in \mathbb{N}$;
(2) $E^2 = N$;
(3) $(E, R) = 2d$;
(4) $\delta = 4d + 2g - 2 - \sum_{i=1}^{r} s_i(n_i - 1)$.

Proof. By Hurwitz formula we have:

$$2 - 2g(E) = 2N - \deg B$$

thus $\deg B$ is even, and (1) is proved; (2) and (3) are trivial.
Using a generic projection onto a line

$$e(B) = 2de(\mathbb{P}^1) - \delta - \sum_{i=1}^{r}(s_i n_i - 1)$$

Thus

$$2 - 2g = e(R) = e(B) + \sum_{i=1}^{r}(s_i - 1) = 4d - \delta - \sum_{i=1}^{r} s_i(n_i - 1)$$

since R is the normalization of B and is obtained by separating locally the irreducible components of B, and this completes the proof. ❏

Lemma 3.4.

$$R^2 = 3d + g - 1$$

Proof. By genus formula

$$2g - 2 = (K_S + R, R) = (-3E + 2R, R) = -6d + 2R^2.$$

❏

Since $\delta \geq 0$, by lemma 3.4 and lemma 3.3, part (4), we have

Corollary 3.5.

$$\sum_{i=1}^{r} s_i(n_i - 1) \leq 2g - 2 + 4d < 2R^2 = 2(3d + g - 1)$$

❏

By Hodge's Index Theorem (E^2 is positive by definition) we have that

$$\begin{vmatrix} E^2 & (E,R) \\ (E,R) & R^2 \end{vmatrix} = N(3d + g - 1) - 4d^2 \leq 0$$

which gives the following

Corollary 3.6.

$$N \leq \frac{4d^2}{3d + g - 1}$$

❏

We can compute the invariants of S:

Lemma 3.7.

$$K_S^2 = 9N - 9d + g - 1$$

$$e(S) = 3N + \delta - 4d = 3N + 2g - 2 - \sum_{i=1}^{r} s_i(n_i - 1)$$

$$\chi(\mathcal{O}_S) = N + \frac{3g - 3 - 9d - \sum_{i=1}^{r} s_i(n_i - 1)}{12}$$

Proof. Since $K_S = -3E + R$ we have $K_S^2 = 9N - 12d + R^2$.
Using a generic pencil of lines in $\mathbb{P}^2$ and its preimage in S we get

$$e(S) = 2e(E) - N + \delta$$

and $e(E) = -(K_S + E, E) = 2N - 2d$.
From Noether's formula $12\chi(\mathcal{O}_S) = K_S^2 + e(S) = 12N + 3g - 3 - 9d - \sum_{i=1}^r s_i(n_i - 1)$ and we have done. ❏

Note that $\sum_{i=1}^r s_i(n_i - 1)$ must be divisible by 3.
Assume that there exist two non-equivalent smooth projective generic covers (S_1, π_1) and (S_2, π_2) with the same branch curve B.
Write N_i =deg π_i and $\pi_i^*(B) = 2R_i + C_i$, for i=1, 2.
Let X be the normalization of the fiber product $S_1 \times_{\mathbb{P}^2} S_2$. Denote by $g_i : X \to S_i$, $\pi_{1,2} : X \to \mathbb{P}^2$ the corresponding natural morphisms, as summarized in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{g_1} & S_1 \\ g_2 \downarrow & \searrow^{\pi_{1,2}} & \downarrow \pi_1 \\ S_2 & \xrightarrow{\pi_2} & \mathbb{P}^2 \end{array}$$

We have deg$g_1 = N_2$ and deg$g_2 = N_1$, so that deg$\pi_{1,2} = N_1N_2$.
The following result is proved in [Ku], proposition 2, section 2. Although Kulikov assumes, at the very beginning, that B has only nodes and cusps as singularities, this proof does not require this hypothesis.

Proposition 3.8. *If (S_1, π_1) and (S_2, π_2) are not equivalent, then X is irreducible.*

Let Y be the set of the points $p \in X$ such that $\pi_{1,2}(p) \subset$ Sing$B$, and π_1 and π_2 restricted to neighborhoods of $g_1(p)$ and $g_2(p)$ respectively, are normal generic covers with different branch loci.

Lemma 3.9. *Sing$X \subset Y$*

Proof. If $g_1(p) \notin R_1$ or $g_2(p) \notin R_2$, then p is clearly smooth.
At a point p such that $p_1 = g_1(p) \in R_1$ and $p_2 = g_2(p) \in R_2$ we can choose small neighborhoods $V_i(p_i) \subset S_i$ and $U(\pi_{1,2}(p)) \subset \mathbb{P}^2$ such that $\pi_i(V_i) = U$ and both $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are equivalent (up to possibly different base changes) to one of the following:
a) (if $n_i = 1$, or $\pi_{1,2}(p)$ is a smooth point of B)
$f_{2,1} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by $(x, y) \longmapsto (x^2, y)$;
b) (if $n_i \geq 2$)
the projection on the (x, y)-plane of the surface in $\mathbb{C}^4$

$$\begin{cases} n_i w + z^{n_i} = (n_i + 1)x \\ zw = y \end{cases}$$

Suppose that the branch loci of $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are the same. We have that, in the first case, $V_1 \times_U V_2$ has equations in $\mathbb{C}^4$

$$\begin{cases} x_1^2 = x_2^2 \\ y_1 = y_2 \end{cases}$$

and the normalization of $V_1 \times_U V_2$ is the disjoint union of two smooth surfaces, namely $x_1 = x_2$, $y_1 = y_2$ and $x_1 = -x_2$, $y_1 = y_2$ in $\mathbb{C}^4$.
In the second case $\tilde{V} = V_1 \times_U V_2$ is the surface in $\mathbb{C}^6$

$$\begin{cases} n_i w_1 + z_1^{n_i} = n_i w_2 + z_2^{n_i} = (n_i + 1)x \\ z_1 w_1 = z_2 w_2 = y \end{cases}$$

which has two irreducible components, namely $\tilde{V}_+$

$$\begin{cases} w_1 = w_2 \\ z_1 = z_2 \\ n_i w_1 + z_1^{n_i} = (n_i + 1)x \\ z_1 w_1 = y \end{cases}$$

which is isomorphic to V_i via g_i, and $\tilde{V}_-$

$$\begin{cases} n_i w_1 = z_2(z_2^{n_i-1} + z_2^{n_i-2} z_1 + \cdots + z_2 z_1^{n_i-2} + z_1^{n_i-1}) \\ n_i w_1 + z_1^{n_i} = n_i w_2 + z_2^{n_i} = (n_i + 1)x \\ z_1 w_1 = z_2 w_2 = y \end{cases}$$

which is expressed by g_1 (resp. g_2) as a normal generic cover of degree $N_2 - 1$ (resp. $N_1 - 1$) branched over C_1 (resp.C_2).
$\tilde{V}_+$ and $\tilde{V}_-$ are both smooth and intersect in $g_1^{-1}(R_1) \cap g_2^{-1}(R_2)$. The normalization will be the disjoint union of these two smooth components. ❏

Now suppose $p \in Y$ and let V_1 (resp. V_2) be the neighborhood of $g_1(p)$ (resp. $g_2(p)$) as in the definition of Y: the branch loci of $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are different.

Proposition 3.10. *X has only R.D.P. as singularities.*
More precisely, for every point $P \in Y$, if $p_i = \pi_{1,2}(P)$, P is a point of X of type A_{m_i-1}, and these are all the singular points of X.

For instance, if $n_i = m_i = 1$ (the case of nodes), we get A_0, i.e. a smooth point.
Proof. By previous lemma, $\pi_{1,2}(P)$=p_i for some i.
If $n_i = 1$, we can assume the two branch loci to be $\{x = 0\}$ and $\{x + y^{m_i} = 0\}$ and we get

$$\begin{cases} x = z_1^2 \\ z_2^2 = x + y^{m_i} \end{cases}$$

i.e. $z_2^2 = z_1^2 + y^{m_i}$ that is clearly a singularity of type A_{m_i-1} (if $m_i = 1 \Rightarrow X$ is smooth at P).

Finally, if $n_i \neq 1$, then $m_i = n_i + 1$, and $V_1 \times_U V_2$ is the surface in $\mathbb{C}^4$

$$\begin{cases} z_1^{m_i} - m_i x z_1 = -(m_i - 1)y \\ z_2^{m_i} - m_i \alpha x z_2 = -(m_i - 1)y \end{cases}$$

with $\alpha^{s_i m_i}{=}1$ but $\alpha^{m_i}{\neq}1$, which is isomorphic to the surface in $\mathbb{C}^3$

$$z_1^{m_i} - m_i x z_1 = z_2^{m_i} - m_i \alpha x z_2$$

which has a double point at the origin.
The Hessian matrix ($m_i \geq 3$), at the origin is

$$\begin{pmatrix} 0 & -m_i & -m_i\alpha \\ -m_i & 0 & 0 \\ -m_i\alpha & 0 & 0 \end{pmatrix}$$

and has rank 2, hence X has in p a singularity of type A_k for some $k \geq 2$. In order to compute k, set $z := m_i(z_1 - \alpha z_2)$. Then, in the coordinate system (x, z, z_1), our equation can be written as $z_1^{m_i} = z(x + f(z, z_1))$ with $f(0,0) = 0$, and, setting $\overline{x} = x + f$, we find that near the origin the triple $(\overline{x}, z, z_1)$ is still a coordinate system in terms of which $V_1 \times_U V_2$ has equation $z_1^{m_i} = \overline{x}z$ that is the standard expression for the singularity $A_{m_i - 1}$. ❏

Note that, if P is singular for X, $g_1^{-1}(R_1) \cap g_2^{-1}(R_2) \cap (V_1 \times_U V_2) = P$. In general, if D_1 and D_2 are two divisors in a normal surface, we define (D_1, D_2) ("the greatest common divisor") as the greatest divisor contained in both. By the local equations for the ramification divisor given in remark 2.3 we notice that the "singular" points in proposition 3.10 are isolated points for $g_1^{-1}(R_1) \cap g_2^{-1}(R_2)$, then

Remark 3.11. *If $R = (g_1^{-1}(R_1), g_2^{-1}(R_2))$, R does not intersect Sing X and, by the local considerations in the proof of lemma 3.9, is smooth and $g_i|_R : R \to R_i$ is a (unramified) double cover.*

Let $F : \tilde{X} \to X$ be the resolution of singularities of X, and let $\tilde{g}_i = g_i \circ F$, $\tilde{\pi}_{1,2} = \pi_{1,2} \circ F$. We define $\tilde{R} := F^*(R)$, $\tilde{C}_1 := F^*((g_1^{-1}(R_1), g_2^{-1}(C_2)))$, $\tilde{C}_2 := F^*((g_1^{-1}(C_1), g_2^{-1}(R_2)))$.

Proposition 3.12. (1) $(\tilde{R}, \tilde{C}_j) = \sum_{i=1}^r (n_i - 1)$
(2) $\tilde{R}^2 = 2(3d + g - 1) - \sum_{i=1}^r (n_i - 1)$
(3) $\tilde{C}_1^2 = (N_2 - 2)(3d + g - 1) - \sum_{i=1}^r (n_i - 1)$
(4) $\tilde{C}_2^2 = (N_1 - 2)(3d + g - 1) - \sum_{i=1}^r (n_i - 1)$

Proof. By remark 3.11 R does not intersect the singular points of X, then we can compute the intersections of $\tilde{C}_1$ and $\tilde{R}$ in X. By definition, $\tilde{C}_1$ and $\tilde{R}$ intersect only at points of the preimage of $R_2 \cap C_2$, in particular over some singular point of B.
But we already noticed that the only points $p \in R$, s.t. $\pi_{1,2}(p) \in$ Sing B are the points such that π_1 near $g_1(p)$ and π_2 near $g_2(p)$ are branched over the same curve, considered in the proof of lemma 3.9.
Let $p \in X$ be such a point. $\pi_{1,2}(p)$ is a singular point of B, so that there

exists i such that $\pi_{1,2}(p) = p_i$.
In case $n_i = 1$, π_1 and π_2 are locally double covers, so C_j does not contain p, and p does not give contribution to the intersection number.
Otherwise, let V_1 (resp. V_2) be a small neighborhood of $g_1(p)$ (resp. $g_2(p)$) as in lemma 3.9. Then, since $g_1|_R$ (and also $g_2|_R$) is an unramified double cover, there are exactly two points over p_i contained both in R and in the normalization of the fiber product of V_1 and V_2, say P_{i+} and P_{i-}.
The two points belong to the two components $\tilde{V}_+$ and $\tilde{V}_-$ respectively (see proof of lemma 3.9), but since C_j does not intersect $\tilde{V}_+$, we may suppose $p = P_{i-} \in \tilde{V}_-$.
If we rewrite the equations for $\tilde{V}_-$ we get

$$\begin{aligned} w_1 &= \frac{z_2(z_1^{n_i} - z_2^{n_i})}{n_i(z_1 - z_2)} \\ w_2 &= w_1 + \frac{z_1^{n_i} - z_2^{n_i}}{n_i} \\ x &= \frac{n_i w_1 + z_1^{n_i}}{n_i + 1} \\ y &= z_1 w_1. \end{aligned}$$

Remark that all the members of these equations are polynomials, and we can take z_1, z_2 as holomorphic coordinates for $\tilde{V}_-$.
Now, $R \cap \tilde{V}_-$ is cut by $w_1 = z_1^{n_i}$, $w_2 = z_2^{n_i}$ i.e.

$$\begin{cases} z_2 \frac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = n_i z_1^{n_i} \\ z_1^{n_i} - z_2^{n_i} + z_2 \frac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = n_i z_2^{n_i}. \end{cases}$$

This implies $z_1^{n_i} = z_2^{n_i}$, i.e. $z_1 = \lambda z_2$, with $\lambda^{n_i} = 1$. But if $\lambda \neq 1$, then the left members in our equations vanish, and we get $z_1 = z_2 = 0$ (which is not a curve), whence $z_1 = z_2$ clearly solve our equations, so it is the local equation we were looking for.
A branch of $F(\tilde{C}_2) \cap \tilde{V}_-$ is given (cf. remark 2.3) by the equations $\alpha w_1 = z_1^{n_i}$, $w_2 = z_2^{n_i}$, where $\alpha = \left(\frac{n_i+\alpha}{n_i+1}\right)^{n_i+1}$, $\alpha \neq 1$, i.e.

$$\begin{cases} z_2 \frac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = \frac{n_i}{\alpha} z_1^{n_i} \\ z_1^{n_i} - z_2^{n_i} + z_2 \frac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = n_i z_2^{n_i} \end{cases}$$

from which $z_1^{n_i} = \frac{\alpha(n_i+1)}{n_i+\alpha} z_2^{n_i} = \left(\frac{n_i+\alpha}{n_i+1}\right)^{n_i} z_2^{n_i}$ so that $z_1 = \lambda \frac{n_i+\alpha}{n_i+1} z_2$ with $\lambda^{n_i} = 1$.
Moreover, setting $t = \frac{n_i+\alpha}{n_i+1}$ (so that $t^{n_i+1} = (n_i+1)t - n_i$), λ must satisfy (by the first equation)

$$(\lambda t)^{n_i - 1} + (\lambda t)^{n_i - 2} + \cdots + 1 = \frac{n_i}{t}$$

Hence
$$t^{n_i} - 1 = (\lambda t)^{n_i} - 1 = (\lambda t - 1)\frac{n_i}{t}$$
or
$$t^{n_i+1} = (\lambda n_i + 1)t - n_i$$
i.e.
$$(n_i + 1)t = (\lambda n_i + 1)t.$$
Thus $\lambda = 1$ and $F(\tilde{C}_2) \cap \tilde{V}_-$ is the union of the $n_i - 1$ curves $z_1 = \frac{n_i+\alpha}{n_i+1} z_2$. Then every component of $\tilde{C}_j$ intersects $\tilde{R}$ transversally, and we conclude $(\tilde{R}, \tilde{C}_j) = \sum_{i=1}^{r}(n_i - 1)$.
Let $E_{\tilde{X}} = F^*\pi_{1,2}^*(\mathcal{O}_{\mathbb{P}^2}(1)) = F^* g_i^*(E_i)$. It is immediate to verify that
$$E_{\tilde{X}}^2 = N_1 N_2$$
$$(E_{\tilde{X}}, \tilde{R}) = 4d$$
$$(E_{\tilde{X}}, \tilde{C}_1) = 2d(N_1 - 2)$$
$$(E_{\tilde{X}}, \tilde{C}_2) = 2d(N_2 - 2).$$
Since the canonical divisor of $\tilde{X}$ is F^*K_X (X has only rational double points as singularities), we have
$$K_{\tilde{X}} = -3E_{\tilde{X}} + \tilde{R} + \tilde{C}_1 + \tilde{C}_2$$
and
$$(K_{\tilde{X}} + \tilde{R}, \tilde{R}) = e(\tilde{R}) = 4g - 4$$
then
$$\tilde{R}^2 = 6d + 2g - 2 - \sum_{i=1}^{r}(n_i - 1).$$
Since $F^* g_1^*(R_1) = \tilde{R} + \tilde{C}_1$ we have
$$N_2 R_1^2 = (\tilde{R} + \tilde{C}_1, \tilde{R} + \tilde{C}_1)$$
from which
$$\tilde{C}_1^2 = N_2(3d+g-1) - 6d - 2g + 2 - \sum_{i=1}^{r}(n_i - 1) = (N_2 - 2)(3d + g - 1) - \sum_{i=1}^{r}(n_i - 1).$$
∎

Finally we can prove

Theorem 3.13. *Let B be the branch locus of a smooth projective generic cover $\pi : S \to \mathbb{P}^2$ having r singular points of type $x^{n_i s_i} = y^{m_i s_i}$ with $n_i \leq m_i$, $(n_i, m_i) = 1$. Then, if*
$$\deg \pi > \frac{4(3d + g - 1)}{2(3d + g - 1) - \sum_{i=1}^{r}(n_i - 1)}$$
where $2d = \deg B$ and $g = g(B)$ is its genus, then π is unique.

In the introduction we wrote the statement in a different notation, that we found better there.

Proof. Since by corollary 3.5 $\tilde{R}^2 > 0$, by Hodge Index Theorem

$$\begin{vmatrix} \tilde{R}^2 & (\tilde{C}_1, \tilde{R}) \\ (\tilde{C}_1, \tilde{R}) & \tilde{C}_1^2 \end{vmatrix} = 2(N_2-2)(3d+g-1)^2 - N_2(3d+g-1)\sum_{i=1}^{r}(n_i-1) \leq 0$$

and the same equation is true replacing $\tilde{C}_1$ by $\tilde{C}_2$ and N_2 by N_1.
So, we get

$$N_j \leq \frac{4(3d+g-1)}{2(3d+g-1)-\sum_{i=1}^{r}(n_i-1)}$$

for $j = 1, 2$. ❏

Following an idea of S. Nemirovski (see [Ne]) we may prove the following

Theorem 3.14. *In the above hypothesis, if deg* $\pi \geq 12$ *then* π *is unique.*

Proof. If S is not an irrational ruled surface of genus $g \geq 2$, it satisfies the Bogomolov-Miyaoka-Yau inequality

$$K_S^2 \leq 3e(S).$$

From lemma 3.7

$$K_S^2 = 9N - 9d + g - 1$$

$$e(S) = 3N + 2g - 2 - \sum_{i=1}^{r}(n_i - 1)$$

so,

$$\sum_{i=1}^{r}(n_i - 1) \leq 3d + \frac{5}{3}(g-1).$$

With this inequality, we can estimate the quantity

$$\frac{4(3d+g-1)}{2(3d+g-1)-\sum_{i=1}^{r}(n_i-1)} \leq \frac{12d+4(g-1)}{3d+\frac{1}{3}(g-1)} = 4 + \frac{8(g-1)}{9d+g-1} < 12$$

If S is an irrational ruled surface, it satisfies

$$K_S^2 \leq 2e(S)$$

then, the same argument shows that

$$\sum_{i=1}^{r}(n_i-1) \leq \frac{-3N+9d+3(g-1)}{2} < \frac{3}{2}(3d+g-1)$$

thus we get the stronger estimate

$$\frac{4(3d+g-1)}{2(3d+g-1)-\sum_{i=1}^{r}(n_i-1)} < 8$$

❏

As a last remark, note that one can rewrite, with the obvious changes, all the results in [Ku], theorems 3-12.

4. A family and a counterexample

In this section we will describe an interesting family of projective generic covers branched over a curve $\bar{B}$ with singularities of type $x^n = y^m$ that will produce a counterexample to Chisini's conjecture if we drop the hypothesis that the ramification divisor is smooth.
Let $\bar{B} \subset \mathbb{P}^2$ be a plane curve of equation $\bar{g}(x,w) = \bar{f}(y,w)$ where $\bar{g}$ and $\bar{f}$ are homogeneous polynomials of degree d, of the form

$$\bar{g}(x,w) = \prod_{i=1}^{r}(x - \alpha_i w)^{n_i}$$

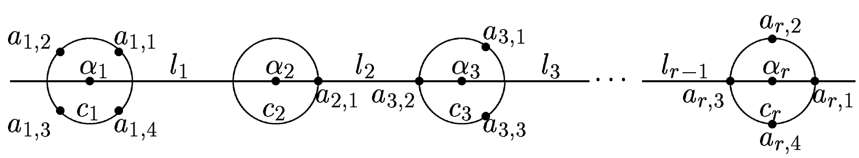
$$\bar{f}(y,w) = \prod_{j=1}^{s}(y - \beta_j w)^{m_j}$$

with $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ mutually distinct.
In a neighborhood $U_{i,j}$ of the point $P_{i,j} = (\alpha_i, \beta_j, 1)$, $\bar{B}$ is analytically equivalent to $x^{n_i} = y^{m_j}$.
Our (open) assumption is that the singular points of $\bar{B}$ are contained in the union of lines $\bar{g}(x,w) = 0$, or, if you prefer, in the set of the P_{ij}'s.
By a classical result (see [De, Fu]), if $\bar{B}$ is a nodal curve then $\pi_1(\mathbb{P}^2 \setminus \bar{B})$ is abelian; since S_d has no center if $d \geq 3$, then, if $\pi_1(\mathbb{P}^2 \setminus \bar{B})$ is abelian, there are no projective generic covers of degree $d \geq 3$ whose branch locus is $\bar{B}$; thus we will suppose that not all $n_i \leq 2$ and not all $m_j \leq 2$.
Note that $p = (0,1,0)$ does not belong to $\bar{B}$, thus, in order to compute $\pi_1(\mathbb{P}^2 \setminus \bar{B})$, we can use the projection from p onto the x-axis.
More precisely, $\bar{B}$ intersects transversally the line at infinity $w=0$ in the d smooth points $(1, \xi, 0)$ with $\xi^d = 1$; then the line at infinity is not tangent to $\bar{B}$. This allows us to compute the fundamental group of the complement of $\bar{B}$ by computing the fundamental group of the complement of the affine curve B in the chart $w \neq 0$, as we will do in proposition 4.6.
Set $g(x) = \bar{g}(x,1)$ and $f(y) = \bar{f}(y,1)$ so that $B = \{g(x) = f(y)\}$.
In order to compute the fundamental group of the complement of B we can do, without lost of generality (by a deformation argument as in [O]), the following assumptions:

1. $\forall i,j$, $\alpha_i, \beta_j \in \mathbb{R}$;
2. $\alpha_1 < \alpha_2 < \cdots < \alpha_r$, and $\beta_1 < \beta_2 < \cdots < \beta_s$;
3. If $\gamma_1, \dots, \gamma_{s-1}$ are the roots of f' such that $f(\gamma_i) \neq 0$, the critical values for f, $f_1 = f(\gamma_1), \dots, f_{s-1} = f(\gamma_{s-1})$ are mutually distinct;
4. For a suitable $\varepsilon_0 > 0$, $\forall x \in (\alpha_1 - \varepsilon_0, \alpha_r + \varepsilon_0)$, $|g(x)| < \min_i |f_i|$

Let us point out (in order to justify assumption 3) that the roots of f' are those β_j for which $m_j \geq 2$ (with multiplicity $m_j - 1$) and the roots of a polynomial of degree $s-1$ that has, by assumption 1, $s-1$ distinct real roots $\gamma_1, \dots, \gamma_{s-1}$ such that $\beta_i < \gamma_i < \beta_{i+1}$.
The critical points of the projection from p onto the x-axis are given by the intersection of B with the union of horizontal lines $\{f'(y) = 0\}$.

Then the critical values are (some of) the α_i (corresponding to points $P_{i,j}$) and the $d(s{-}1)$ distinct points $\delta_{j,h}$ for $h{=}1,\dots,d$ and $j{=}1,\dots,s{-}1$ where $g(\delta_{j,h}) = f_j$ (smooth points with vertical tangent). By assumption 4, no $\delta_{j,h}$ is contained in the interval $[\alpha_1, \alpha_r]$.
Choose $\varepsilon{>}0$ small enough such that, $\forall j$ (resp. $\forall i$), for every t s.t. $0 < |t-\beta_j| \le \varepsilon$ (resp. $0 < |t-\alpha_i| \le \varepsilon$), $f^{-1}(t)$ (resp. $g^{-1}(t)$) is given by m_j (resp. n_i) distinct points. We denote by $b_{j,1},\dots,b_{j,m_j}$, (resp. $a_{i,1},\dots,a_{i,n_i}$) the points in $f^{-1}(\varepsilon)$ (resp. $g^{-1}(\varepsilon)$) ordered by their argument.
We fix now a free basis for $\Pi = \pi_1(\{y{=}0\} \setminus \{\alpha_i, \delta_{j,k}\}, a_{1,1})$, in terms of which we will describe the braid monodromy of the projection.
Let $C_\varepsilon(z_0) \subset \mathbb{C}$ be the circle of center z_0 and radius ε; we define by c_i the closed path supported on the connected component of $g^{-1}(C_\varepsilon(0))$ near α_i, with starting point the unique real point bigger than α_i, with the counter-clockwise orientation; c_i^+ the "subpath" contained in the positive half plane (imaginary part bigger than 0), c_i^- the "subpath" in the negative half plane.
Let l_i (for $i{=}1,\dots,r{-}1$) be the (positively oriented) path contained in the real line connecting $c_i \cap \mathbb{R}$ and $c_{i+1} \cap \mathbb{R}$ but not containing any of the α_j's.

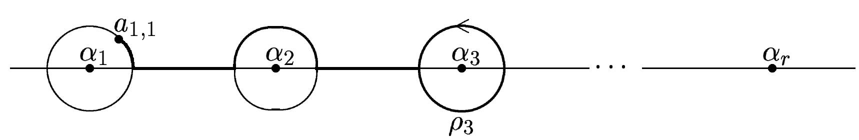


$n_1{=}n_r{=}4,\ n_2{=}1,\ n_3{=}3$

Let ω be the small path supported on c_1 connecting $a_{1,1}$ with the base point of c_1 (in the clockwise direction).
Consider the paths $\rho_1,\dots,\rho_r$ based at $a_{1,1}$ defined by

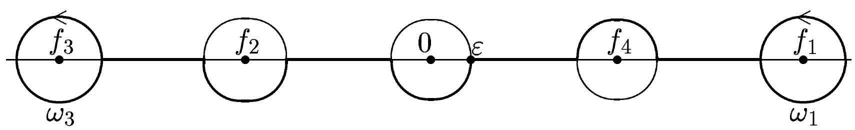
$$\rho_i = (\omega l_1 (c_2^+)^{-1} \cdots l_{i-2}(c_{i-1}^+)^{-1} l_{i-1}) c_i (\omega l_1 (c_2^+)^{-1} \cdots l_{i-2}(c_{i-1}^+)^{-1} l_{i-1})^{-1}$$

$(\rho_1 = \omega c_1 \omega^{-1})$.



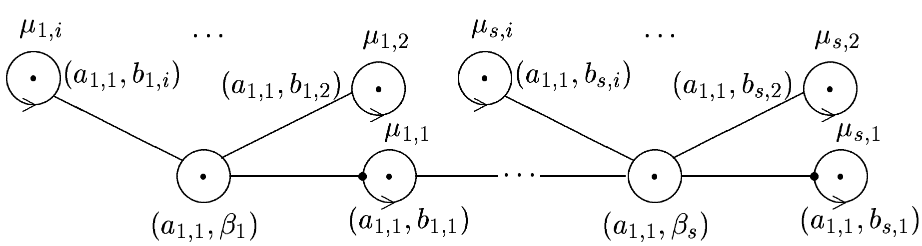
These are path around the α_i's. To complete the free basis of Π, we need some paths around the $\delta_{j,k}$'s.
Consider the (real) critical values for f, f_i, defined before.
Let ω_i be a loop around f_i based at ε contained in the union of the paths

$C_\varepsilon(f_i)$ and the real line constructed by the following algorithm: follow the real line in direction of f_i until you meet the first $C_\varepsilon(f_j)$; if $j \neq i$, follow $C_\varepsilon(f_j)$ clockwise until you meet again the real line, then follow the real line again till a new $C_\varepsilon(f_j)$ and repeat the algorithm; if $i = j$ follow counter-clockwise the whole $C_\varepsilon(f_i)$ and come back to ε from the way you arrived (and end the algorithm). Here you find two examples, were we defined C_ε^+ and C_ε^- in the natural way as we did for the c_i.



$$\omega_1 = TC_\varepsilon(f_1)T^{-1} \; T = [\varepsilon, f_4 - \varepsilon]C_\varepsilon^+(f_4)^{-1}[f_4 + \varepsilon, f_1 - \varepsilon]$$
$$\omega_3 = T'C_\varepsilon(f_3)T'^{-1} \; T' = C_\varepsilon^-(0)^{-1}[-\varepsilon, f_2 + \varepsilon]C_\varepsilon^-(f_2)^{-1}[f_2 - \varepsilon, f_3 + \varepsilon]$$

For every fixed pair i, h, we can uniquely lift ω_j to a (closed) path $\tilde{\Delta}_{j;i,h}$, based at $a_{i,h}$, s.t. $g(\tilde{\Delta}_{j;i,h}) = f(\omega_j)$, that is in fact a loop around some $\delta_{j,\bar{h}}$. Finally we define $\Delta_{j;i,h} \in \Pi$ the path based in $a_{1,1}$ obtained conjugating $\tilde{\Delta}_{j;i,h}$ by a path connecting $a_{1,1}$ and $a_{i,h}$, obtained following the orientation of each real interval and the reverse orientation of each circle.
The paths ρ_i's, $\Delta_{j;i,h}$'s give clearly a free basis for Π.
Now we can compute $\pi_1(\mathbb{C}^2 \setminus B)$ (and $\pi_1(\mathbb{P}^2 \setminus \bar{B})$).
We can take as generators of $\pi_1(\mathbb{C}^2 \setminus B)$ (and of $\pi_1(\mathbb{P}^2 \setminus \bar{B})$) a geometric basis $\mu_{j,k}$ (for j=1, ..., s, k=1, ..., m_j) of $\pi_1(\{x = a_{1,1}\} \setminus B) \cong F_d$ in such a way that $\mu_{j,1}, \dots, \mu_{j,m_j}$ are (conjugated to) the "standard generators" of $\pi_1(U_{1,j} \setminus B)$ (cf. [MP]) as in the following picture.



We recall now the following definition and theorem from [O].
Definition 4.1[O]

$$G_{m,n} := < g_1, \dots, g_m | \; g_k^{-1}(g_1 \cdots g_n)g_{k+n}(g_1 \cdots g_n)^{-1}, \; \forall k = 1, \dots, m >,$$

where the indices in the relators are taken to be cyclical mod m.

Theorem 4.2 ([O]).

$$\pi_1(\mathbb{C}^2 \setminus \{x^m = y^n\}) \cong G_{m,n}$$

.

Proposition 4.3. *If $B = \{f(x) = g(y)\}$ as above,*

$$\pi_1(\mathbb{C}^2 \setminus B) \cong G_{m,n}$$

where $n=(n_1, \dots, n_r)$, $m=(m_1, \dots, m_s)$ (the greatest common divisors).

Proof. Every path in Π induces a braid (acting on $p^{-1}(a_{1,1})$) that is its *braid monodromy*: we compute the relations in $\pi_1(\mathbb{C}^2 \setminus B)$ by the braid monodromy of the generators of Π, following the method introduced in [Mo]. In order to express the braid monodromy of a path we use the standard generators of the braid group on d strands given by the positive half-twists σ_i, $1 \leq i \leq d-1$ exchanging the ith and the $(i+1)$th strands counterclockwise (the reader unexperienced with the braid group can find precise definitions and more, e.g., in [Bi]).
In order to compute the braid monodromy of the generators we chose for Π, we think the points $a_{i,h}$ lying on a line following the lexicographical order in their indices, i.e. $a_{i,h}>a_{i',h'} \iff i>i'$ or, if $i=i'$, $h>h'$.
The braid monodromy of ρ_1 is

$$\tilde{\sigma}_1^{n_1} \cdots \tilde{\sigma}_s^{n_1}$$

where $\tilde{\sigma}_j = \sigma_{m_0+\dots+m_j-1} \cdots \sigma_{m_0+\dots+m_{j-1}+1}$, $(m_0=0)$, and gives us the relations

$$\mu_{j,k} = T_{j;1,n_1} \mu_{j,k+n_1} T_{j;1,n_1}^{-1}$$

for all j, k, where $T_{j;1,l} = \mu_{j,1}\mu_{j,2} \cdots \mu_{j,l}$ and the second index of the $\mu_{j,k}$'s is taken to be cyclical $(\mathrm{mod} m_j)$.
Since by condition 4, lifting the path l_i gives the identity braid for all i, then the braid monodromy of ρ_i is similar, i.e.

$$\tilde{\sigma}_1^{n_i} \cdots \tilde{\sigma}_s^{n_i}$$

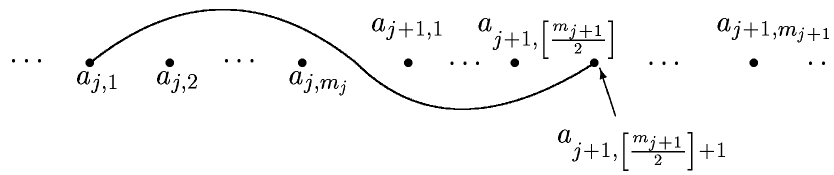
inducing in $\pi_1(\mathbb{C}^2 \setminus B)$ (and $\pi_1(\mathbb{P}^2 \setminus \bar{B})$), the relations

$$\mu_{j,k} = T_{j;1,n_i} \mu_{j,k+n_i} T_{j;1,n_i}^{-1}$$

for all i, j, k. It is easy to see (cf. proposition 1.1 in [MP]) that these relations are equivalent to

$$\mu_{j,k} = T_{j;1,n} \mu_{j,k+n} T_{j;1,n}^{-1}$$

for all j, k, where $n = (n_1, \dots, n_r)$.
The monodromy of $\Delta_{j;1,1}$ is retrieved from the braid $f^{-1}(\omega_j)$ and gives the positive half twist



i.e.

$$T^{-1}\sigma_{m_0+\cdots+m_j}T \tag{4.4}$$

$$T = (\sigma_{m_0+\cdots+m_j+1}\cdots\sigma_{m_0+\cdots+m_{\left[\frac{j+1}{2}\right]}})(\tilde{\sigma}_j)$$

and gives the relation

$$\mu_{j,1} = (\mu_{j+1,1}\cdots\mu_{j+1,\left[\frac{m_{j+1}}{2}\right]})\mu_{j+1,\left[\frac{m_{j+1}}{2}\right]+1}(\mu_{j+1,1}\cdots\mu_{j+1,\left[\frac{m_{j+1}}{2}\right]})^{-1}.$$

This relation is best understood in terms of the "minimal standard generators" (cf. [MP])

$$\gamma_{j,k} = (\mu_{j,1}\cdots\mu_{j,k-1})\mu_{j,k}^{-1}(\mu_{j,1}\cdots\mu_{j,k-1})^{-1}.$$

The relation above becomes the simpler

$$\gamma_{j,1} = \gamma_{j+1,\left[\frac{m_{j+1}}{2}\right]+1}.$$

The braid monodromies of the other $\Delta_{j;i,h}$ are a conjugate of 4.4 by a multiple of $\tilde{\sigma}_j\tilde{\sigma}_{j+1}$ and give the relations

$$\gamma_{j,k} = \gamma_{j+1,\left[\frac{m_{j+1}}{2}\right]+k}$$

for all j, k.
These are cancellation relations, since we can express each $\mu_{j,k}$ in terms of the $\mu_{1,k}$'s, and moreover give us the relations

$$\mu_{1,k} = \mu_{1,k+m_j}$$

for all j, k.
So, if $m := (m_1, \ldots, m_s)$, the paths $\mu_1{=}\mu_{1,1}, \cdots\ \mu_m{=}\mu_{1,m}$, generate $\pi_1(\mathbb{C}^2 \setminus B)$, and between them we have only the relations

$$\mu_k = T_{1,n}\mu_{k+n}T_{1,n}^{-1}$$

where $T_{1,n} = \mu_1\cdots\mu_n$ with cyclical indices mod m. ❑

Remark 4.5. *By theorem 4.2, $\pi_1(U_{i,j} \setminus B) \cong G_{n_i,m_j}$ and $\mu_1, \dots, \mu_{m_j}$ are (conjugated to) the standard generators for this group: in particular, the map $\pi_1(U_{i,j} \setminus B) \to \pi_1(\mathbb{C}^2 \setminus B)$ induced by the inclusion coincides with the map $(f_{\frac{n_i}{n}, \frac{m_j}{m}})_*$ we introduced immediately after remark 2.3.*
This implies that if B is the branch curve of a normal generic cover with monodromy $\mu : G_{m,n} \to \mathcal{S}_d$, the graph representing the local monodromy at $P_{i,j}$ is the pullback by $(f_{\frac{n_i}{n}, \frac{m_j}{m}})$ of the graph representing the global monodromy μ.

Proposition 4.6.

$$\pi_1(\mathbb{P}^2 \setminus \bar{B}) \cong G_{m,n} / < (\mu_1 \cdots \mu_m)^{\frac{d}{m}} >$$

where $n=(n_1, \dots, n_r)$, $m=(m_1, \dots, m_s)$. ❏

Proof. To compute $\pi_1(\mathbb{P}^2 \setminus B)$ we use the standard remark that the kernel of the surjective map

$$\pi_1(\mathbb{C}^2 \setminus B) \to \pi_1(\mathbb{P}^2 \setminus \bar{B}) \to 0$$

is infinite cyclic and is generated by a loop L around the line at infinity.
In our case this loop is

$$L = (\mu_{1,1} \cdots \mu_{1,m_1})(\mu_{2,1} \cdots \mu_{2,m_2}) \cdots (\mu_{s,1} \cdots \mu_{s,m_s})$$

or in terms of the generators μ_i

$$L = (\mu_1 \cdots \mu_m)^{\frac{m_1}{m}} \cdots (\mu_1 \cdots \mu_m)^{\frac{m_s}{m}} = (\mu_1 \cdots \mu_m)^{\frac{d}{m}}$$

❏

Assume now $\bar{B}$ irreducible, i.e. $(n, m)=1$.
In this case the monodromy of the cover lifts to a generic (geometric loops map to transpositions) homomorphism $\mu : G_{n,m} \to \mathcal{S}_d$ for which $\mu(T_{1,n}^{\frac{d}{n}}) = 1$.
By the classification of generic homomorphisms in theorem 2.8, the monodromy graph is (exchanging n and m if necessary) a polygon.
We know that in this case there exist h, k, a, b s.t. $n=a(h+k)$ and $m=bkh$ with $(h, k)=1$. Now we can introduce our family: we define

$$\bar{g}_l(x, w) = (x - w)(x - 2w) \cdots (x - lw)$$

$$\bar{f}_l(y, w) = (y - w)(y - 2w) \cdots (y - lw)$$

and, given h, k coprime, we consider the generic cover of degree $h + k$ branched over

$$\bar{g}_{h+k}(x, w)^{hk} = \bar{f}_{hk}(y, w)^{h+k}$$

with monodromy graph a polygon with m edges, valence 1 and increment h.
Here all the singularities have the same form $x^{hk} = y^{h+k}$, so, by remark 4.5, all the local monodromy graphs have to coincide with the global one.
In order to ensure the existence of the cover we have only to check that the monodromy of $(\mu_1 \cdots \mu_{h+k})^{hk}$ is trivial, which was clear since the very

beginning because it belongs to the center of the (local) fundamental group (in fact, the order of the monodromy of $\mu_1 \cdots \mu_{h+k}$ is exactly hk).
Moreover, by corollary 2.10, having all the singular points $a = b = 1$ the surface we defined is smooth, whence the cover is smooth if and only if $h = 1$; in this last case one can easily check that the cover is given by the projection on the plane $z = 0$ from the point $(0, 0, 1, 0)$ of the surface

$$z^{k+1} - (k+1)z\bar{f}_k(x, w) + k\bar{g}_{k+1}(y, w) = 0$$

Finally we can state the counterexample we were looking for:

Proposition 4.7. *Let B be the projective plane curve of degree 30 given by the equation*

$$\bar{g}_5(x, w)^6 = \bar{f}_6(y, w)^5,$$

Then there are two generic covers $S' \xrightarrow{\pi'} \mathbb{P}^2$, $S'' \xrightarrow{\pi''} \mathbb{P}^2$, with

1. *S', S'' smooth;*
2. *deg $\pi' = 6$;*
3. *deg $\pi'' = 5$;*
4. *the ramification divisor of π' is smooth;*
5. *the ramification divisor of π'' has (exactly) 30 ordinary cusps as singularities.*

Proof. The covers π', π'' are the covers of the family we just constructed for, respectively, $h = 1, k = 5$ and $h = 2, k = 3$. We check quickly the 5 properties:
(1) holds for every surface in our family;
(2) and (3) follows because the degree is the number of vertices of the graph, i.e. $h + k$.
Finally, (4) and (5) come directly from remark 2.3. ❏

This is a counterexample to the Chisini's conjecture if we drop the assumption that the ramification divisor is non-singular.
This family does not produce counterexamples in higher degrees: in fact the pair $(5, 6)$ is the only one which can be expressed as sum and product of two coprime integers in two different ways.
Indeed, suppose we have $h+k=h'k'$ and $hk=h'+k'$ with $(h, k) = (h', k') = 1$ and, say, $h<k$, $h'<k'$, $h+k<hk$.
From $h'+k'>h'k'$ we have that $h'=1$ and $k'=hk-1$.
But now $k(h-1)=h+1=h-1+2$ and it must be $(h-1)|2$, so that $h=2$ and $k=3$ which gives $k'=5$.
In order to find counterexamples to a Chisini-Kulikov-Nemirovski's type result in arbitrarily large degrees, we have to consider a slightly different family:

Proposition 4.8. *Let $t \in \mathbb{N}$, $t \geq 2$ B be the projective plane curve given by the equation*

$$\bar{g}_{4t+1}(x, w)^{2t(2t+1)} = \bar{f}_{2t(2t+1)}(y, w)^{4t+1}.$$

Then there are two generic covers $S' \xrightarrow{\pi'} \mathbb{P}^2$, $S'' \xrightarrow{\pi''} \mathbb{P}^2$, with S', S'' smooth, degrees respectively $4t+1$ and $4t+2$, and both singular ramification divisor.

In fact, the case $t = 1$ is exactly the case of previous proposition, so the statement still holds except for the singularities of the ramification divisor.

Proof. The cover of degree $4t + 1$ is the cover in our family for $h = 2t$, $k = 2t + 1$.

The cover of degree $4t + 2$ is simply the cover constructed in the same way as we did for our family, starting from the monodromy graph given by the polygon with $4t + 2$ vertices, valence t and increment 1.

The smoothness comes from corollary 2.10 observing that locally we have $h = b = 1$.

The other verifications are exactly as in the previous case and we leave them to the reader. ❏

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