THREEFOLDS ON THE NOETHER LINE AND THEIR MODULI SPACES

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Abstract. In this paper, we completely classify the canonical threefolds on the Noether line with geometric genus $p_g \geq 11$ by studying their moduli spaces. For every such moduli space, we establish an explicit stratification, estimate the number of its irreducible components and prove a dimension formula. A new and unexpected phenomenon is that the number of irreducible components grows linearly with the geometric genus, while the moduli space of canonical surfaces on the Noether line with any prescribed geometric genus has at most two irreducible components.

The key idea in the proof is to relate the canonical threefolds on the Noether line to simple fibrations in $(1, 2)$-surfaces by proving a conjecture stated by two of the authors in [CP23].

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1. Introduction

1.1. **Background.** One of the most fundamental problems in algebraic geometry is to classify algebraic varieties, with probably the ultimate goal to understand the moduli space of varieties with prescribed discrete numerical invariants. As a typical example, the moduli spaces $\mathcal{M}_g$ of smooth curves of genus $g \geq 2$ have been extensively studied since the seminal work of Mumford. In the moduli theory for higher dimensional varieties of general type, the main objects are varieties with ample canonical class and with canonical singularities [Kol23, §1.2]. Geometric invariant theory (GIT) can be applied to construct a quasi-projective coarse moduli space of such varieties [Vie95] (see also [Gie77] for surfaces). An alternative construction using the minimal model program (MMP) was outlined for surfaces in [KSB88] (see also

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[Ale96]), and it gives a projective moduli space by adding stable varieties (see [Kol23] for details including the higher dimensional case). However, the geometry of these moduli spaces seems far from being understood, even without considering the locus parametrizing strictly stable varieties. The basic questions include, for example:

- the non-emptiness of the moduli space of varieties of general type with prescribed birational invariants;
- the dimension and the number of irreducible/connected components of the moduli space, if it is non-empty.

In this paper, we describe the explicit geometry of moduli spaces of a class of threefolds with ample canonical class, which are of special importance from the viewpoint of the geography of algebraic varieties. To motivate our result, in the following, we assume that X is a variety of general type of dimension $n \geq 2$ with at worst canonical singularities. If the canonical class K_X is ample, then X is called *canonical*. Let

$$p_g(X) := h^0(X, K_X)$$

denote the *geometric genus* of X, and let

$$\mathrm{Vol}(X) := \limsup_{m\to\infty} \frac{h^0(X, mK_X)}{m^n/n!}$$

denote its *canonical volume*. These two numerical invariants are fundamental in the study of the birational geometry of X. Note that if K_X is nef, then $\mathrm{Vol}(X) = K_X^n$.

When $n = 2$, the famous inequality due to M. Noether [Noe75] states that

$$\mathrm{Vol}(X) \geq 2p_g(X) - 4.$$

In his celebrated paper [Hor76], Horikawa completely described for each $p_g \geq 3$ the moduli space parametrizing all canonical surfaces "on the Noether line" (i.e., $K^2 = 2p_g - 4$) with geometric genus p_g. More precisely, he showed loc. cit. that the moduli space is either irreducible and unirational, or it has two irreducible and unirational components of the same dimension that do not intersect. The "second" component appears if and only if K^2 is divisible by 8. Moreover, when $p_g \geq 7$, the dimension of the moduli space is $7p_g + 14$.

When $n = 3$, the corresponding Noether inequality, conjectured around the end of the last century, is now "essentially" proved. More precisely, Chen et al. proved in [CCJ20b, CCJ20a] that the inequality

$$\mathrm{Vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3} \tag{1.1}$$

holds for every threefold X of general type, possibly with some exceptions with $5 \leq p_g(X) \leq 10$.[1] The inequality is optimal due to known examples found by Kobayashi [Kob92] for infinitely many p_g. As in the surface case, we say that a threefold X with $p_g(X) \geq 11$ is *on the Noether line* if $\mathrm{Vol}(X) =$

[1] In a very recent preprint [CHJ24], Chen et al. proved that (1.1) holds when $p_g(X) = 5$.

$\frac{4}{3}p_g(X) - \frac{10}{3}$. In other words, given the geometric genus $p_g \geq 11$, threefolds on the Noether line have the smallest possible canonical volume. Recently, more examples of threefolds on the Noether line have been constructed in [CH17, CJL24, CP23], but it remains an open question whether there is a classification for all threefolds on the Noether line (see [CCJ20b, Question 1.5]).

1.2. **Main theorem.** The main result in this paper is an explicit description of the moduli spaces of canonical threefolds on the Noether line with geometric genus $p_g \geq 11$. It is a three dimensional version of Horikawa's work [Hor76] and provides a complete answer to the above question. We summarize it as the following.

Theorem 1.1. *For an integer $p_g \geq 11$, let $\mathcal{M}_{K^3,p_g}$ be the coarse moduli space parametrizing all canonical threefolds with geometric genus p_g and canonical volume $K^3 = \frac{4}{3}p_g - \frac{10}{3}$. Then $\mathcal{M}_{K^3,p_g}$ is non-empty if and only if $p_g \equiv 1 \pmod 3$.*

Suppose that $\mathcal{M}_{K^3,p_g}$ is non-empty. Then

(1) *$\mathcal{M}_{K^3,p_g}$ is a union of $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ unirational strata.*
(2) *The number of irreducible components is at most $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ and at least $\left\lfloor \frac{p_g+6}{4} \right\rfloor - \left\lfloor \frac{p_g+8}{78} \right\rfloor$. In particular, this number grows linearly with p_g.*
(3) *We have*

$$\dim \mathcal{M}_{K^3,p_g} = \frac{169}{3}p_g - 56\left\lceil \frac{p_g+2}{12} \right\rceil + \frac{386}{3},$$

where the dimension means the maximal one among all irreducible components of $\mathcal{M}_{K^3,p_g}$.

Theorem 1.1 (2) implies that the number of irreducible components of $\mathcal{M}_{K^3,p_g}$ is exactly $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ when $p_g < 70$. In contrast with Horikawa's result [Hor76] (and rather surprisingly for us), it shows that the number of irreducible components is unbounded as p_g tends to infinity. Note that the number of irreducible components of $\mathcal{M}_{K^3,p_g}$ containing smooth canonical threefolds is "only" one or two, the exact number depending whether $p_g + 2$ is divisible by 8. All other components contain only singular canonical threefolds.

Moreover, we not only obtain the dimension of $\mathcal{M}_{K^3,p_g}$ as in Theorem 1.1 (3), but also obtain dimensions of all strata of those in Theorem 1.1 (1) (see Proposition 4.3). As a consequence, a general canonical threefold with $p_g \geq 11$ on the Noether line (i.e., those parameterized by the irreducible component of $\mathcal{M}_{K^3,p_g}$ with the maximal dimension), has singularities of type cE_8. In contrast, a general canonical surface on the Noether line is smooth.

We remark that when $p_g \geq 11$, it was already known that the moduli space $\mathcal{M}_{K^3,p_g}$ in Theorem 1.1 is empty unless $p_g \equiv 1 \pmod 3$ [HZ24, Theorem

1.2]. Thus the novelty of Theorem 1.1 is the complete description of the non-empty moduli spaces.

1.3. **Idea of the proof.** The proof of Theorem 1.1 starts from investigating the following conjecture stated in [CP23, Introduction].

Conjecture 1.2. *Every canonical threefold on the Noether line with p_g sufficiently large birationally admits a simple fibration in $(1,2)$-surfaces over $\mathbb{P}^1$.*

Here and throughout this paper, a $(1,2)$-*surface* is a surface S with at worst canonical singularities, $\mathrm{Vol}(S) = 1$ and $p_g(S) = 2$. A key feature of a $(1,2)$-surface is that its canonical ring is generated by four elements of respective degree 1, 1, 2 and 5 and related by a single equation of degree 10. Simple fibrations in $(1,2)$-surfaces were introduced and studied in [CP23] (see Definition 2.5 for a precise definition). They are fibrations $f\colon X \to B$ from a threefold X with canonical singularities to a smooth curve B with K_X being f-ample such that the canonical ring of each fibre is "algebraically" like that of a $(1,2)$-surface. An enlightening result proved in [CP23, Theorem 1.11] is that every Gorenstein minimal threefold X admitting a simple fibration in $(1,2)$-surfaces over $\mathbb{P}^1$ is isomorphic to a divisor in a toric fourfold. Moreover, such an X is on the Noether line if it is Gorenstein. Thus Conjecture 1.2 is the converse of this result. Moreover, in all but a handful of cases, the canonical model itself admits the simple fibration and no birational map is needed.

As a key step in the proof of Theorem 1.1, we show in Theorem 3.1 that

Theorem 1.3. *Conjecture 1.2 holds true for $p_g \geq 11$.*

As a corollary, the description in [CP23] via simple fibrations holds for all threefolds on the Noether line with $p_g \geq 11$. To put this result into perspective, by the result of Horikawa [Hor76, Hor77], every canonical surface on the Noether line with $p_g \geq 7$ admits a "simple" fibration in genus 2 curves over $\mathbb{P}^1$. That is, the canonical ring of each fibre is "algebraically" like that of a smooth genus 2 curve, which is generated by three elements of respective degree 1, 1 and 3 and related by a single equation of degree 6 (see also [Xia85, Rei90, CP06]). Hence Theorem 1.3 is an analogue of this result in dimension three.

By Theorem 1.3, we are able to show that canonical threefolds on the Noether line with geometric genus $p_g \geq 11$ are determined by two integers d, d_0 with $p_g = 3d - 2$ and $\frac{1}{4} \leq \frac{d_0}{d} \leq \frac{3}{2}$. Each pair (d, d_0) gives a different unirational stratum $V_d(d_0)$ of the moduli space $\mathcal{M}_{K^3,p_g}$ in Theorem 1.1. Explicit deformations show that $V_d(d_0)$ is on the boundary of $V_d\left(\left\lfloor \frac{3}{2}d \right\rfloor\right)$ for every $d_0 \geq d$. On the other hand, we prove by a dimensional argument that $V_d(d_0)$ is dense in an irreducible component of $\mathcal{M}_{K^3,p_g}$ for each $d_0 \leq \frac{25d-3}{26}$ and that $V_d\left(\left\lceil \frac{d}{4} \right\rceil\right)$ has the largest dimension. Thus Theorem 1.1 is proved.

It is worth mentioning that most of the results in this paper are proved not just for $p_g \geq 11$, but under the weaker assumption that $p_g \geq 7$ and the image of the canonical map of the threefold has dimension 2. If one could prove that there are no canonical threefolds with $p_g = 7, 8, 9, 10$, $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ and 1-dimensional canonical image, then Theorem 1.1 would automatically extend to $p_g \geq 7$.

We remark that at the moment we cannot determine if the remaining $\left\lfloor \frac{p_g+8}{78} \right\rfloor$ strata, those $V_d(d_0)$ with $\frac{25d-3}{26} < d_0 < d$, are dense in an irreducible component or contained in the boundary of $V_d\left(\left\lfloor \frac{3}{2}d \right\rfloor\right)$ (see §4.3 for more details).

1.4. **Structure of the paper.** The paper is structured as follows.

In Section 2, we recall all known results we need in the paper, mostly from our previous papers [HZ24] and [CP23]. The novelty here is Proposition 2.11, that is a refinement of a result in [CP23].

Section 3 is devoted to Theorem 1.3 (see Theorem 3.1 for details), that is the proof of the aforementioned conjecture 1.2.

In Section 4, we study the moduli space of canonical threefolds on the Noether line with $p_g \geq 11$. More precisely, we compute in Proposition 4.3 the dimension of each stratum $V_d(d_0)$ and finally establish Theorem 1.1.

1.5. **Notation.** Throughout this paper, we work over the complex number field $\mathbb{C}$, and all varieties are projective, with at worst canonical singularities.

- A variety X is *minimal* if it has at worst $\mathbb{Q}$-factorial terminal singularities and K_X is nef.
- A variety X is *Gorenstein* if the canonical class K_X is Cartier.
- For a variety X, the irregularity is defined as $q(X) = h^1(X, \mathcal{O}_X)$. We say that X is *regular* if $q(X) = 0$.
- For a variety X, if $p_g(X) \geq 2$, then the global sections of the canonical class induce a rational map, called the *canonical map*, from X to $\mathbb{P}^{p_g(X)-1}$. The closure of the image of X under its canonical map is called the *canonical image* of X.

Given two variables t_0, t_1, we denote by $S^n(t_0, t_1)$ the set of monomials of degree n in the variables t_0, t_1. In particular, $S^n(t_0, t_1)$ is empty if n is negative.

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2. Preliminary results

In this section, we collect some known results about threefolds with small volume and simple fibrations in $(1,2)$-surfaces that we are going to use in the rest of the paper.

We are interested in the moduli space of canonical threefolds. Some of the results we use are stated in the original papers for minimal threefolds of general type. These results extend to canonical threefolds by the obvious use of a terminalisation. Indeed, for a canonical threefold X, there exists a crepant birational morphism $\tau\colon \tilde{X} \to X$ such that $\tilde{X}$ is minimal by [Kaw88] or [KM98, Theorem 6.25]. Note that if X is Gorenstein, then so is $\tilde{X}$ (see [Rei87, §3] or [KM98, Theorem 6.23]). So we reformulate those results directly here for canonical threefolds.

2.1. **Threefolds with small volume.** The starting point is the Noether inequality for threefolds, first proved in a weaker version in [CCJ20b, Theorem 1.1] and then in the following version in [CCJ20a, Theorem 1].

Theorem 2.1 (The Noether inequality for threefolds). *Let X be a canonical threefold with either $p_g(X) \le 4$ or $p_g(X) \ge 11$. Then the Noether inequality (1.1) holds for X.*

As defined in §1.1, canonical threefolds for which the equality in (1.1) holds are said to be on the Noether line. For them we know

Proposition 2.2. [HZ24, Theorem 1.2 (3)] *Let X be a canonical threefold with $p_g(X) \ge 11$ and $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$. Then the canonical image of X has dimension 2.*

In particular all the statements in the paper that have the assumption "$p_g(X) \ge 7$ and canonical image of dimension 2" hold for $p_g(X) \ge 11$ without further assumptions on the canonical image.

If a canonical threefold X lies on the Noether line, then it is Gorenstein. In fact, we know a bit more:

Proposition 2.3. [HZ24, Theorem 4.7 and Proposition 4.3] *Let X be a canonical threefold with $p_g(X) \ge 7$ and the canonical image of dimension 2. Then*

- *If $p_g \equiv 1 \pmod 3$, then either $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and X is Gorenstein or $K_X^3 \ge \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{3}{6}$.*
- *If $p_g \equiv 2 \pmod 3$, then $K_X^3 \ge \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{1}{6}$ and it is optimal.*
- *If $p_g \equiv 0 \pmod 3$, then $K_X^3 \ge \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{2}{6}$ and it is optimal.*

Furthermore, all threefolds "close to the Noether line" have a fibration over $\mathbb{P}^1$ whose general fibre is a $(1,2)$-surface. More precisely, we have

Proposition 2.4. [HZ24, Proposition 2.1 and Lemma 3.4] *Let X be a canonical threefold with $p_g(X) \geq 7$ and canonical image of dimension 2. If $K_X^3 < \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{4}{6}$, then X has a birational model X_1 such that*

- *X_1 is minimal;*
- *there is a fibration $\pi_1 \colon X_1 \to \mathbb{P}^1$ whose general fibre is a smooth $(1,2)$-surface.*

Moreover, $q(X) = h^2(X, \mathcal{O}_X) = 0$.

2.2. **Simple fibrations in $(1,2)$-surfaces.** Recall the definition of a simple fibration in $(1,2)$-surfaces from [CP23].

Definition 2.5. A *simple fibration in $(1,2)$-surfaces* is a surjective morphism $\pi \colon X \to B$ such that

- B is a smooth curve;
- X is a threefold with at worst canonical singularities;
- K_X is π-ample;
- for all $p \in B$, the canonical ring $R(X_p, K_{X_p}) := \bigoplus_d H^0(X_p, dK_{X_p})$ of the surface $X_p := \pi^* p$ is generated by four elements of respective degree $1, 1, 2$ and 5 and related by a single equation of degree 10, where $K_{X_p} = K_X|_{X_p}$.

For brevity, if a threefold X admits a simple fibration in $(1,2)$-surfaces $\pi \colon X \to B$, we often write that X *is a simple fibration* as in [CP23].

The simple fibrations in $(1,2)$-surfaces that are both *Gorenstein* and *regular* can be canonically embedded in a toric 4-fold as follows.

Choose integers d, d_0 and define $\mathbb{F} = \mathbb{F}(d; d_0)$ to be the toric 4-fold with weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & d-d_0 & d_0-2d & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix} \tag{2.1}$$

and irrelevant ideal $I = (t_0, t_1) \cap (x_0, x_1, y, z)$. Set $e = 3d - 2d_0$.

The following result is a combination of [CP23, Theorem 4.23, Definition 1.4 and Theorem 1.11].

Theorem 2.6. *Each Gorenstein regular simple fibration in $(1,2)$-surfaces is a divisor in a unique $\mathbb{F}(d; d_0)$ defined by a bihomogeneous equation of bidegree $(0, 10)$ respect to the weights given by the rows of the matrix (2.1), in other words an equation of the form*

$$z^2 = y^5 + \cdots .$$

Conversely, each divisor as above with at worst canonical singularities is a Gorenstein regular simple fibration in $(1,2)$-surfaces.

Then [CP23] introduced the following

Definition 2.7. We say that a Gorenstein regular simple fibration X contained in $\mathbb{F}(d; d_0)$ is *of type* (d, d_0) and sometimes denote it by $X(d; d_0)$.

By [CP23, Proposition 1.6], Gorenstein regular simple fibrations in $(1, 2)$-surfaces of type (d, d_0) exist if and only if

$$\frac{1}{4}d \leq d_0 \leq \frac{3}{2}d. \tag{2.2}$$

Remark 2.8. Here we correct a small inaccuracy in the proof of [CP23, Proposition 1.6]. In the proof of the inequality $\frac{1}{4}d \leq d_0$, it was implicitly assumed that $d \geq 0$, which had not yet been proved. In fact, this can be easily shown as follows: since $d_0 \leq \frac{3}{2}d$ (that was proven right before), it is enough to show that $d_0 \geq 0$. This is an immediate consequence of Fujita semipositivity, since by definition $\mathcal{O}_{\mathbb{P}^1}(d_0)$ is a direct summand of $f_*\omega_{X/\mathbb{P}^1}$.

We may assume in the following that d_0 and d are both strictly positive. This is because the Gorenstein regular simple fibrations of type $(0, 0)$ are products of a $(1, 2)$-surface and $\mathbb{P}^1$, and these products have no interest for us since they are not of general type.

For $X = X(d; d_0)$, a basis of $H^0(X, K_X)$ is given by the monomials in $S^{d_0-2}(t_0, t_1) \cdot x_0$ and $S^{3d-d_0-2}(t_0, t_1) \cdot x_1$ (see the proof of [CP23, Proposition 1.9]). This shows that the integer d_0 is strictly related to the canonical image Σ of X in the sense that

- if $d_0 = 1$, then Σ is a rational normal curve of degree $3d - 3$;
- if $d_0 = 2$, then Σ is a cone over a rational normal curve of degree $3d - 4$;
- if $d_0 \geq 3$, then Σ is isomorphic to the Hirzebruch surface $\mathbb{F}_e$.

Moreover, we have $p_g = 3d - 2$, so the integer d is a deformation invariant. More precisely,

Proposition 2.9. [CP23, Theorem 1.11] *Gorenstein regular simple fibrations X of type (d, d_0) have*

$$p_g(X) = 3d - 2, \quad q(X) = 0, \quad K_X^3 = 4d - 6 = \frac{4}{3}p_g(X) - \frac{10}{3}.$$

In particular, if a Gorenstein regular simple fibration in $(1, 2)$-surfaces is a canonical threefold, then it is on the Noether line. In fact this is almost always the case.

Proposition 2.10. [CP23, Lemma 1.8 and Section 6] *A Gorenstein regular simple fibration X of type (d, d_0) has ample canonical class if and only if* $\min(d, d_0) \geq 3$.

If $\min(d, d_0) = 2$*, then X is minimal of general type but not canonical, and the morphism from X onto its canonical model is crepant.*

If $\min(d, d_0) = 1$*, then either X is not of general type, or its minimal model is not on the Noether line.*

As a result, when $\min(d, d_0) \geq 2$, the Gorenstein regular simple fibrations X of type (d, d_0) give pairwise disjoint unirational subvarieties of the moduli space of threefolds of general type on the Noether line. We will see in Section 3 that this is a stratification of the moduli space when $d \geq 5$ (equivalently $p_g \geq 11$).

Later it will be useful to know the singular locus of the general element in each of these unirational families. For that we need to recall some standard notation for toric varieties: for each variable $\rho \in \{t_0, t_1, x_0, x_1, y, z\}$ we set D_ρ for the corresponding torus invariant divisor of $\mathbb{F}(d; d_0)$, i.e.,

$$D_\rho := \{\rho = 0\}. \tag{2.3}$$

Then we prove a refined version of [CP23, Proposition 1.6].

Proposition 2.11. *The singular locus of the general $X(d; d_0)$ is contained in the torus invariant curve $\mathfrak{s}_0 := D_{x_1} \cap D_y \cap D_z$. More precisely,*

(1) *$X(d; d_0)$ is nonsingular if and only if $1 \leq \frac{d_0}{d} \leq \frac{3}{2}$ or $\frac{d_0}{d} = \frac{7}{8}$;*
(2) *$X(d; d_0)$ has $8d_0 - 7d$ terminal singularities (counted with multiplicity) if and only if $\frac{7}{8} < \frac{d_0}{d} < 1$;*
(3) *$X(d; d_0)$ has canonical singularities along $\mathfrak{s}_0$ of type*
    (a) *cA_1 if and only if $\frac{5}{6} \leq \frac{d_0}{d} < \frac{7}{8}$;*
    (b) *cA_3 if and only if $\frac{3}{4} \leq \frac{d_0}{d} < \frac{5}{6}$;*
    (c) *cA_4 if and only if $\frac{2}{3} \leq \frac{d_0}{d} < \frac{3}{4}$;*
    (d) *cD_6 if and only if $\frac{1}{2} \leq \frac{d_0}{d} < \frac{2}{3}$;*
    (e) *cE_8 if and only if $\frac{1}{4} \leq \frac{d_0}{d} < \frac{1}{2}$.*

Proof. Part (1) and (2) are in [CP23, Proposition 1.6]. We prove the refinement of part (3) using the same approach.

We assume that $d_0 < \frac{7}{8}d$. Denote a general $X(d; d_0)$ by X. After coordinates changes, the hypersurface X is defined by a polynomial of the form

$$z^2 + y^5 + \sum_{\substack{a_0+a_1+2a_2=10 \\ a_2 \neq 5}} c_{a_0,a_1,a_2}(t_0, t_1) x_0^{a_0} x_1^{a_1} y^{a_2}, \tag{2.4}$$

where $c_{a_0,a_1,a_2}(t_0, t_1)$ is a homogeneous polynomial whose degree is

$$\deg c_{a_0,a_1,a_2} = -a_0(d - d_0) - a_1(d_0 - 2d) = \frac{(a_0 + a_1)d + (a_1 - a_0)e}{2}. \tag{2.5}$$

Since we have assumed that $d_0 < \frac{7}{8}d$, it follows that the coefficients $c_{10,0,0}$, $c_{8,0,1}$, $c_{6,0,2}$, $c_{4,0,3}$, $c_{2,0,4}$ and $c_{9,1,0}$ vanish. Hence the polynomial (2.4) has the form

$$z^2 + y^5 + x_1(c_{8,2,0}x_0^8 x_1 + c_{7,1,1}x_0^7 y + c_{7,3,0}x_0^7 x_1^2 + c_{6,2,1}x_0^6 x_1 y + c_{5,1,2}x_0^5 y^2 + g)$$

where g vanishes at $\mathfrak{s}_0$ with multiplicity at least 3. So X is singular along $\mathfrak{s}_0$. The five coefficients appearing above are the critical coefficients. Here we list them with their degrees

$$\deg c_{7,1,1} = 6d_0 - 5d, \quad \deg c_{5,1,2} = 4d_0 - 3d, \quad \deg c_{8,2,0} = 6d_0 - 4d,$$
$$\deg c_{6,2,1} = 4d_0 - 2d, \quad \deg c_{7,3,0} = 4d_0 - d.$$

Applying [Rei87, §4.6, §4.9] and [Rei83, §1.14], we know that X will have canonical singularities along $\mathfrak{s}_0$ if and only if at least one of the critical coefficients is non-zero (see [CP23, §1.4] for further details). It remains to determine the type of singularities for each case (a), . . . , (e).

(a) If $\frac{5}{6}d \le d_0 < \frac{7}{8}d$, then for degree reasons, all critical coefficients are nonzero for X. It is then easy to see that X has cA_1 singularities along $\mathfrak{s}_0$, because the local analytic equation is $z^2 + c_{7,1,1}x_1y$.
(b) If $\frac{3}{4}d \le d_0 < \frac{5}{6}d$, then $c_{7,1,1}$ has negative degree, so X has cA_3 singularities along $\mathfrak{s}_0$ and local analytic equation $z^2 + c_{8,2,0}x_1^2 + c_{5,1,2}x_1y^2$.
(c) If $\frac{2}{3}d \le d_0 < \frac{3}{4}d$, then $c_{5,1,2}$ has negative degree, so X has cA_4 singularities along $\mathfrak{s}_0$ and local analytic equation $z^2 + c_{8,2,0}x_1^2 + y^5$.
(d) If $\frac{1}{2}d \le d_0 < \frac{2}{3}d$, then $c_{8,2,0}$ has negative degree, so X has cD_6 singularities along $\mathfrak{s}_0$ and local analytic equation $z^2 + c_{6,2,1}x_1^2y + y^5$.
(e) If $\frac{1}{4}d \le d_0 < \frac{1}{2}d$, then $c_{6,2,1}$ has negative degree, so X has cE_8 singularities along $\mathfrak{s}_0$ and local analytic equation $z^2 + c_{7,3,0}x_1^3 + y^5$.

This concludes the proof. □

3. Proof of Conjecture 1.2

This section is devoted to the proof of the following result, answering affirmatively Conjecture 1.2 stated in [CP23].

Theorem 3.1. *Suppose that X is a canonical threefold with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$, $p_g(X) \ge 7$ and the canonical image of dimension 2. Then there is a crepant birational morphism $X_0 \to X$ such that X_0 is a Gorenstein regular simple fibration in $(1,2)$-surfaces. If $p_g(X) \ge 23$, then $X_0 \cong X$.*

Proof. By Proposition 2.4, we can choose a minimal model X_1 of X so that X_1 admits a fibration $\pi_1 \colon X_1 \to \mathbb{P}^1$ whose general fibre is a smooth $(1,2)$-surface. We know that X_1 is Gorenstein by Proposition 2.3.

Let X_0 be the relative canonical model of X_1 over $\mathbb{P}^1$, that is, $X_0 = \mathbf{Proj} \bigoplus (\pi_1)_*(n(K_{X_1} - \pi_1^* K_{\mathbb{P}^1}))$. So we have a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \epsilon\ } & X_0 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_0} & \\ & \mathbb{P}^1 & \end{array}$$

Since the general fibre of π_1 is a smooth $(1,2)$-surface, its canonical model is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$ with at worst canonical singularities.

Now let F_p be the fibre of $X_0 \to \mathbb{P}^1$ over any point $p \in \mathbb{P}^1$. Since F_p is a Cartier divisor on X_0 and X_0 is Gorenstein, we get that F_p is Gorenstein. Thus K_{F_p} is Cartier. For any given integer $n \ge 1$, consider the exact sequence

$$
\begin{aligned}
&0 \to H^0(X_0, nK_{X_0}) \to H^0(X_0, nK_{X_0} + F_p) \to H^0(F_p, nK_{F_p}) \\
&\to H^1(X_0, nK_{X_0}) \to H^1(X_0, nK_{X_0} + F_p) \to H^1(F_p, nK_{F_p}) \\
&\to H^2(X_0, nK_{X_0}).
\end{aligned} \tag{3.1}
$$

Now $H^i(X_0, nK_{X_0})$ vanishes for $i = 1, 2$ when $n = 1$ by Proposition 2.4 and Serre duality, and when $n \geq 2$ by the Kawamata–Viehweg vanishing theorem. Thus we have $h^1(X_0, nK_{X_0} + F_p) = h^1(F_p, nK_{F_p})$, and this does not depend on p. Therefore, since $h^1(F_p, nK_{F_p}) = 0$ for a general F_p which is a canonical $(1, 2)$-surface, we have $h^1(X_0, nK_{X_0} + F_p) = h^1(F_p, nK_{F_p}) = 0$ for all F_p. Moreover all plurigenera $h^0(F_p, nK_{F_p}) = h^0(X_0, nK_{X_0} + F_p) - h^0(X_0, nK_{X_0})$ do not depend on p. We conclude that every F_p is a Gorenstein surface with $h^0(F_p, K_{F_p}) = 2$ and $K^2_{F_p} = 1$.

If we could assume that all F_p are stable as in [Kol13, §5.1–5.3], then [FPR17, Theorem 3.3 (1)] implies that all F_p are hypersurfaces of degree 10 in $\mathbb{P}(1, 1, 2, 5)$ and we could conclude that X_0 is a simple fibration in $(1, 2)$-surfaces.

On the other hand, in the proof of [FPR17, Theorem 3.3 (1)], stability is only used to prove that:

(i) $h^1(F_p, nK_{F_p}) = 0$ for any $n \geq 1$;
(ii) there is an integral curve $C \in |K_{F_p}|$.

We have already shown (i) using (3.1). So to conclude that X_0 is a simple fibration in $(1, 2)$-surfaces (without assuming stability of fibres), we need to check that (ii) holds for every fibre of π_0.

In fact this is proven in Lemma [HZ24, Lemma 5.6]. An alternative and similar proof uses the argument of [FPR15, Lemma 4.1]: since F_p is Gorenstein, a general $C \in |K_{F_p}|$ is Gorenstein and is of arithmetic genus 2 by adjunction. Since K_{F_p} is ample and $K^2_{F_p} = (K_{F_p} \cdot C) = 1$, we get that C is reduced and irreducible, i.e., integral.

Then X_0 is a Gorenstein regular simple fibration in $(1, 2)$-surfaces. Set (d, d_0) for its type as defined in Definition 2.7.

The birational morphism $X_0 \to X$ is the morphism of X_0 onto its canonical model. By Proposition 2.10, it is an isomorphism unless $\min(d, d_0) = 2$. Since we are assuming $p_g(X) \geq 7$, then $d \geq 3$, so in this last case $d_0 = 2$. Then by (2.2), we have $d \leq 8$, so $p_g(X) \leq 22$. This concludes the proof. □

Remark 3.2. The proof shows that X_0 and X are not isomorphic if and only if the canonical image Σ is singular, that is, a cone over a rational normal curve. In particular, the assumption $p_g(X) \geq 23$ in the last claim of Theorem 3.1 is sharp.

Remark 3.3. Proposition 2.4 is a key building block of the proof of Theorem 3.1. We need to assume that $p_g(X) \geq 7$ and that the canonical image of X has dimension 2 to apply it. If one could weaken this assumption in Proposition 2.4, a similar generalization of Theorem 3.1 should hold too.

One cannot hope to completely remove these assumptions, because $X_{10} \subset \mathbb{P}(1,1,1,1,5)$ is a threefold of general type with $p_g = 4$, $K^3 = 2$ that is not birational to any simple fibration in $(1,2)$-surfaces. Its canonical map is a double cover of $\mathbb{P}^3$, a threefold.

Combining Proposition 2.2 and Theorem 3.1, we conclude the following.

Corollary 3.4. *Suppose that X is a canonical threefold with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 11$. Then there is a crepant birational morphism $X_0 \to X$ such that X_0 is a Gorenstein regular simple fibration in $(1,2)$-surfaces. If $p_g(X) \geq 23$, then $X_0 \cong X$.*

So classifying canonical threefolds on the Noether line with $p_g \geq 11$ is equivalent to classifying Gorenstein regular simple fibrations in $(1,2)$-surfaces with $d \geq 5$ and $d_0 \geq 2$. Using Proposition 2.9 and 2.10 as well as [CP23, Example 1.13], we can rewrite Corollary 3.4 as

Corollary 3.5. *The canonical threefolds with $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ and $p_g \geq 11$ are, up to a birational transformation, the Gorenstein regular simple fibrations in $(1,2)$-surfaces of type (d, d_0) with $d = 3p_g - 2 \geq 5$ and $d_0 \geq 2$.*

The birational transformation is an isomorphism unless $d_0 = 2$, in which case $p_g \leq 22$ and the birational transformation is described in [CP23, Example 1.13].

4. Moduli spaces of threefolds on the Noether line

In this section, we describe the moduli space $\mathcal{M}_{K^3,p_g}$ of the canonical threefolds on the Noether line with geometric genus $p_g \geq 11$, so by Corollary 3.5 we have to consider the Gorenstein regular simple fibrations in $(1,2)$-surfaces of type (d, d_0) with $d \geq 5$ and $d_0 \geq 2$.

Let $\mathcal{M}_d(d_0)$ denote the corresponding modular family of hypersurfaces $X(d; d_0)$ in $\mathbb{F}(d; d_0)$ as in Definition 2.7. Then it is unirational. Let $\mathcal{M}_{K^3,p_g}$ be the moduli space of canonical threefolds with $p_g = 3d-2$ and $K^3 = 4d-6$. By (2.2), there is a non-trivial morphism

$$\Phi_{d,d_0} : \mathcal{M}_d(d_0) \to \mathcal{M}_{K^3,p_g}$$

when $\frac{1}{4} \leq \frac{d_0}{d} \leq \frac{3}{2}$. Moreover, by Proposition 2.10, Φ_{d,d_0} is an isomorphism onto its image for $d_0 \geq 3$.

If $d_0 = 2$, then $d \leq 8$, and $X(d; d_0)$ is not a canonical model, since K_X is not ample. However the map onto the canonical model, described in [CP23, Example 1.13], is a projective crepant birational morphism. By [KM87, Main Theorem] on the finiteness of minimal models for threefolds, each canonical model admits only finitely many such maps. Hence Φ_{d,d_0}, if not one-to-one, is at least finite-to-one onto its image.

4.1. **The dimension of $\mathcal{M}_d(d_0)$.** From now on, we set $\Delta_d(d_0)$ for the dimension of $\mathcal{M}_d(d_0)$. In the following, we will use the notation for divisors on $\mathbb{F}(d;d_0)$ that was introduced in (2.3). We also define

$$H := D_{x_0} + (d_0 - d)D_{t_0}. \tag{4.1}$$

Note that the definition of H used here differs slightly from that used in [CP23, §1.1].

Analogous to [CP23, §1.1], we have the following relations:

$$D_{t_0} \sim D_{t_1}, \quad D_{x_0} \sim H + (d - d_0)D_{t_0}, \quad D_{x_1} \sim H + (d_0 - 2d)D_{t_0},$$

$$D_y \sim 2H, \quad D_z \sim 5H.$$

As noticed in [CP23, Section 4 and 5, see also the proof of Theorem 5.2], the bicanonical map of $X(d;d_0)$ is the restriction of the projection from $\mathbb{F}(d;d_0)$ onto the $\mathbb{P}(1,1,2)$-bundle

$$D_z = \mathbb{F}(d;d_0) \cap (z = 0).$$

It is a finite morphism of degree 2 whose branch locus B is cut out by an element of $H^0(D_z, 10H_{D_z})$, where $H_{D_z} = H|_{D_z}$. The dimension of $\mathcal{M}_d(d_0)$ is therefore equal to the dimension of the family of pairs (D_z, B), i.e.,

$$\begin{aligned} \dim \mathcal{M}_d(d_0) &= \dim |10H_{D_z}| - \dim \operatorname{Aut} D_z \\ &= h^0(D_z, 10H_{D_z}) - \dim \operatorname{Aut} D_z - 1. \end{aligned} \tag{4.2}$$

Before computing the dimension, we define D'_ρ to be the torus invariant divisors in D_z given by $\rho = 0$ for each $\rho \in \{t_0, t_1, x_0, x_1, y\}$. Then $D'_\rho = D_\rho|_{D_z}$. Write $F = D'_{t_0}$.

We compute the dimension of the automorphism group of D_z first.

Lemma 4.1. *The dimension of the automorphism group of D_z is*

$$\dim \operatorname{Aut} D_z = \begin{cases} 3d + 10 & \text{if } d_0 = \frac{3}{2}d; \\ 6d - 2d_0 + 9 & \text{if } d \le d_0 < \frac{3}{2}d; \\ 8d - 4d_0 + 8 & \text{if } \frac{1}{4}d \le d_0 < d. \end{cases}$$

Proof. By [Cox95, §4] and the above relations among D_ρ and H, we have the formula

$$\begin{aligned} \dim \operatorname{Aut} D_z &= \sum_{\rho \in \{t_0, t_1, x_0, x_1, y\}} h^0(D_z, D'_\rho) - 2 \\ &= 2h^0(D_z, F) + h^0(D_z, (d - d_0)F + H_{D_z}) \\ &\quad + h^0(D_z, (d_0 - 2d)F + H_{D_z}) + h^0(D_z, 2H_{D_z}) - 2. \end{aligned} \tag{4.3}$$

It is easy to decompose these vector spaces in terms of monomials on D_z using the weight matrix (2.1):

$$\begin{aligned} H^0(D_z, F) &= S^1(t_0, t_1), \\ H^0(D_z, (d-d_0)F + H_{D_z}) &= \mathbb{C}x_0 \oplus S^{3d-2d_0}(t_0,t_1)x_1, \\ H^0(D_z, (d_0-2d)F + H_{D_z}) &= S^{2d_0-3d}(t_0,t_1)x_0 \oplus \mathbb{C}x_1, \\ H^0(D_z, 2H_{D_z}) &= S^{2(d_0-d)}(t_0,t_1)x_0^2 \oplus S^d(t_0,t_1)x_0x_1 \\ &\quad \oplus S^{2(2d-d_0)}(t_0,t_1)x_1^2 \oplus \mathbb{C}y. \end{aligned}$$

It is clear that $h^0(D_z, F) = 2$. The sum of the dimensions of the next two terms is

$$h^0((d-d_0)F + H_{D_z}) + h^0((d_0-2d)F + H_{D_z}) = \begin{cases} 4 & \text{if } d_0 = \frac{3}{2}d; \\ 3d - 2d_0 + 3 & \text{otherwise.} \end{cases}$$

In fact, if $d_0 = \frac{3}{2}d$, then $3d - 2d_0 = 0$, and hence $H^0(D_z, (d-d_0)F + H_{D_z}) = H^0(D_z, (d_0-2d)F + H_{D_z})$ are both two dimensional with basis x_0, x_1. On the other hand, if $d_0 < \frac{3}{2}d$, then $2d_0 - 3d < 0$, and hence $h^0(D_z, (d-d_0)F+H_{D_z}) = 3d-2d_0+2$ whereas $h^0(D_z, (d_0-2d)F+H_{D_z}) = 1$.

Finally, we have

$$h^0(D_z, 2H_{D_z}) = \begin{cases} 3d+4 & \text{if } d_0 \geq d; \\ 5d - 2d_0 + 3 & \text{otherwise.} \end{cases}$$

In fact, note first that both d and $2d - d_0$ are positive. If $d_0 \geq d$, then $2(d_0 - d) \geq 0$, and hence

$$h^0(D_z, 2H_{D_z}) = (2(d_0-d)+1) + (d+1) + (2(2d-d_0)+1) + 1 = 3d+4.$$

If $d_0 < d$, then x_0^2 does not appear in any section of $2H$, and hence

$$h^0(D_z, 2H_{D_z}) = (d+1) + (2(2d-d_0)+1) + 1 = 5d - 2d_0 + 3.$$

Combining the above computations, we get the following three cases.

(1) If $d_0 = \frac{3}{2}d$, then
$$\dim \operatorname{Aut} D_z = 2\cdot 2 + 4 + (3d+4) - 2 = 3d + 10.$$

(2) If $d \leq d_0 < \frac{3}{2}d$, then
$$\dim \operatorname{Aut} D_z = 2\cdot 2 + (3d - 2d_0 + 3) + (3d+4) - 2 = 6d - 2d_0 + 9.$$

(3) If $\frac{1}{4}d \leq d_0 < d$, then
$$\dim \operatorname{Aut} D_z = 2\cdot 2 + (3d - 2d_0 + 3) + (5d - 2d_0 + 3) - 2 = 8d - 4d_0 + 8.$$

This concludes the proof. □

Next we count parameters for the branch divisor B in D_z, which is an element of $H^0(D_z, 10H_{D_z})$ of the form

$$\sum_{a_0+a_1+2a_2=10} c_{a_0,a_1,a_2}(t_0,t_1)x_0^{a_0}x_1^{a_1}y^{a_2}.$$

Each monomial $x_0^{a_0}x_1^{a_1}y^{a_2}$ contributes by adding $1+\deg c_{a_0,a_1,a_2}$ to the dimension $h^0(D_z, 10H_{D_z})$, unless $\deg c_{a_0,a_1,a_2} < 0$, in which case the contribution is zero. The formula for the degree of each $c_{a_0,a_1,a_2}(t_0,t_1)$ is in (2.5).

In the proof of Proposition 2.11 we noticed that the negativity of the degree of c_{a_0,a_1,a_2} depends on the ratio d_0/d: the smaller d_0/d is, the more monomials there are, whose coefficient has negative degree.

We summarize the result of that computation in the following Table 1.

Table 1. Table of vanishing monomials

d_0/d	monomials with vanishing coefficient	stratum
<1	$x_0^{10},\ x_0^8y,\ x_0^6y^2,\ x_0^4y^4,\ x_0^2y^4$	terminal
$<\frac{7}{8}$	$x_0^9x_1$	cA_1
$<\frac{5}{6}$	$x_0^7x_1y$	cA_3
$<\frac{3}{4}$	$x_0^5x_1y^2$	cA_4
$<\frac{2}{3}$	$x_0^8x_1^2$	cD_6
$<\frac{1}{2}$	$x_0^6x_1^2y,\ x_0^3x_1y^3$	cE_8

The last column reminds us what singularities the general $X(d;d_0)$ has, when d/d_0 approaches the upper bound in the first column. When $d_0/d \geq \frac{1}{4}$, all the other coefficients have non-negative degree.

Lemma 4.2. *The vector space $H^0(D_z, 10H_{D_z})$ has dimension*

$$h^0(D_z,10H_{D_z}) = \begin{cases} 125d+36 & \text{if } d \leq d_0 \leq \frac{3}{2}d;\\ 155d-30d_0+31 & \text{if } \frac{7}{8}d \leq d_0 < d;\\ 162d-38d_0+30 & \text{if } \frac{5}{6}d \leq d_0 < \frac{7}{8}d;\\ 167d-44d_0+29 & \text{if } \frac{3}{4}d \leq d_0 < \frac{5}{6}d;\\ 170d-48d_0+28 & \text{if } \frac{2}{3}d \leq d_0 < \frac{3}{4}d;\\ 174d-54d_0+27 & \text{if } \frac{1}{2}d \leq d_0 < \frac{2}{3}d;\\ 177d-60d_0+25 & \text{if } \frac{1}{4}d \leq d_0 < \frac{1}{2}d. \end{cases}$$

Proof. We first observe that

$$H^0(D_z,10H_{D_z}) = \bigoplus_{a_0+a_1+2a_2=10} S^{\deg c_{a_0,a_1,a_2}}(t_0,t_1)x_0^{a_0}x_1^{a_1}y^{a_2} \tag{4.4}$$

If $d \leq d_0 \leq \frac{3}{2}d$, then all the coefficients c_{a_0,a_1,a_2} have non-negative degree. The number of monomials is $\sum_{a_2=0}^{5} h^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(10-2a_2)\right) = 11+9+7+5+3+1 = 36$. Hence

$$\begin{aligned} h^0(D_z,10H_{D_z}) &= \sum_{a_0+a_1+2a_2=10}(1+\deg c_{a_0,a_1,a_2})\\ &= 36 + \sum_{a_0+a_1+2a_2=10} \deg c_{a_0,a_1,a_2}. \end{aligned}$$

Now we replace $\deg c_{a_0,a_1,a_2}$ with its expression in (2.5). By symmetry,

$$\sum_{a_0+a_1+2a_2=10} (a_1 - a_0) = 0,$$

and then

$$\begin{aligned} \sum_{a_0+a_1+2a_2=10} \deg c_{a_0,a_1,a_2} &= d\sum \frac{a_0+a_1}{2} = d\sum a_1 \\ &= d\sum_{a_2=0}^{5}\sum_{a_1=0}^{10-2a_2} a_1 = d\left[\binom{11}{2}+\binom{9}{2}+\binom{7}{2}+\binom{5}{2}+\binom{3}{2}\right] = 125d. \end{aligned} \tag{4.5}$$

This concludes the proof of the case $d \le d_0$.

If $\frac{7}{8}d \le d_0 < d$, then the monomials x_0^{10}, $x_0^8 y$, $x_0^6 y^2$, $x_0^4 y^3$, $x_0^2 y^4$ no longer appear in the equation of the branch divisor because their coefficients have negative degree. We modify the computation of $h^0(D_z, 10H_{D_z})$ to correct for these missing summands of (4.4), to get

$$\begin{aligned} h^0(D_z, 10H_{D_z}) &= 125d + 36 - \sum_{k=0}^{4}(1 + \deg c_{10-2k,0,k}) \\ &= 125d + 36 - (5 + 30(d_0 - d)) = 155d - 30d_0 + 31. \end{aligned}$$

If $\frac{5}{6}d \le d_0 < \frac{7}{8}d$, then we also lose $x_0^9 x_1$, and so the dimension is

$$h^0(D_z, 10H_{D_z}) = 155d - 30d_0 + 31 - (1 + \deg c_{9,1,0}) = 162d - 38d_0 + 30.$$

If $\frac{3}{4}d \le d_0 < \frac{5}{6}d$, then we also lose $x_0^7 x_1 y$, and so the dimension is

$$h^0(D_z, 10H_{D_z}) = 162d - 38d_0 + 30 - (1 + \deg c_{7,1,1}) = 167d - 44d_0 + 29.$$

If $\frac{2}{3}d \le d_0 < \frac{3}{4}d$, then we also lose $x_0^5 x_1 y^2$, and so the dimension is

$$h^0(D_z, 10H_{D_z}) = 167d - 44d_0 + 29 - (1 + \deg c_{5,1,2}) = 170d - 48d_0 + 28.$$

If $\frac{1}{2}d \le d_0 < \frac{2}{3}d$, then we also lose $x_0^8 x_1^2$, and so the dimension is

$$h^0(D_z, 10H_{D_z}) = 170d - 48d_0 + 28 - (1 + \deg c_{8,2,0}) = 174d - 54d_0 + 27.$$

If $\frac{1}{4}d \le d_0 < \frac{1}{2}d$, then we also lose $x_0^6 x_1^2 y$ and $x_0^3 x_1 y^3$, and so the dimension is

$$\begin{aligned} h^0(D_z, 10H_{D_z}) &= 174d - 54d_0 + 27 - (1 + \deg c_{6,2,1}) - (1 + \deg c_{3,1,3}) \\ &= 177d - 60d_0 + 25. \end{aligned}$$

This concludes the proof. □

Using the dimensions of $H^0(D_z, 10H_{D_z})$ and $\operatorname{Aut} D_z$ which were computed by the preceding lemmas and the formula (4.2), we get

Proposition 4.3. *For each $d \geq 5$, the modular family $\mathcal{M}_d(d_0)$ is unirational and has dimension*

$$\Delta_d(d_0) = \dim \mathcal{M}_d(d_0) = \begin{cases} 122d + 25 & \text{if } d_0 = \frac{3}{2}d; \\ 119d + 2d_0 + 26 & \text{if } d \leq d_0 < \frac{3}{2}d; \\ 147d - 26d_0 + 22 & \text{if } \frac{7}{8}d \leq d_0 < d; \\ 154d - 34d_0 + 21 & \text{if } \frac{5}{6}d \leq d_0 < \frac{7}{8}d; \\ 159d - 40d_0 + 20 & \text{if } \frac{3}{4}d \leq d_0 < \frac{5}{6}d; \\ 162d - 44d_0 + 19 & \text{if } \frac{2}{3}d \leq d_0 < \frac{3}{4}d; \\ 166d - 50d_0 + 18 & \text{if } \frac{1}{2}d \leq d_0 < \frac{2}{3}d; \\ 169d - 56d_0 + 16 & \text{if } \frac{1}{4}d \leq d_0 < \frac{1}{2}d. \end{cases}$$

In Proposition 4.3, d_0 is assumed to be an integer, but it is natural to view Δ_d as a function in one real variable. From this point of view, we have the following proposition.

Proposition 4.4. *Fix $d \geq 5$. Then there exists a piecewise linear real-valued function*

$$\Delta_d \colon \left[\frac{1}{4}d, \frac{3}{2}d\right] \to \mathbb{R}$$

whose component linear functions are given in Proposition 4.3 such that

(i) *the set of discontinuities of Δ_d is composed of the following seven points*
$$\left\{ d_0 = \lambda d \text{ with } \lambda = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, 1, \frac{3}{2} \right\};$$
(ii) *Δ_d is linear in each connected component of the domain of continuity;*
(iii) *for each integer d_0 in the domain of Δ_d, we have*
$$\dim \mathcal{M}_d(d_0) = \Delta_d(d_0).$$

Moreover,

(1) *The restriction of Δ_d to $\left[\frac{1}{4}d, d\right] \cap \mathbb{N}$ is strictly decreasing;*
(2) *The restriction of Δ_d to $\left[d, \frac{3}{2}d\right] \cap \mathbb{N}$ is strictly increasing;*
(3) $\Delta_d\left(\frac{3}{2}d\right) = \Delta_d\left(\frac{25d-3}{26}\right)$.

Proof. The statements about the monotonicity of Δ_d follow from the formulae of Proposition 4.3 by looking at the sign of the coefficient of d_0. We only need to check what happens at the discontinuities.

Let us first emphasize that both monotonicity statements (1) and (2) do not concern the function Δ_d as a whole, but only its restriction to the integers. Indeed, such statements do not generalize to the whole function Δ_d, exactly because of the points of discontinuity. More precisely, the discontinuities are as follows

λ	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{7}{8}$	1	$\frac{3}{2}$
$\Delta_d(\lambda d) - \lim_{x \to \lambda d^-} \Delta_d(x)$	2	1	1	1	1	4	-1

whereas the aforementioned generalization, to be true, would require all the "gaps" in the second row to have the opposite sign to that which is displayed.

Since the statements are for integers, we only need to check the sign of $\Delta_d(\lambda d) - \Delta_d(\lambda d - \varepsilon_0)$ where ε_0 is the smallest strictly positive number such that $\lambda d - \varepsilon_0 \in \mathbb{N}$.

The discontinuity at $\frac{3}{2}d$ is only relevant if d is even, and then we have $\varepsilon_0 = 1$, and Δ_d is linear with derivative $\Delta'_d = 2$ on the interval $[d, \frac{3}{2}d)$. Thus $\lim_{x\to\lambda d^-} \Delta_d(x) - \Delta_d(\lambda d - \varepsilon_0) = \Delta'_d \cdot \varepsilon_0 = 2 \cdot 1$ is positive enough to compensate the -1 from the above table. So $\Delta_d(\frac{3}{2}d) - \Delta_d(\frac{3}{2}d - 1) = -1 + 2 > 0$. This proves the monotonicity statement (2).

To prove (1), we use an argument very similar to the last one. Now $\lambda \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, 1\right\}$. If $\lambda = \frac{p}{q}$, then $\varepsilon_0 \geq \frac{1}{q}$, and so in all cases $\varepsilon_0 \geq \frac{1}{8}$. Then $\lim_{x\to\lambda d^-} \Delta_d(x) - \Delta_d(\lambda d - \varepsilon_0) = \Delta'_d \cdot \varepsilon_0 \leq -26 \cdot \frac{1}{8} < -3$. Since the above table shows that $\Delta_d(\lambda d) - \lim_{x\to\lambda d^-} \Delta_d(x) \leq 2$, we conclude that $\Delta_d(\lambda d) - \Delta_d(\lambda d - \varepsilon_0) < -3 + 2 < 0$, proving (1).

To prove (3), we first notice that since $d \geq 5$, then $\frac{25d-3}{26} - \frac{7}{8}d = \frac{9d-12}{13\cdot 8} > 0$. So $\frac{25d-3}{26} \in \left(\frac{7}{8}d, d\right)$. Hence by Proposition 4.3, we have

$$\Delta_d\left(\frac{25d-3}{26}\right) = 147d - 26 \cdot \frac{25d-3}{26} + 22 = 122d + 25 = \Delta_d\left(\frac{3}{2}d\right).$$

This completes the proof. □

4.2. The moduli space $\mathcal{M}_{K^3,p_g}$.

We can now prove the description of the moduli space of threefolds on the Noether line with $p_g \geq 11$.

Write $V_d(d_0) = \Phi_{d,d_0}(\mathcal{M}_d(d_0))$. Since Φ_{d,d_0} is always finite-to-one, we have $\dim V_d(d_0) = \Delta_d(d_0)$. Recall that d is a deformation invariant, so if the closures of $V_d(d_0)$ and $V_{d'}(d'_0)$ intersect, then $d = d'$.

Theorem 4.5. *For each $d \geq 5$, the moduli space $\mathcal{M}_{K^3,p_g}$ of the canonical threefolds with $p_g = 3d - 2$ and $K^3 = 4d - 6$ stratifies as the disjoint union of the unirational strata $V_d(d_0)$, where $d_0 \in \mathbb{N}$ and $\frac{1}{4}d \leq d_0 \leq \frac{3}{2}d$. Moreover,*

(1) *$V_d\left(\left\lfloor\frac{3}{2}d\right\rfloor\right)$ is dense in an irreducible component of $\mathcal{M}_{K^3,p_g}$.*
(2) *If $d_0 \geq d$, then $V_d(d_0)$ is contained in the closure of $V_d\left(\left\lfloor\frac{3}{2}d\right\rfloor\right)$.*
(3) *If $d_0 \leq \frac{25d-3}{26}$, then $V_d(d_0)$ is dense in an irreducible component of $\mathcal{M}_{K^3,p_g}$.*

Proof. Since we are assuming $d \geq 5$, by Corollary 3.5, the unirational subvarieties $V_d(d_0)$ stratify $\mathcal{M}_{K^3,p_g}$. Part (1) is [CP23, Proposition 2.2]. Part (2) has been proved in [CP23, Proposition 2.2 and 2.4] borrowing a technique from [Pig12].

It remains to prove (3). Arguing by contradiction, we assume the existence of an integer $d_0 \leq \frac{25d-3}{26}$ such that $V_d(d_0)$ is contained in the closure of $V_d(d'_0)$ for some $d'_0 \neq d_0$. In other words, for each $X = X(d; d_0)$ we have a flat family $\mathcal{X} \to \Lambda$ over a small open disc Λ with central fibre X and general fibre of type (d, d'_0).

We claim that $d_0' \geq d_0$. Otherwise, we have $d_0 > d_0' \geq 2$. Thus by the discussion before Proposition 2.9, the canonical image of X is a Hirzebruch surface $\mathbb{F}_{3d-2d_0}$. It follows that the relative canonical sheaf $\omega_{\mathcal{X}/\Lambda}$ induces a rational map $\mathcal{X}/\Lambda \dashrightarrow \mathcal{F}/\Lambda$ where $\mathcal{F}/\Lambda$ is a flat family of Hirzebruch surfaces, with central fibre isomorphic to $\mathbb{F}_{3d-2d_0}$ and general fibre isomorphic to $\mathbb{F}_{3d-2d_0'}$ (see [CP23, Proof of Theorem 5.4]). This implies that $d_0' \geq d_0$, which is a contradiction. The claim is proved.

On the other hand, if $V_d(d_0)$ is contained in the closure of $V_d(d_0')$, then $\Delta_d(d_0) < \Delta_d(d_0')$ which by Proposition 4.4 implies $d_0' < d_0$, a contradiction. This completes the proof. □

Now we are ready to proof Theorem 1.1.

Proof of Theorem 1.1. For $p_g \geq 11$, the non-emptiness of $\mathcal{M}_{K^3,p_g}$ follows from Proposition 2.3.

Suppose that $\mathcal{M}_{K^3,p_g}$ is non-empty. By Theorem 4.5, all $X(d;d_0)$ with $d_0 \geq d$ are in a single irreducible component, while the others may each be a different component. Note that all the possible irreducible components are unirational. So an upper bound for the number of irreducible components is the number of integers between $\frac{d}{4}$ and d, which is $\left\lfloor \frac{3}{4}d+1 \right\rfloor = \left\lfloor \frac{p_g+6}{4} \right\rfloor$.

Similarly, a lower bound is obtained by removing all the integers strictly bigger then $\frac{25d-3}{26}$ and strictly smaller than d. That is, we remove $\left\lfloor \frac{d+2}{26} \right\rfloor = \left\lfloor \frac{p_g+8}{78} \right\rfloor$ integers.

To prove the dimension formula, note that by Proposition 4.4, the stratum $V_d(d_0)$ with the maximal dimension is the one with $d_0 = \left\lceil \frac{d}{4} \right\rceil = \left\lceil \frac{p_g+2}{12} \right\rceil$. Hence the result follows from Proposition 4.3. □

Remark 4.6. Though the moduli space of canonical surfaces on the Noether line has at most two irreducible components, recently Rana and Rollenske [RR24] studied the moduli space of stable surfaces of general type on the Noether line, also obtaining several components.

4.3. **Final remark.** The statement of Theorem 4.5 does not say anything about the strata $V_d(d_0)$ with $\frac{25d-3}{26} < d_0 < d$, and there are $\left\lfloor \frac{d+2}{26} \right\rfloor = \left\lfloor \frac{p_g+8}{78} \right\rfloor$ of them. For these strata, the argument in the proof of Theorem 4.5 leaves two possibilities: either $V_d(d_0)$ is dense in an irreducible component of $\mathcal{M}_{K^3,p_g}$ or $V_d(d_0)$ is contained in the closure of $V_d\left(\left\lfloor \frac{3}{2}d \right\rfloor\right)$.

For numerical reasons, there is no such stratum when $p_g \leq 69$, (equivalently $d < 24$). The case $d = 24$ (so $p_g = 70$ and $K^3 = 90$) is the first case in which we cannot decide if a certain stratum is dense in an irreducible component or not. As an illustration, the dimensions $\Delta_{24}(d_0)$ of the relevant strata $V_{24}(d_0)$ of the moduli space $\mathcal{M}_{90,70}$ are given in Table 2.

In this case, we do not know whether $V_{24}(23)$, that has dimension 2952, is dense in an irreducible component of $\mathcal{M}_{90,70}$, or lies in the boundary of $V_{24}(36)$, whose dimension is 2953.

TABLE 2. Table of $\Delta_{24}(d_0)$

d_0	$h^0(D_z, 10H_{D_z})$	$\dim \mathrm{Aut}\, D_z$	$\Delta_{24}(d_0)$
36	3036	82	2953
35	3036	83	2952
34	3036	85	2950
33	3036	87	2948
⋮	⋮	⋮	⋮
25	3036	103	2932
24	3036	105	2930
23	3061	108	2952
22	3091	112	2978
21	3121	116	3004
⋮	⋮	⋮	⋮
8	3793	168	3624
7	3853	172	3680
6	3913	176	3736

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