QUOTIENTS OF PRODUCTS OF CURVES, NEW SURFACES WITH $p_g = 0$ AND THEIR FUNDAMENTAL GROUPS.

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Abstract. We construct many new surfaces of general type with $q = p_g = 0$ whose canonical model is the quotient of the product of two curves by the action of a finite group G, constructing in this way many new interesting fundamental groups which distinguish connected components of the moduli space of surfaces of general type.

We indeed classify all such surfaces whose canonical model is singular (the smooth case was classified in an earlier work).

As an important tool we prove a structure theorem giving a precise description of the fundamental group of quotients of products of curves by the action of a finite group G.

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Date: June 8, 2010.

After the few initial examples, constructed 'by hands' by the first two named authors, were presented at a workshop in Pisa in may 2006, the present work became a central project of the collaboration of the first three authors in the DFG Forschergruppe 790 'Classification of algebraic surfaces and compact complex manifolds'. In particular the visit of the fourth author to Bayreuth was supported by the DFG. Complete results were presented in february 2008 at Warwick University, then at Tokyo University, and at the Centro De Giorgi, Scuola Normale di Pisa; we thank these institutions for their hospitality and a referee for helpful comments.

† Fritz Grunewald passed away on March 21, 2010: we lost a very close friend and collaborator. We will miss him, his contagious enthusiasm for mathematics, his deep knowledge and his dear friendhsip.

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Introduction

The first main purpose of this paper is to contribute to the existing knowledge about the complex projective surfaces S of general type with $p_g(S) = 0$ and their moduli spaces, constructing 19 new families of such surfaces with hitherto unknown fundamental groups.

Minimal surfaces of general type with $p_g(S) = 0$ are known to have invariants $p_g(S) = q(S) = 0, 1 \leq K_S^2 \leq 9$, and to yield a finite number of irreducible components of the moduli space of surfaces of general type.

They represent for algebraic geometers a very difficult test case about the possibility of extending the Enriques classification of special surfaces to surfaces of general type.

They are also very interesting in view of the Bloch conjecture ([Blo75]), predicting that for surfaces with $p_g(S) = q(S) = 0$ the group of zero cycles modulo rational equivalence is isomorphic to $\mathbb{Z}$.

In this paper, using the beautiful results of Kimura ([Kim05], see also [GP03]), the present results, and those of the previous paper [BCG08], we produce more than 40 families of surfaces for which Bloch's conjecture holds.

Surfaces with $p_g(S) = q(S) = 0$ have a very old history, dating back to 1896 ([Enr96], see also [EnrMS], I, page 294) when Enriques constructed the so called Enriques surfaces in order to give a counterexample to the conjecture of Max Noether that any such surface should be rational.

The first surfaces of general type with $p_g = q = 0$ were constructed in the 1930' s by Luigi Campedelli and Lucien Godeaux (cf. [Cam32], [God35]): in their honour minimal surfaces of general type with $K_S^2 = 1$ are called numerical Godeaux surfaces, and those with $K_S^2 = 2$ are called numerical Campedelli surfaces.

In the 1970's there was a big revival of interest in the construction of these surfaces and in a possible classification.

After rediscoveries of these and other old examples a few new ones were found through the efforts of several authors, in particular Rebecca Barlow ([Bar85a]) found a simply connected numerical Godeaux surface, which played a decisive role in the study of the differential topology of algebraic surfaces and 4-manifolds (and also in the discovery of Kähler Einstein metrics of opposite sign on the same manifold, see [CatLB97]).

A (relatively short) list of the existing examples appeared in the book [BPV84], (see [BPV84], Vii, 10 and references therein, and see also [BHPV04] for an updated longer list).

There has been recently important progress on the topic, and the current situation is as follows:

- $K_S^2 = 9$: these surfaces have the unit ball in $\mathbb{C}^2$ as universal cover, and their fundamental group is an arithmetic subgroup Γ of $SU(2,1)$.

  This case seems to be completely classified through exciting new work of Prasad and Yeung and of Steger and Cartright ([PY07], [PY09]) asserting that the moduli space consists exactly of 50 pairs of complex conjugate surfaces.
- $K_S^2 = 8$: we pose the question whether in this case the universal cover must be the bidisk in $\mathbb{C}^2$.

  Assuming this, a complete classification should be possible.

  The classification has already been accomplished in [BCG08] for the reducible case where there is a finite étale cover which is isomorphic to a product of curves: in this case there are exactly 17 irreducible connected components of the moduli space.

  There are many examples, due to Kuga and Shavel ([Kug75], [Sha78]) for the irreducible case, which yield (as in the case $K_S^2 = 9$) rigid surfaces; but a complete classification of this second case is still missing.

The constructions of minimal surfaces of general type with $p_g = 0$ and with $K_S^2 \leq 7$ available in the literature (to the best of the authors' knowledge, and excluding the results of this paper and of the forthcoming [BP10]) are listed in table 1.

We proceed to a description, with the aim of putting the results of the present paper in proper perspective.

- $K_S^2 = 1$, i.e., *numerical Godeaux surfaces*: it is conjectured by Miles Reid that the moduli space should have exactly five irreducible connected components, distinguished by the fundamental group, which should be a cyclic group $\mathbb{Z}_m$ of order $1 \leq m \leq 5$ ([Rei78] settled the case where the order m of the first homology group is at least 3; [Bar85a], [Bar84] and [Wer94] were the first to show the occurrence of the two other groups).
- $K_S^2 = 2$, i.e., *numerical Campedelli surfaces*: here, it is known that the order of the algebraic fundamental group is at most 9, and the cases of order $8, 9$ have been classified by Mendes Lopes, Pardini and Reid ([MP08], [MPR08], [Rei]), who show in particular that the fundamental group equals the algebraic fundamental group and cannot be a dihedral group D_4 of order 8. Naie ([Nai99]) showed that the group D_3 of order 6 cannot occur as the fundamental group of a numerical Campedelli surface. By the work of Lee and Park ([LP07]), one knows that there exist simply connected numerical Campedelli surfaces.

Table 1. Minimal surfaces of general type with $p_g = 0$ and $K^2 \leq 7$ available in the literature

K^2	π_1	π_1^{alg}	H_1	References
1	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[God34][Rei78][Miy76]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[Rei78][OP81][Bar84][Nai94]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[Rei78]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Bar84][Ino94]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Wer94][Wer97]
	$\{1\}$	$\{1\}$	$\{0\}$	[Bar85a][LP07]
	?	$\{1\}$	$\{0\}$	[CG94][DW99]
2	$\mathbb{Z}_9$	$\mathbb{Z}_9$	$\mathbb{Z}_9$	[MP08]
	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	[Xia85][MP08]
	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	[Cam32][Rei][Pet76][Ino94] [Nai94]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[Rei][Nai94][Keu88]
	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	[Rei]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[Rei] [Beau96]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[Rei91]
	?	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[NP09]
	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[Cat81][Sup98]
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	[Ino94][Keu88]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[LP09]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[LP09]
	$\{1\}$	$\{1\}$	$\{0\}$	[LP07]
3	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[Nai94] [Keu88] [MP04a]
	$Q_8 \times \mathbb{Z}_2$	$Q_8 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$	[Bur66][Pet77] [Ino94]
	?	?	$\mathbb{Z}_2$	[PPS08a]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS07]
4	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	[Nai94][Keu88]
	$Q_8 \times \mathbb{Z}_2^2$	$Q_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	[Bur66][Pet77][Ino94]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS08b]
5	$Q_8 \times \mathbb{Z}_2^3$	$Q_8 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	[Bur66][Pet77][Ino94]
	?	?	?	[Ino94]
6	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^6$	[Bur66][Pet77][Ino94]
	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_3^3 \lhd H_1$	[Kul04]
	?	?	?	[Ino94][MP04b]
7	$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	?	[Ino94][MP01a] [BCC10]

Our first result here is the construction of two families of numerical Campedelli surfaces with fundamental group $\mathbb{Z}_3$. Recently Neves and Papadakis ([NP09]) constructed a numerical Campedelli surface with algebraic fundamental group $\mathbb{Z}_6$, while

Lee and Park ([LP09]) constructed one with algebraic fundamental group $\mathbb{Z}_2$, and one with algebraic fundamental group $\mathbb{Z}_3$ was added in the second version of the same paper.

Open conjectures are:

Conjecture 0.1. *Is the fundamental group $\pi_1(S)$ of a numerical Campedelli surface finite?*

Question 0.2. *Does every group of order ≤ 9 except D_4 and D_3 occur?*

The answer to question 0.2 is completely open for $\mathbb{Z}_4$; for $\mathbb{Z}_6, \mathbb{Z}_2$ one suspects that these fundamental groups are realized by the the Neves-Papadakis surfaces, respectively by the Lee-Park surfaces.

Note that the existence of the case where $\pi_1(S) = \mathbb{Z}_7$ is shown in the paper [Rei91] (where the result is slightly hidden).

- $K_S^2 = 3$: here there were two examples of non trivial fundamental groups, the first one due to Burniat and Inoue, the second one to Keum and Naie ([Bur66], [Ino94], [Keu88] [Nai94]).

  It is conjectured that for $p_g(S) = 0, K_S^2 = 3$ the algebraic fundamental group is finite, and one can ask as in 1) above whether also $\pi_1(S)$ is finite. Park, Park and Shin ([PPS07]) showed the existence of simply connected surfaces, and of surfaces with torsion $\mathbb{Z}_2$ ([PPS08a]).

  Other constructions were given in [Cat99], together with two more examples with $p_g(S) = 0, K^2 = 4, 5$: these turn out however to be the same as the Burniat surfaces.
- $K_S^2 = 4$: there were known up to now three examples of fundamental groups, the trivial one (Park, Park and Shin , [PPS08b]), a finite one, and an infinite one.

  We show in this paper the existence of 7 new groups, 3 finite and 4 infinite: thus minimal surfaces with $K_S^2 = 4$, $p_g(S) = q(S) = 0$ realize at least 10 distinct topological types.
- $K_S^2 = 5$: there was known up to now only one example of fundamental group, the one of the Burniat surfaces.
- $K_S^2 = 6$: there were known up to now two examples of fundamental groups, distinguished by their abelianization. We mention here (a proof will appear elsewhere) that the fundamental group of Kulikov's surface has a similar presentation to the one of the Burniat surfaces with $K_S^2 = 6$, namely

$$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1.$$

  We show in this paper the existence of 6 new groups, three of which finite: thus minimal surfaces with $K_S^2 = 6$, $p_g(S) = q(S) = 0$ realize at least 7 distinct topological types.

- $K_S^2 = 7$: there was known up to now only one example of such surfaces, constructed by Inoue in [Ino94].

  We shall show elsewhere ([BCC10]) that the fundamental group of these Inoue surfaces fits into an exact sequence

$$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1.$$

  This motivates the following further question

Question 0.3. *Is it true that fundamental groups of surfaces of general type with $q = p_g = 0$ are finite for $K_S^2 \leq 3$, and infinite for $K_S^2 \geq 7$?*

Let us observe that, as we just saw, for $4 \leq K_S^2 \leq 6$ both finite and infinite groups occur; however, there is no reason to exclude the existence of such surfaces with $K_S^2 = 7$ and with finite fundamental group.

In a forthcoming paper ([BP10]) the first and the fourth author shall show the existence, for $K_S^2 = 5$, of 7 new fundamental groups, 3 of which infinite, and also of 3 new finite fundamental groups for $K_S^2 = 4$, and of 4 new finite fundamental groups for $K_S^2 = 3$.

One of the reasons why we succeed in constructing so many new families of surfaces with $K_S^2 = 2, 4, 6$, thereby showing that 14 new fundamental groups are realized, is the fact that we use a systematic method and classify completely the following situation.

We consider the algebraic surfaces whose canonical models arise as quotients $X = (C_1 \times C_2)/G$ of the product $C_1 \times C_2$ of two curves of genera $g_1 := g(C_1), g_2 := g(C_2) \geq 2$, by the action of a finite group G. In other words, we make the restriction that X has only rational double points as singularities.

We achieve a complete classification of the surfaces X as above which have $p_g(X) = q(X) = 0$.

An interesting corollary of our classification is that then either

i) G acts freely, i.e., equivalently, X is smooth (hence $K_S^2 = 8$, and we are in the case previously classified in [BCG08]), or

ii) X has only nodes (A_1-singularities) as singular points: in this case moreover their number is even and equal to $8 - K_S^2$.

This result explains why our construction only produces new surfaces with $K_S^2 = 2, 4, 6$, as illustrated by the following three main theorems concerning surfaces with $p_g(S) = q(S) = 0$.

For convenience of notation we shall use the following

Definition 0.4. *A surface which is birational to the quotient of a product of curves by the action of a finite group will be called a* product-quotient *surface.*

A surface with rational double points which is the quotient of a product of curves by the action of a finite group will be called a canonical product-quotient *surface.*

Number 2 will mean: ‘ number 2 in the list given in Table 2’.

Theorem 0.5. *There exist eight families of canonical product-quotient surfaces yielding numerical Campedelli surfaces (i.e., minimal surfaces with $K_S^2 = 2, p_g(S) = 0$).*

Two of them (numbers 2 and 5, with group $G = \mathfrak{S}_5$, resp. $G = \mathfrak{S}_3 \times \mathfrak{S}_3$) yield fundamental group $\mathbb{Z}_3$.

Our classification also shows the existence of a family of canonical product-quotient surfaces yielding numerical Campedelli surfaces with fundamental group $\mathbb{Z}_5$ (but numerical Campedelli surfaces with fundamental group $\mathbb{Z}_5$ had already been constructed in [Cat81]), respectively with fundamental group $\mathbb{Z}_2^2$ (but such fundamental group already appeared in [Ino94]).

Theorem 0.6. *There exist precisely eleven families of canonical product-quotient surfaces yielding minimal surfaces with $K_S^2 = 4, p_g(S) = 0$.*

Three of these families (numbers 9, 12, 15) realize 3 new finite fundamental groups, $\mathbb{Z}_{15}$, $G(32, 2)$ (see rem. 5.10) and $(\mathbb{Z}_3)^3$.

Six of these families (numbers 8 and 14, 10 and 13, 11, 16) realize 4 new infinite fundamental groups.

The two families 17, 18 realize instead a fundamental group Γ which is isomorphic to the one of the surfaces constructed by Keum ([Keu88]) and later by Naie ([Nai94]). It was not clear to us whether these four families of surfaces all belonged to a unique irreducible component of the moduli space. This question has motivated (in the time between the first and the final version of this paper) work by the first two authors ([BC09a]): it is shown there a construction of Keum-Naie surfaces yielding a unique irreducible connected component of the moduli space, to which all minimal surfaces S with fundamental group Γ, $K_S^2 = 4, p_g(S) = 0$ belong, provided either

a) they are homotopically equivalent to a Keum-Naie surface, or

b) they admit a deformation to a surface with ample canonical divisor.

We come here to an important difference between the case of surfaces isogenous to a product of curves with $K_S^2 = 8, p_g(S) = 0$, classified in [BCG08], and the case classified in the present paper. Namely, unlike the case of surfaces isogenous to a product of curves, our product-quotient surfaces do not necessarily form a connected, or even an irreducible component of the moduli space. For instance, our families of product-quotient Keum-Naie surfaces form two subvarieties of respective codimensions 2,4 in the moduli space.

Actually, by the Enriques-Kuranishi inequality for the number of moduli $M(S)$ (the local dimension of the moduli space), $M(S) \geq 10\chi(S) - 2K_S^2$, our families number 1-10 and 14 necessarily do not contain an open set of the moduli space.

It is then an interesting question to see first of all whether the nodes of our product-quotient surfaces can be smoothed, or even independently smoothed (see [Cat89] for the consequences about singularities of the moduli space), and more generally to investigate the local structure of the moduli space at the sublocus corresponding to our families of product-quotient surfaces.

A second harder problem is to try to describe the irreducible (resp.: connected) components of the moduli space containing these subloci.

Still referring to the previously shown table 1, this is possible in some cases. Note first of all that Mendes Lopes and Pardini ([MP01b]) proved that the 'primary' Burniat surfaces (those with $K^2 = 6$) form a connected component of the moduli space. This result has been reproven in the meantime by the first two authors in [BC09b], showing more generally that any surface homotopically equivalent to a primary Burniat surface is indeed a primary Burniat surface.

The positive results of [BC09c], showing that also Burniat surfaces with $K_S^2 = 5, 4$ determine 3 connected components of the moduli space, seem to depend heavily on the fact that these surfaces are realized as finite Galois covers with group $(\mathbb{Z}_2)^2$ of Del Pezzo surfaces.

In other words, our constructions offer several interesting challenges concerning the investigation of the moduli space, but these cannot be solved all at once, and for each family of surfaces one has to resort very much to the special geometric properties which the surfaces possess, and which are not being lost by deformation.

The next case is all the more challenging, since the Enriques-Kuranishi inequality gives no information.

For instance a referee asked the question whether the surfaces corresponding to the numbers 19-20-21 are rigid surfaces.

Theorem 0.7. *There exist eight families (numbers 19-24) of canonical product-quotient surfaces yielding minimal surfaces with* $K_S^2 = 6, p_g(S) = 0$ *and realizing 6 new fundamental groups, three of them finite and three of them infinite.*

In particular, there exist minimal surfaces of general type with $p_g = 0$, $K^2 = 6$ *and with finite fundamental group.*

Since a main new contribution of our paper is the calculation of new fundamental groups (as opposed to new algebraic fundamental groups), our second main purpose is to describe in greater generality the fundamental groups of smooth projective varieties which occur as the minimal resolutions of the quotient of a product of curves by the action of a finite group.

More precisely, in this paper we are concerned with the following rather (but not completely) general situation.

Let $C_1, \dots, C_n$ be smooth projective curves of respective genera $g_i \geq 2$ and suppose that there is a finite group G acting faithfully on each of the n curves.

Then we consider the diagonal action of G on the Cartesian product $C_1 \times \dots, C_n$, and the quotient $X := (C_1 \times \dots \times C_n)/G$.

Recall that if the action of G on $C_1 \times \dots \times C_n$ is free, then X is said to be a *variety isogenous to a (higher) product of curves (of unmixed type)*. These varieties were introduced and studied by the second author in [Cat00], mainly in the case $n = 2$. In this case the universal cover of X is the product of n copies of the upper half plane $\mathbb{H} \times \dots \times \mathbb{H}$.

We drop here the hypothesis that G acts freely on $C_1 \times \dots \times C_n$. Then X has singularities, but since they are cyclic quotient singularities, they can be resolved, in the case where they are isolated singularities, by a simple normal crossing divisor whose components are smooth rational varieties ([Fuj74]). By van Kampen's theorem this implies that the fundamental group of X is equal to the fundamental group of a minimal desingularisation S of X.

One of the preliminary observations of [Cat00] was the following:

Proposition 0.8. *Let $X := (C_1 \times \dots \times C_n)/G$ be isogenous to a product as above. Then the fundamental group of X sits in an exact sequence*

$$1 \to \Pi_{g_1} \times \dots \times \Pi_{g_n} \to \pi_1(X) \to G \to 1,$$

where $\Pi_{g_i} := \pi_1(C_i)$, and this extension is determined by the associated maps $G \to \mathfrak{Map}_{g_i} := Out(\Pi_{g_i})$ to the respective Teichmüller modular groups.

If one drops the assumption about the freeness of the action of G on $C_1 \times \dots \times C_n$, there is no reason that the behaviour of the fundamental group of the quotient should be similar to the above situation. Nevertheless it turns out that, as an abstract group, the fundamental group admits a very similar description.

Before giving the main result of the first part of our paper, which is a structure theorem for the fundamental group of $X = (C_1 \times \dots \times C_n)/G$, we need the following

Definition 0.9. *We shall call the fundamental group $\Pi_g := \pi_1(C)$ of a smooth compact complex curve of genus g a* (genus g) surface group.

Note that we admit also the "degenerate cases" $g = 0, 1$.

Theorem 0.10. *Let $C_1, \dots, C_n$ be compact complex curves of respective genera $g_i \geq 2$ and let G be a finite group acting faithfully on each C_i as a group of biholomorphic transformations.*

Let $X = (C_1 \times \dots \times C_n)/G$, and denote by S a minimal desingularisation of X. Then the fundamental group $\pi_1(X) \cong \pi_1(S)$ has a normal subgroup $\mathcal{N}$ of finite index which is isomorphic to the product of surface groups, i.e., there are natural numbers $h_1, \dots, h_n \geq 0$ such that $\mathcal{N} \cong \Pi_{h_1} \times \dots \times \Pi_{h_n}$.

Remark 0.11. In the case of dimension $n = 2$ there is no loss of generality in assuming that G acts faithfully on each C_i (see [Cat00]). In the general case there will be a group G_i, quotient of G, acting faithfully on C_i, hence the strategy should slightly be changed in the general case.

Remark 0.12. The fundamental groups of certain 3-manifolds which arise as quotients of hyperbolic 3-space by groups which act discontinuously but not necessarily fixpoint freely are discussed in [GrMe80]. Unlike in the case treated the present paper, here the groups do not contain CAT(0) groups of finite index.

We shall now give a short description of the proof of theorem 0.10 in the case $n = 2$. The case where n is arbitrary is exactly the same.

We need to recall the definition of an orbifold surface group

Definition 0.13. *An* orbifold surface group *of genus g' and multiplicities $m_1, \dots m_r$ is the group presented as follows:*

$$\mathbb{T}(g'; m_1, \dots, m_r) := \langle a_1, b_1, \dots, a_{g'}, b_{g'},\ c_1, \dots, c_r | \\ c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{g'} [a_i, b_i] \cdot c_1 \cdot \ldots \cdot c_r \rangle.$$

In the case $g' = 0$ it is called a polygonal group.

The sequence $(g'; m_1, \dots m_r)$ is called the signature of the orbifold surface group.

The above definition shows that an orbifold fundamental group is the factor group of the fundamental group of the complement, in a complex curve C' of genus g', of a finite set of r points $\{p_1, \dots, p_r\}$, obtained by dividing modulo the normal subgroup generated by $\gamma_1^{m_1}, \dots, \gamma_r^{m_r}$, where for each i γ_i is a simple geometric loop starting from the base point and going once around the point p_i counterclockwise (cf. [Cat00]).

Hence, by Riemann's existence theorem, the action of a finite group G on a curve C of genus $g \geq 2$ is determined by the following data:

1) the quotient curve $C' := C/G$

2) the branch point set $\{p_1, \dots p_r\} \subset C'$

3) a surjection of the fundamental group $\pi_1(C' \setminus \{p_1, \dots p_r\})$ onto $\mathbb{T}(g'; m_1, \dots, m_r)$, such that the given generators of $\mathbb{T}(g'; m_1, \dots, m_r)$ are image elements of a *standard basis* of $\pi_1(C' \setminus \{p_1, \dots p_r\})$.

This means that $a_1, b_1, \dots, a_{g'}, b_{g'}$ correspond to a symplectic basis of the fundamental group of C', while each c_i is the image of a simple geometric loop around the point p_i.

4) an *appropriate* homomorphism

$$\varphi \colon \mathbb{T}(g'; m_1, \dots, m_r) \to G,$$

i.e., a surjective homomorphism such that

5) $\varphi(c_i)$ is an element of order exactly m_i and
6) *Hurwitz' formula* holds:

$$2g-2=|G|\left(2g'-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right).$$

Therefore in our situation we have two surjective homomorphisms

$$\varphi_1\colon \mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \ldots, m_r) \to G,$$
$$\varphi_2\colon \mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \ldots, n_s) \to G.$$

We define the fibre product $\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2)$ as

(1) $$\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2) := \{\, (x, y) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid \varphi_1(x) = \varphi_2(y) \,\}.$$

Then the exact sequence

(2) $$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{T}_1 \times \mathbb{T}_2 \to G \times G \to 1,$$

where $\Pi_{g_i} := \pi_1(C_i)$, induces an exact sequence

(3) $$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{H}(G; \varphi_1, \varphi_2) \to G \cong \Delta_G \to 1.$$

Here $\Delta_G \subset G \times G$ denotes the diagonal subgroup.

Definition 0.14. Let H be a group. Then its *torsion subgroup* $\mathrm{Tors}(H)$ is the normal subgroup generated by all elements of finite order in H.

The first observation is that one can calculate our fundamental groups via a simple algebraic recipe: $\pi_1((C_1 \times C_2)/G) \cong \mathbb{H}(G; \varphi_1, \varphi_2)/\mathrm{Tors}(\mathbb{H})$.

In this algebraic setup, we need to calculate $\mathbb{H}(G; \varphi_1, \varphi_2)/\mathrm{Tors}(\mathbb{H})$. Our proof is rather indirect, and we have no direct geometric construction leading to our result (even if we can then explain its geometric meaning).

Our strategy is now the following: using the structure of orbifold surface groups we construct an exact sequence

$$1 \to E \to \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to \Psi(\hat{\mathbb{H}}) \to 1,$$

where

i) E is finite,
ii) $\Psi(\hat{\mathbb{H}})$ is a subgroup of finite index in a product of orbifold surface groups.

Condition *ii*) implies that $\Psi(\hat{\mathbb{H}})$ is residually finite and "good" according to the following

Definition 0.15 (J.-P. Serre)**.** *Let $\mathbb{G}$ be a group, and let $\tilde{\mathbb{G}}$ be its profinite completion. Then $\mathbb{G}$ is said to be* good *iff the homomorphism of cohomology groups*

$$H^k(\tilde{\mathbb{G}}, M) \to H^k(\mathbb{G}, M)$$

is an isomorphism for all $k \in \mathbb{N}$ and for all finite $\mathbb{G}$ - modules M.

Then we use the following result due to F. Grunewald, A. Jaikin-Zapirain, P. Zalesski.

Theorem 0.16. ([GJZ08]) *Let G be residually finite and good, and let $\varphi\colon H \to G$ be surjective with finite kernel. Then H is residually finite.*

The above theorem implies that $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$ is residually finite, whence there is a subgroup $\Gamma \leq \mathbb{H}/\mathbf{Tors}(\mathbb{H})$ of finite index such that

$$\Gamma \cap E = \{1\}.$$

Now, $\Psi(\Gamma)$ is a subgroup of $\Psi(\hat{\mathbb{H}})$ of finite index, whence of finite index in a product of orbifold surface groups, and $\Psi|\Gamma$ is injective. This easily implies our result.

In the second part of the paper, as already explained at length, we use the above results in order to study the moduli spaces of surfaces S of general type with $p_g(S) = q(S) = 0$.

While the investigation of the moduli space of surfaces S of general type with $p_g(S) = q(S) = 0$ and $K_S^2 \leq 7$ is (as our research indicates) an extremely difficult task, for instance since there could be over a hundred irreducible components, it makes sense to try first to ask some 'easier' problems:

Problem 0.17. *Determine the pairs $(K_S^2, \pi_1(S))$ for all the surfaces with $p_g(S) = q(S) = 0$.*

Note that the pair $(K_S^2, \pi_1(S))$ is an invariant of the connected component of the moduli space, hence we know that there are only a finite number of such pairs.

Several question marks in our previous table would be removed by a positive answer to the following

Conjecture 0.18. *The fundamental groups $\pi_1(S)$ of surfaces with $p_g(S) = q(S) = 0$ are residually finite.*

Remark 0.19. There are algebraic surfaces with non residually finite fundamental groups, as shown by Toledo in [Tole93] (see also [Cat-Kol92]), answering thus a question attributed to J.P. Serre.

Since in both types of examples one takes general hyperplane sections of varieties of general type, then necessarily we get surfaces with $p_g > 0$. Evidence for the conjecture is provided by the existing examples, but a certain wishful thinking is also involved on our side.

One could here ask many more questions (compare [Rei79]), but as a beginning step it is important to start providing several examples.

Here we give a contribution to the above questions, determining the fundamental groups of the surfaces whose canonical models are the quotient of a product of curves by the action of a finite group.

This follows from the following complete classification result:

Theorem 0.20. *All the surfaces $X := (C_1 \times C_2)/G$, where G is a finite group with an unmixed action on a product $C_1 \times C_2$ of smooth projective curves C_1, C_2 of respective genera $g_1, g_2 \geq 2$ such that:*

i) *X has only* rational double points *as singularities,*

ii) $p_g(S) = q(S) = 0$

are obtained by a pair of appropriate epimorphisms of polygonal groups $\mathbb{T}_1, \mathbb{T}_2$ to a finite group G as listed in table 2, for an appropriate choice of respective branch sets in $\mathbb{P}^1$.

Let us briefly illustrate the strategy of the proof of the above theorem. The first step is to show that, under the above hypotheses, the singularities of X are either $0, 2, 4$ or 6 nodes (A_1-singularities). And, according to the number of nodes, $K_X^2 = K_S^2 \in \{8, 6, 4, 2\}$. Since all surfaces isogenous to a product (i.e., with 0 nodes) and with $p_g(S) = q(S) = 0$ have been completely classified in [BCG08], we restrict ourselves here to the case of 2, 4, resp. 6 nodes.

Note furthermore that the assumption $q = 0$ implies that C_i/G is rational, i.e., $g'_1 = g'_2 = 0$.

We know that X determines the following data

- a finite group G,
- two polygonal groups $\mathbb{T}_1 := \mathbb{T}(0; m_1, \dots, m_r)$, $\mathbb{T}_2 := \mathbb{T}(0; n_1, \dots, n_s)$, of respective signatures $T_1 = (m_1, \dots, m_r)$, $T_2 = (n_1, \dots, n_s)$
- two appropriate homomorphisms (i.e., surjective and preserving the order of the generators) $\varphi_i \colon \mathbb{T}_i \to G$, such that the stabilizers fulfill certain conditions ensuring that X has only $2, 4$ or 6 nodes respectively.

Our second main result, summarized in table 2, contains moreover the description of the first homology group $H_1(S, \mathbb{Z})$ (the abelianization of $\pi_1(S)$), and an indication of the number of irreducible families that we construct in each case.

Note that, in order to save space in table 2, we have used a particular notation for the signatures: for example in the first row 4^3 stands for $(4, 4, 4)$. A more precise explanation of the table is to be found in subsection 5.4, including the definition of all groups listed in remark 5.10. More details on the constructed surfaces are in section 6.

Let's spend here some words in order to give an idea of the methods used for the proof.

First of all we use the combinatorial restriction imposed by the further assumption $p_g(S) = 0$, and the conditions on the singularities of X.

This allows, for each possible value of $K^2 := K_S^2$, to restrict to a finite list of possible signatures T_1, T_2 of the respective polygonal groups.

A finite but rather big list is provided by lemma 5.9; moreover a MAGMA ([BCP97]) script allows us to shorten the list to at most 24 signatures for each value of K^2.

TABLE 2. The surfaces

	K^2	T_1	T_2	g_1	g_2	G	# fams	H_1	$\pi_1(S)$
1)	2	2, 3, 7	4^3	3	22	PSL(2,7)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2)	2	2, 4, 5	2, 6^2	4	11	$\mathfrak{S}_5$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
3)	2	2, 5^2	2^3, 3	4	6	$\mathfrak{A}_5$	1	$\mathbb{Z}_5$	$\mathbb{Z}_5$
4)	2	2, 4, 6	2^3, 4	3	7	$\mathfrak{S}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
5)	2	2, 6^2	2^3,3	4	4	$\mathfrak{S}_3 \times \mathfrak{S}_3$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
6)	2	4^3	4^3	3	3	$\mathbb{Z}_4^2$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
7)	2	2^3, 4	2^3, 4	3	3	$D_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
8)	4	2, 4, 5	3, 6^2	4	21	$\mathfrak{S}_5$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$
9)	4	2, 5^2	2^2, 3^2	4	11	$\mathfrak{A}_5$	1	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$
10)	4	2, 4, 6	2^2, 4^2	3	13	$\mathfrak{S}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$
11)	4	2, 4, 6	2^5	3	13	$\mathfrak{S}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$
12)	4	2^3, 4	2^3, 4	5	5	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	1	$\mathbb{Z}_4^2$	$G(32, 2)$
13)	4	3, 4^2	2^5	3	7	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$
14)	4	3, 6^2	2^2, 3^2	4	4	$\mathfrak{S}_3 \times \mathbb{Z}_3$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$
15)	4	2^2, 3^2	2^2, 3^2	4	4	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$	1	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$
16)	4	2^3, 4	2^5	3	5	$D_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \hookrightarrow \pi_1 \twoheadrightarrow D_4$
17)	4	2^2, 4^2	2^2, 4^2	3	3	$\mathbb{Z}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}_2^2$
18)	4	2^5	2^5	3	3	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}_2^2$
19)	6	2, 5^2	3^2, 4	19	16	$\mathfrak{A}_6$	2	$\mathbb{Z}_{15}$	$\mathfrak{A}_4 \times \mathbb{Z}_5$
20)	6	2, 4, 6	2, 4, 10	11	19	$\mathfrak{S}_5 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathfrak{S}_3 \times D_{4,5,-1}$
21)	6	2, 7^2	3^2, 4	19	8	PSL(2,7)	2	$\mathbb{Z}_{21}$	$\mathfrak{A}_4 \times \mathbb{Z}_7$
22)	6	2, 5^2	2, 3^3	4	16	$\mathfrak{A}_5$	1	$\mathbb{Z}_3 \times \mathbb{Z}_{15}$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_{15}$
23)	6	2, 4, 6	2^4, 4	3	19	$\mathfrak{S}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$\Pi_2 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4$
24)	6	2^3, 4	2^4, 4	3	7	$D_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$\mathbb{Z}^2 \times \Pi_2 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}_2^2$

The order of G is now determined by T_1, T_2 and by K^2: it follows that there are only finitely many groups to consider.

A second MAGMA script computes, for each K^2, all possible triples (T_1, T_2, G), where G is a quotient of both polygonal groups (of respective signatures T_1, T_2) and has the right order. Note that our code skips a few pairs of signatures giving rise to groups of large order, either not covered by the MAGMA SmallGroup database, or causing extreme computational complexity. These cases left out by our program are then excluded via a case by case argument.

For each of the triples (T_1, T_2, G) which we have then found there are the corresponding pairs of surjections each giving a family of surfaces of the form $(C_1 \times C_2)/G$. Some of these surfaces have singularities which violate the hypotheses of theorem 0.20.

A third MAGMA script produces the final list of surfaces, discarding the ones which are too singular.

A last script calculates, using prop. 3.4, the fundamental groups.

As we already remarked, in the case $K_S^2 = 2$ (of the so called *numerical Campedelli surfaces*) all fundamental groups that we get are finite;

but for $K_S^2 = 4, 6$ both cases occur: finite and infinite fundamental groups.

The case of infinite fundamental groups is the one where the structure theorem proven in the first part of the paper turns out to be extremely helpful to give an explicit description of these groups (since in general a presentation of a group does not say much about it).

1. Notation

In this section we collect the notation we use throughout the paper.

G: a finite group.

For lack of space in the tables we denote by $\mathbb{Z}_d := \mathbb{Z}/d = \mathbb{Z}/d\mathbb{Z}$ the cyclic group of order d, $\mathfrak{S}_n$ is the symmetric group in n letters, $\mathfrak{A}_n$ is the alternating group.

Q_8 is the quaternion group of order 8.

$PSL(2,7)$ is the group of 2×2 matrices over $\mathbb{F}_7$ with determinant 1 modulo the subgroup generated by $-Id$.

$D_{p,q,r} = \langle x, y | x^p, y^q, xyx^{-1}y^{-r} \rangle$, and $D_n = D_{2,n,-1}$ is the usual dihedral group of order $2n$.

$G(32,2)$ is for instance the second group of order 32 in the MAGMA database.

C_i: a smooth compact (connected) curve of genus $g_i \geq 2$.

Given natural numbers $g \geq 0$; $m_1, \ldots, m_r \geq 1$ the *orbifold surface group* of *signature* $(g; m_1, \ldots, m_r)$ is defined as follows:

$$\mathbb{T}(g; m_1, \ldots, m_r) := \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r | \\ c_1^{m_1}, \ldots, c_r^{m_r}, \prod_{i=1}^{g} [a_i, b_i] \cdot c_1 \cdot \ldots \cdot c_r \rangle.$$

For $g = 0$ we get the *polygonal group*

$$\mathbb{T}(0; m_1, \ldots, m_r) = \langle c_1, \ldots, c_r \mid c_1^{m_1}, \ldots, c_r^{m_r}, c_1 \cdot \ldots \cdot c_r \rangle. \tag{4}$$

For $r = 0$ we get the *surface group* of genus g

$$\Pi_g := \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

An *appropriate orbifold homomorphism* $\varphi \colon \mathbb{T}(g'; m_1, \ldots, m_r) \to G$ is a surjective homomorphism such that $\gamma_i := \varphi(c_i)$ has order *exactly* m_i.

Given two surjective homomorphisms

$$\varphi_1 \colon \mathbb{T}_1 \to G, \;\; \varphi_2 \colon \mathbb{T}_2 \to G.$$

we define the fibre product $\mathbb{H}$ as

$$\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2) := \{ (x, y) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid \varphi_1(x) = \varphi_2(y) \}. \tag{5}$$

2. Finite group actions on products of curves

The following facts will be frequently used without explicit mention in the subsequent chapters.

The following is a reformulation of *Riemann's existence theorem*:

Theorem 2.1. *A finite group G acts as a group of automorphisms on a compact Riemann surface C of genus g if and only if there are natural numbers $g', m_1, \ldots, m_r$, and an appropriate orbifold homomorphism*

$$\varphi\colon \mathbb{T}(g'; m_1, \ldots, m_r) \to G$$

such that the Riemann - Hurwitz relation holds:

$$2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

If this is the case, then g' is the genus of $C' := C/G$. The G-cover $C \to C'$ is branched in r points $p_1, \ldots, p_r$ with branching indices $m_1, \ldots, m_r$, respectively.

Moreover, if we denote by $\gamma_i \in G$ the image of c_i under φ, then

$$\Sigma(\gamma_1, \ldots, \gamma_r) := \cup_{a \in G} \cup_{i=0}^{\infty} \{a\gamma_1^i a^{-1}, \ldots a\gamma_r^i a^{-1}\},$$

is the set of stabilizers for the action of G on C.

Assume now that there are two homomorphisms

$$\varphi_1\colon \mathbb{T}(g_1'; m_1, \ldots, m_r) \to G,$$

$$\varphi_2\colon \mathbb{T}(g_2'; n_1, \ldots, n_s) \to G,$$

determined by two Galois covers $\lambda_i\colon C_i \to C_i'$, $i = 1, 2$.
We will assume in the following that $g(C_1),\ g(C_2) \geq 2$, and we consider the diagonal action of G on $C_1 \times C_2$.

If G acts freely on $C_1 \times C_2$, then $S := (C_1 \times C_2)/G$ is smooth and is said to be *isogenous to a product*. These surfaces were introduced and extensively studied by the second author in [Cat00], where the following crucial *weak rigidity* of surfaces isogenous to a product was proved (see also [Cat03]).

Theorem 2.2. *Let $S = C_1 \times C_2/G$ be a surface isogenous to a product, $g(C_1),\ g(C_2) \geq 2$. Then every surface with the same*

- *topological Euler characteristic and*
- *fundamental group*

is diffeomorphic to S. The corresponding moduli space $\mathcal{M}_S^{top} = \mathcal{M}_S^{diff}$ of surfaces (orientedly) homeomorphic (resp. diffeomorphic) to S is either irreducible and connected or consists of two irreducible connected components exchanged by complex conjugation.

In particular, any flat deformation of a surface isogenous to a product is again isogenous to a product. Observe that this property does not hold any longer if the action is not free.

Moreover, the fundamental group of $S = C_1 \times C_2/G$ sits inside an exact sequence

$$1 \to \pi_1(C_1) \times \pi_1(C_2) \to \pi_1(X) \to G \to 1.$$

This extension is determined by the associated maps to the Teichmüller modular groups.

Remark 2.3. In [BC04] and [BCG08] a complete classification of surfaces S isogenous to a product with $p_g(S) = q(S) = 0$ is given.

In this paper we drop the condition that the action of G on $C_1 \times C_2$ is free and we are mainly interested in the following two questions:

- what is the fundamental group of $X := (C_1 \times C_2)/G$?
- is it still possible to classify these quotients under suitable restrictions on the invariants of a minimal resolution of the singularities of X?

Remark 2.4. If the diagonal action of G on $C_1 \times C_2$ is not free, then G has a finite set of fixed points. The quotient surface $X := (C_1 \times C_2)/G$ has a finite number of (finite) cyclic quotient singularities, which are rational singularities.

Since, as we will shortly recall, the minimal resolution $S \to X$ of the singularities of X replaces each singular point by a tree of smooth rational curves, we have, by van Kampen's theorem, that $\pi_1(X) = \pi_1(S)$.

Note that by a result of A. Fujiki (cf. [Fuj74]), the exceptional divisors of a resolution of the singularities $\tilde{X}$ of $(C_1 \times \ldots \times C_n)/G$ consists, in the case where the singularities are isolated points, of a union of irreducible rational varieties intersecting with simple normal crossings. Therefore

$$\pi_1((C_1 \times \ldots \times C_n)/G) \cong \pi_1(\tilde{X}).$$

Remark 2.5. 1) Assume that $x \in X$ is a singular point. Then it is a cyclic quotient singularity of type $\frac{1}{n}(1, a)$ with $g.c.d(a, n) = 1$, i.e., X is locally around x the quotient of $\mathbb{C}^2$ by the action of a diagonal linear automorphism with eigenvalues $\exp(\frac{2\pi i}{n})$, $\exp(\frac{2\pi i a}{n})$. This follows since the tangent representation is faithful on both factors.

The particular case where $a = -1$, i.e., the stabilizer has a tangent representation with determinant $= 1$, is precisely the case where the singularity is a RDP (Rational Double Point) of type A_{n-1}.

2) We denote by K_X the canonical (Weil) divisor on the normal surface corresponding to $i_*(\Omega^2_{X^0})$, $i\colon X^0 \to X$ being the inclusion of the smooth locus of X. According to Mumford we have an intersection

product with values in $\mathbb{Q}$ for Weil divisors on a normal surface, and in particular we consider the selfintersection of the canonical divisor,

$$K_X^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} \in \mathbb{Q}, \tag{6}$$

which is not necessarily an integer.

K_X^2 is however an integer (equal indeed to K_S^2) if X has only RDP's as singularities (cf. [Rei87]).

The resolution of a cyclic quotient singularity of type $\frac{1}{n}(1,a)$ with $g.c.d(a,n)=1$ is well known. These singularities are resolved by the so-called *Hirzebruch-Jung strings*. More precisely, let $\pi\colon S \to X$ be a minimal resolution of the singularities and let $E = \bigcup_{i=1}^m E_i = \pi^{-1}(x)$. Then E_i is a smooth rational curve with $E_i^2 = -b_i$ and $E_i \cdot E_{i+1} = 1$ for $i \in \{1, \dots, m-1\}$ and zero otherwise. The b_i's are given by the formula

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}.$$

Moreover, we have (around x)

$$K_S = \pi^* K_X + \sum_{i=1}^m a_i E_i,$$

where the rational numbers a_i are determined by the conditions

$$(K_S + E_j)E_j = -2, \quad (K_S - \sum_{i=1}^m a_i E_i)E_j = 0, \quad \forall j = 1, \dots, m.$$

The *index* r of the singularity x is now given by

$$r = \min\{\lambda \in \mathbb{N} | \lambda a_i \in \mathbb{Z}, \ \forall i = 1, \dots, m\}.$$

Observe that the above formulae allow to calculate the self intersection number of the canonical divisor K_S of a minimal resolution of the singularities of X.

The next result gives instead a formula for the topological Euler characteristic $e(S)$ of a minimal resolution S of singularities of X.

Proposition 2.6. *Assume that $\{p_1, \dots p_k\}$ are the cyclic quotient singularities of $X = C_1 \times C_2/G$, of respective types $\frac{1}{n_i}(1,a_i)$. Let $p\colon C_1 \times C_2 \to X$ be the quotient morphism and let S be a minimal resolution of the singularities of X. Denote by l_i the length of the resolution tree of the singularity p_i. Then*

$$e(S) = \frac{K_X^2}{2} - \frac{|p^{-1}(\{p_1, \dots p_k\})|}{|G|} + \sum_{i=1}^k (l_i + 1).$$

Proof. Let $X^* := X \setminus \{p_1, \dots, p_k\}$. Then, using the additivity of the Euler number for a stratification given by orientable manifolds (see e.g. [Cat00]), we obtain

$$e(S) = e(X^*) + \sum_{i=1}^{k}(2l_i - (l_i - 1)) = e(X^*) + \sum_{i=1}^{k}(l_i + 1).$$

Let $Z^* := (C_1 \times C_2) \setminus p^{-1}(\{p_1, \dots p_k\})$: then $p|Z^* \colon Z^* \to X^*$ is an étale Galois covering with group G. Therefore:

$$e(X^*) = \frac{e(Z^*)}{|G|} = \frac{e(C_1 \times C_2) - |p^{-1}(\{p_1, \dots p_k\})|}{|G|} =$$
$$\frac{4(g(C_1) - 1)(g(C_2) - 1) - |p^{-1}(\{p_1, \dots p_k\})|}{|G|} = \frac{K_X^2}{2} - \frac{|p^{-1}(\{p_1, \dots p_k\})|}{|G|}.$$

□

An immediate consequence of the previous proposition is the following:

Corollary 2.7. *Assume that the singular points $p_1, \dots p_k$ of $X = (C_1 \times C_2)/G$ are ordinary double points (i.e., A_1 singularities). Let S be a minimal resolution of the singularities of X. Then*

$$e(S) = \frac{K_X^2}{2} + \frac{3}{2}k.$$

Proof. Here, for each i, $l_i = 1$ and $|p^{-1}(p_i)| = \frac{|G|}{2}$.

□

3. From the geometric to the algebraic set up for calculating the fundamental group.

Assume that $X = (C_1 \times C_2)/G$, where G is a finite group of automorphisms of each factor C_i and acts diagonally on $C_1 \times C_2$ (for short: the action of G on $C_1 \times C_2$ is *unmixed*).

Definition 3.1. *Let X be as above and consider the minimal resolution S of the singularities of X. The holomorphic map $f_1 \colon S \to C_1' := C_1/G$ is called a* standard isotrivial fibration *if it is a relatively minimal fibration.*

In the general case one lets $f' \colon S' \to C_1'$ be the relatively minimal model of f_1, and says that f' is an isotrivial fibration.

As already observed in remark 2.4, we have $\pi_1(X) = \pi_1(S)$.

The aim is now to determine the fundamental group of X in terms of the following algebraic data:
i) the group G together with
ii) the two surjective homomorphisms

$$\varphi_1 \colon \mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \dots, m_r) \to G, \quad \varphi_2 \colon \mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \dots, n_s) \to G.$$

Remark 3.2. The surjectivity of the homomorphisms φ_1 and φ_2 implies that for each $h_1 \in \mathbb{T}_1$ there exists an element $h_2 \in \mathbb{T}_2$ such that $(h_1, h_2) \in \mathbb{H}$, where $\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2)$ is the fibre product as defined in (5).

The exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{T}_1 \times \mathbb{T}_2 \to G \times G \to 1, \tag{7}$$

where $\Pi_{g_i} := \pi_1(C_i)$, induces an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{H} \to G \cong \Delta_G \to 1. \tag{8}$$

Here $\Delta_G \subset G \times G$ denotes the diagonal.

Definition 3.3. Let H be a group. Then its *torsion subgroup* $\mathrm{Tors}(H)$ is the (normal) subgroup generated by all elements of finite order in H.

We have the following

Proposition 3.4. *Let G be a finite group and let*

$$\varphi_1 \colon \mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \ldots, m_r) \to G, \quad \varphi_2 \colon \mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \ldots, n_s) \to G$$

be two surjective homomorphisms. Consider the induced action of G on $C_1 \times C_2$.

Then $\pi_1((C_1 \times C_2)/G) \cong \mathbb{H}/\mathrm{Tors}(\mathbb{H})$.

Proof. It follows from the main theorem of [Arm65], [Arm68] since the elements of finite order are precisely those elements of $\mathbb{H}$ which have fixed points.

□

4. The structure theorem for fundamental groups of quotients of products of curves

The aim of this section is to prove the following result

Theorem 4.1. *Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ be orbifold surface groups and assume that there are surjective homomorphisms $\varphi_i \colon \mathbb{T}_i \to G$ to a finite group G. Let*

$$\begin{aligned} \mathbb{H} &:= \mathbb{H}(G; \varphi_1, \ldots, \varphi_n) \\ &:= \{ (x_1, \ldots, x_n) \in \mathbb{T}_1 \times \ldots \times \mathbb{T}_n \mid \varphi_1(x_1) = \ldots = \varphi_n(x_n) \} \end{aligned}$$

be the fibre product of $\varphi_1, \ldots, \varphi_n$.

Then $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$ has a normal subgroup $\mathcal{N}$ of finite index isomorphic to the product of surface groups $\Pi_{h_1} \times \ldots \times \Pi_{h_n}$.

In particular, we have an exact sequence

$$1 \to \Pi_{h_1} \times \ldots \times \Pi_{h_n} \to \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to G' \to 1.$$

where G' is a finite group.

Remark 4.2. If G acts freely on $C_1 \times \ldots \times C_n$, or in other words, if $\mathrm{Tors}(\mathbb{H}) = \{1\}$, this is just the exact sequence (8) (here $G = G'$). Morally speaking, our theorem means that even admitting fixed points of the action of G the structure of the fundamental group of the quotient of a product of curves by G (or equivalently of a minimal resolution of singularities) is not different from the étale case.

In order to keep the notation down to a reasonable level we shall give the proof of theorem (4.1) only for the case $n = 2$. The proof for the general case is exactly the same.

Let now $g', m_1, \ldots, m_r$ be natural numbers and consider an orbifold surface group

$$\mathbb{T} := \mathbb{T}(g'; m_1, \ldots, m_r).$$

In the sequel we will frequently use the following well known properties of orbifold surface groups:

Proposition 4.3. *i) Every element of finite order in $\mathbb{T}$ is equal to a conjugate of a suitable power of one of the generators c_i of finite order.*

ii) Every orbifold surface group contains a surface group of finite index.

Proof. The group $\mathbb{T}$ is isomorphic to a cocompact Fuchsian group acting on the upper half plane with quotient the curve C' (of genus g'). Its elements of finite order are exactly the elliptic elements, i.e., the transformations with fixed points. These fixed points map then to one of the 'branch' points $p_1, \ldots, p_r$: this shows i) (cf. [Bear83], Theorem 10.3.2, page 263).

The second assertion is a direct consequence of the fact that $\mathbb{T}$ is residually finite (observed by Magnus as a consequence of a theorem of Malcev, see [Mag69], and [Mal40]): since then there is an epimorphism of $\mathbb{T}$ into a finite group G sending each c_i to an element of order precisely m_i. The fundamental group of the corresponding branched covering C of C' is the desired subgroup.

□

Definition 4.4. Let $R \lhd \mathbb{T}$ be a normal subgroup of finite index and let L be an arbitrary subset of $\mathbb{T}$. We define

$$\mathrm{N}(R, L) := \langle\langle \{ hkh^{-1}k^{-1} \mid h \in L,\ k \in R \} \rangle\rangle_{\mathbb{T}} \tag{9}$$

to be the normal subgroup in $\mathbb{T}$ generated by the set $\{ hkh^{-1}k^{-1} \mid h \in L,\ k \in R \}$, and denote the corresponding quotient by

$$\hat{\mathbb{T}} := \hat{\mathbb{T}}(R, L) := \mathbb{T}/\mathrm{N}(R, L).$$

The centralizer of the image of L in $\hat{\mathbb{T}}(R, L)$ is a finite index subgroup of $\hat{\mathbb{T}}$.

Proposition 4.5. *Let $R \lhd \mathbb{T}$ be a normal subgroup of finite index and let L be a finite subset of $\mathbb{T}$ consisting of elements of finite order. Then the normal subgroup $\langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$, generated in $\hat{\mathbb{T}}$ by the image of the set L, is finite.*

The proof of proposition 4.5 follows easily from the following lemma. Even though this lemma is a consequence of a theorem of Schur (see [Hup67], Satz 2.3 page 417), we prefer to give a short and self contained proof.

Lemma 4.6. *Let H be a group such that*

i) *H is generated by finitely many elements of finite order,*
ii) *the center $\mathrm{Z}(H)$ has finite index in H.*

Then H is finite.

Proof. By ii) it suffices to show that $\mathrm{Z}(H)$ is finite. Observe first that, since $\mathrm{Z}(H)$ has finite index, it is finitely generated.

Assume to the contrary that $\mathrm{Z}(H)$ is infinite. Writing the multiplication of $\mathrm{Z}(H)$ additively, we denote by $m\mathrm{Z}(H)$ the subgroup of $\mathrm{Z}(H)$ consisting of m-th powers. Since $\mathrm{Z}(H)$ is an infinite, finitely generated abelian group we infer that $\mathrm{Z}(H)/m\mathrm{Z}(H)$ is non trivial for every $m \in \mathbb{Z} \setminus \{1, -1\}$. Note also that $m\mathrm{Z}(H)$ is normal in H.

We choose m now coprime to the order of $H/\mathrm{Z}(H)$ and also coprime to the orders of the generators of H. Consider the exact sequence

$$1 \to \mathrm{Z}(H)/m\mathrm{Z}(H) \to H/m\mathrm{Z}(H) \to H/\mathrm{Z}(H) \to 1. \tag{10}$$

By our choice of m this sequence splits (cf. [Hup67], Satz 17.5 , page 122, or [Jac80], Theorems 6.15 page 365 and 6.16 page 367), whence $H/m\mathrm{Z}(H)$ is a semi-direct product of $\mathrm{Z}(H)/m\mathrm{Z}(H)$ and $H/\mathrm{Z}(H)$.

Since $\mathrm{Z}(H)/m\mathrm{Z}(H)$ is central in $H/m\mathrm{Z}(H)$ we infer that $H/m\mathrm{Z}(H)$ is isomorphic to the direct product of $\mathrm{Z}(H)/m\mathrm{Z}(H)$ and $H/\mathrm{Z}(H)$. Therefore there is a surjective homomorphism $H/m\mathrm{Z}(H) \to \mathrm{Z}(H)/m\mathrm{Z}(H)$, a contradiction to the fact that m is coprime to the order of each generator of H.

□

Proof. (of prop. (4.5)). Denote by $\hat{R}$ the image of R in $\hat{\mathbb{T}}$, and recall that $c_1, \dots, c_r$ are the generators of finite order of $\mathbb{T}$.

L consists of finitely many elements, each a conjugate of a $c_i^{l_i}$. Their image in $\hat{\mathbb{T}}$ is centralised by $\hat{R}$. Since $\hat{R}$ has finite index in $\hat{\mathbb{T}}$, finitely many conjugates of the elements of L suffice to generate $\langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$.

Hence the group $\langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$ is generated by finitely many elements of finite order.

Now, $\hat{R} \cap \langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$ has finite index in $\langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$. By construction $\hat{R} \cap \langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$ is a central subgroup of $\langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$. Therefore the second condition of lemma 4.6 is fulfilled, and we conclude that $\langle\langle L\rangle\rangle_{\hat{\mathbb{T}}}$ is finite.

□

Lemma 4.7. *Let $\mathbb{T}(g'; m_1, \dots, m_r)$ be an orbifold surface group and let L be a subset of $\mathbb{T}(g'; m_1, \dots, m_r)$ consisting of elements of finite order.*

Then there are $k_1, \dots, k_r \in \mathbb{N}$, such that $k_i | m_i$ $\forall i = 1, \dots r$ such that

$$\mathbb{T}(g'; m_1, \dots, m_r)/\langle\langle L \rangle\rangle_{\mathbb{T}(g'; m_1, \dots, m_r)} \cong \mathbb{T}(g'; k_1, \dots, k_r). \tag{11}$$

In particular, the quotient group $\mathbb{T}(g'; m_1, \dots, m_r)/\langle\langle L \rangle\rangle_{\mathbb{T}(g'; m_1, \dots, m_r)}$ is again an orbifold surface group.

Proof. The normal subgroup $\langle\langle L \rangle\rangle_{\mathbb{T}(g'; m_1, \dots, m_r)}$ is normally generated by a set of the form $\{c_1^{k_1}, \dots, c_r^{k_r}\}$ with $k_1, \dots, k_r \geq 1$, $k_i \leq m_i$, $k_i | m_i$. □

Remark 4.8. Let $R \lhd \mathbb{T}$ be a normal subgroup of finite index and let L be an arbitrary subset of $\mathbb{T}$ consisting of elements of finite order.

Note that

$$E(R, L) := \langle\langle L \rangle\rangle_{\hat{\mathbb{T}}(R, L)}$$

is a finite normal subgroup of $\hat{\mathbb{T}}(R, L)$ and

$$N(R, L) \lhd \langle\langle L \rangle\rangle_{\mathbb{T}}.$$

Hence we have

$$\hat{\mathbb{T}}(R, L)/\langle\langle L \rangle\rangle_{\hat{\mathbb{T}}(R, L)} \cong \mathbb{T}/\langle\langle L \rangle\rangle_{\mathbb{T}}.$$

We want to apply the above general considerations to our situation. For this purpose we have to fix some more notation.

We fix two orbifold surface groups

$$\mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \dots, m_r) =$$
$$= \langle a_1, b_1, \dots, a_{g_1'}, b_{g_1'}, c_1, \dots, c_r | c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{g_1'} [a_i, b_i] \cdot c_1 \cdot \ldots \cdot c_r \rangle,$$

and

$$\mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \dots, n_s) =$$
$$= \langle a_1', b_1', \dots, a_{g_2'}', b_{g_2'}', d_1, \dots, d_s | d_1^{n_1}, \dots, d_s^{n_s}, \prod_{i=1}^{g_2'} [a_i', b_i'] \cdot d_1 \cdot \ldots \cdot d_s \rangle,$$

together with two surjective homomorphisms $\varphi_1 \colon \mathbb{T}_1 \to G$, $\varphi_2 \colon \mathbb{T}_2 \to G$ to a (nontrivial) finite group G.
Denote by

$$R_1 := \mathrm{Ker}(\varphi_1) \lhd \mathbb{T}_1, \qquad R_2 := \mathrm{Ker}(\varphi_2) \lhd \mathbb{T}_2$$

the respective kernels. If φ_1 and φ_2 are appropriate orbifold homomorphisms, then R_1 and R_2 are both isomorphic to surface groups (else, they are just orbifold surface groups).

Define further

$$C := \{ c_1, \dots, c_r \} \subset \mathbb{T}_1, \qquad D := \{ d_1, \dots, d_s \} \subset \mathbb{T}_2.$$

Lemma 4.9. *Let G be a finite group and let $\varphi_1 \colon \mathbb{T}_1 \to G$, $\varphi_2 \colon \mathbb{T}_2 \to G$ be two surjective group homomorphisms.*
1) Then there is a finite set $\mathcal{N}_1 \subset \mathbb{T}_1 \times \mathbb{T}_2$ of elements of the form

$$(c^l, zd^n z^{-1}) \in \mathbb{T}_1 \times \mathbb{T}_2 \ , \qquad c \in C,\ d \in D,\ l,\ n \in \mathbb{N},\ z \in \mathbb{T}_2,$$

which have the property that

a) *$\mathcal{N}_1 \subset \mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2)$,*
b) *the normal closure of $\mathcal{N}_1$ in $\mathbb{H}$ is equal to* $\mathrm{Tors}(\mathbb{H})$.

2) Similarly there is $\mathcal{N}_2 \subset \mathbb{T}_1 \times \mathbb{T}_2$ as in 1) exchanging the roles of $\mathbb{T}_1$ and $\mathbb{T}_2$. Then the two sets $\mathcal{N}_1$, $\mathcal{N}_2$ can be moreover chosen in such a way that the following further condition holds:

if $(c^l, zd^n z^{-1})$ is an element of $\mathcal{N}_1$, there is a $z_1 \in \mathbb{T}_1$ such that $(z_1 c^l z_1^{-1}, d^n)$ is in $\mathcal{N}_2$, and conversely, if $(z_1 c^l z_1^{-1}, d^n)$ is in $\mathcal{N}_2$, then there is $z_2 \in \mathbb{T}_2$, such that $(c^l, z_2 d^n z_2^{-1})$ is an element of $\mathcal{N}_1$.

Proof. Every element of finite order in $\mathbb{H}$ is of the form

$$(z_1 c^l z_1^{-1}, z_2 d^n z_2^{-1}),$$

where $c \in C$, $d \in D$, $l, n \in \mathbb{N}$ and $z_i \in \mathbb{T}_i$. There is an element in $\mathbb{H}$ of the form (z_1, f), hence every element of finite order in $\mathbb{H}$ is conjugate in $\mathbb{H}$ to an element of the form $(c_i^l, zd_j^n z^{-1})$. Since C and D are finite sets of elements of finite orders, finitely many of them suffice to normally generate $\mathrm{Tors}(\mathbb{H})$.

□

Lemma 4.10. *Under the same assumptions as in lemma (4.9), the following holds:*

i) *if $(c^l, zd^n z^{-1}) \in \mathcal{N}_1$ (for $c \in C$, $d \in D$, l, $n \in \mathbb{N}$, $z \in \mathbb{T}_2$), then*
$$(c^l k c^{-l} k^{-1}, 1) \in \mathrm{Tors}(\mathbb{H})$$
for all $k \in R_1$;

ii) *if $(zc^l z^{-1}, d^n) \in \mathcal{N}_2$ (for $c \in C$, $d \in D$, l, $n \in \mathbb{N}$, $z \in \mathbb{T}_1$), then*
$$(1, d^n k d^{-n} k^{-1}) \in \mathrm{Tors}(\mathbb{H})$$
for all $k \in R_2$.

Proof. We have $(k, 1) \in \mathbb{H}$ for every $k \in R_1$. From

$$(c^l k c^{-l} k^{-1}, 1) = (c^l, zd^n z^{-1}) \cdot ((k, 1) \cdot (c^l, zd^n z^{-1})^{-1} \cdot (k, 1)^{-1})$$

the first claim follows. The second claim follows by symmetry.

□

Definition 4.11. We define $L_i \subset \mathbb{T}_i$ as follows: L_1 is the set of first components of elements of $\mathcal{N}_1$, L_2 is the set of second components of elements of $\mathcal{N}_2$.

Now the two homomorhisms $\varphi_1\colon \mathbb{T}_1 \to G$, $\varphi_2\colon \mathbb{T}_2 \to G$ induce surjective homomorphisms

$$\hat{\varphi_1}\colon \hat{\mathbb{T}}_1 := \hat{\mathbb{T}}(R_1, L_1) \to G, \quad \hat{\varphi_2}\colon \hat{\mathbb{T}}_2 := \hat{\mathbb{T}}(R_2, L_2) \to G.$$

Let's take then the fibre product

$$\hat{\mathbb{H}} := \hat{\mathbb{H}}(G; \hat{\varphi}_1, \hat{\varphi}_2) := \{\, (x, y) \in \hat{\mathbb{T}}_1 \times \hat{\mathbb{T}}_2 \mid \hat{\varphi}_1(x) = \hat{\varphi}_2(y) \,\}.$$

We shall define now a homomorphism

$$\Phi\colon \hat{\mathbb{H}} \to \mathbb{H}/\mathrm{Tors}(\mathbb{H})$$

as follows: for $(x_0, y_0) \in \hat{\mathbb{H}}$ choose $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that x maps to x_0 under the quotient homomorphism $\mathbb{T}_1 \to \hat{\mathbb{T}}_1$ and y maps to y_0 under the quotient homomorphism $\mathbb{T}_2 \to \hat{\mathbb{T}}_2$.

We have $\varphi_1(x) = \hat{\varphi}_1(x_0)$, $\varphi_2(y) = \hat{\varphi}_2(y_0)$, hence $(x, y) \in \mathbb{H}$. We set

$$\Phi((x_0, y_0)) := (x, y) \bmod \mathrm{Tors}(\mathbb{H}).$$

Lemma 4.10 shows that Φ is well defined. Obviously, Φ is a homomorphism.

We have

Lemma 4.12. $\Phi\colon \hat{\mathbb{H}} \to \mathbb{H}/\mathrm{Tors}(\mathbb{H})$ *is surjective. Its kernel is contained in* $E(R_1, L_1) \times E(R_2, L_2)$.

In particular, we have an exact sequence

$$1 \to E \to \hat{\mathbb{H}} \to \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to 1, \tag{12}$$

where E is a finite group.

Proof. Φ is clearly surjective. Let $(x_0, y_0) \in \ker(\Phi)$. Then $(x, y) \in \mathrm{Tors}(\mathbb{H})$, hence x_0 is an element of $E(R_1, L_1)$ by the definition of $E(R_1, L_1)$. Analogously, $y_0 \in E(R_2, L_2)$.

□

We are now ready to prove theorem (4.1).

The quotient homomorphisms $\hat{\mathbb{T}}_1 \to \hat{\mathbb{T}}_1/\langle\langle L_1\rangle\rangle_{\hat{\mathbb{T}}_1} \cong \mathbb{T}_1/\langle\langle L_1\rangle\rangle_{\mathbb{T}_1}$, $\hat{\mathbb{T}}_2 \to \mathbb{T}_2/\langle\langle L_2\rangle\rangle_{\mathbb{T}_2}$ both have finite kernel. They may be put together to give a a homomorphism

$$\Psi\colon \hat{\mathbb{T}}_1 \times \hat{\mathbb{T}}_2 \to \mathbb{T}_1/\langle\langle L_1\rangle\rangle_{\mathbb{T}_1} \times \mathbb{T}_2/\langle\langle L_2\rangle\rangle_{\mathbb{T}_2},$$

again with finite kernel. Notice that

$$E \leq \ker(\Psi) \tag{13}$$

where E is the kernel of Φ, as in (12).

We need the following:

Remark 4.13. The subgroup

$$\mathbf{H} := \Psi(\hat{\mathbb{H}}) \leq \mathbb{T}_1/\langle\langle L_1\rangle\rangle_{\mathbb{T}_1} \times \mathbb{T}_2/\langle\langle L_2\rangle\rangle_{\mathbb{T}_2}$$

is of finite index.

Hence, we obtain an exact sequence

$$1 \to E_1 \to \hat{\mathbb{H}} \to \mathbf{H} \to 1, \tag{14}$$

where E_1 is finite.

By (13) $E \leq E_1$, and by (12) we obtain a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & E \arrow[r, equal] \arrow[d] & E \arrow[d] & & \\
1 \arrow[r] & E_1 \arrow[r] \arrow[d] & \hat{\mathbb{H}} \arrow[r] \arrow[d] & \mathbf{H} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & E_2 \arrow[r] \arrow[d] & \mathbb{H}/\operatorname{Tors}(\mathbb{H}) \arrow[r] \arrow[d] & \mathbf{H} \arrow[r] & 1 \\
 & 1 & 1 & & .
\end{tikzcd}$$

Therefore we have

$$1 \to E_2 \to \mathbb{H}/\operatorname{Tors}(\mathbb{H}) \to \mathbf{H} \to 1 \tag{15}$$

where E_2 is finite.

Moreover, $\mathbf{H}$, as a subgroup of finite index in the direct product $\mathbb{T}_1/\langle\langle L_1 \rangle\rangle_{\mathbb{T}_1} \times \mathbb{T}_2/\langle\langle L_2 \rangle\rangle_{\mathbb{T}_2}$ of orbifold surface groups, is residually finite. But this property is not sufficient for our purposes. We need the following notion of *cohomological goodness* which was introduced by J.P. Serre in [Serre94] and is very important for the comparison of a group with its profinite completion.

Definition 4.14. *Let H be a group and let $\hat{H}$ be its profinite completion. The group H is called* good*, if, for each $k \in \mathbb{N}, k \geq 0$, and for each finite H-module M the natural homomorphism of cohomology groups (induced by the homomorphism $H \to \hat{H}$)*

$$H^k(\hat{H}, M) \to H^k(H, M)$$

is an isomorphism.

In [GJZ08] it is shown that a direct product of good groups is good, that a subgroup of finite index of a good group is again good, whence it follows that $\mathbf{H}$ is also good (since orbifold surface groups are good). This allows us to apply the following result.

Proposition 4.15. *(*[GJZ08, Prop. 6.1]*). Let F be a residually finite, good group, and let $\phi\colon H \to F$ be a surjective homomorphism with finite kernel K. Then H is residually finite.*

This implies that $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$ is residually finite. Therefore (and because E_2 is finite) there is a normal subgroup of finite index

$$\Gamma \lhd \mathbb{H}/\mathrm{Tors}(\mathbb{H})$$

such that $\Gamma \cap E_2 = \{1\}$.
Let $\Psi_1 \colon \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to \mathbf{H}$ be the surjective homomorphism in the exact sequence (15). Then $\Psi_1|\Gamma$ is injective and clearly $\Psi_1(\Gamma)$ has finite index in $\mathbf{H}$, which in turn is a subgroup of finite index in a direct product of orbifold surface groups.
From the general properties of orbifold surface groups, we find a normal subgroup $\mathbf{H}_1 \lhd \mathbf{H}$ of finite index, such that $\mathbf{H}_1$ is contained in $\Psi_1(\Gamma)$ and isomorphic to the direct product of two surface groups.
We set $\Gamma_1 := \Psi_1^{-1}(\mathbf{H}_1) \cap \Gamma$. Then Γ_1 satisfies:

i) Γ_1 is a normal subgroup of finite index in $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$,
ii) Γ_1 is isomorphic to the direct product of two surface groups.

This proves theorem (4.1).

□

We want to explain briefly the geometry underlying the above structure theorem for the fundamental group, and its significance for the investigation of the moduli space of the product-quotient surfaces.

We have the quotient map $C_1 \times C_2 \to X = (C_1 \times C_2)/G$, and the minimal model S of X is such that $\pi_1(X) \cong \pi_1(S)$.

Let $\hat{X}$ be the unramified covering of X corresponding to the normal subgroup $\mathcal{N}$ of $\pi_1(X)$ of finite index such that

$$\pi_1(\hat{X}) = \mathcal{N} \cong \Pi_{h_1} \times \Pi_{h_2}.$$

The fibre product $\hat{X} \times_X (C_1 \times C_2)$ is an unramified Galois covering of $(C_1 \times C_2)$ with group $G' := \pi_1(X)/\mathcal{N}$, which is connected if $\pi_1(C_1 \times C_2) \cong \Pi_{g_1} \times \Pi_{g_2}$ surjects onto $\pi_1(X)$. Let Y be a connected component of $\hat{X} \times_X (C_1 \times C_2)$: it is a Galois covering of $(C_1 \times C_2)$ with Galois group G'', the subgroup of G' stabilizing Y.

In turn the surjection $\psi \colon \Pi_{g_1} \times \Pi_{g_2} \to G''$ determines on the one hand two normal subgroups $G''_i := \psi(\Pi_{g_i})$ of G'' and two Galois unramified coverings

$$C''_i \to C_i = (C''_i)/G''_i.$$

On the other hand it shows that each element of G''_1 commutes with every element of G''_2, and G''_1, G''_2 generate G''.

Hence, setting $G''' := G''_1 \cap G''_2$, we have a central extension:

$$1 \to G''' \to G'' \to (G''_1/G''') \times (G''_2/G''') \to 1.$$

We analyse now the surjective morphism $(C''_1 \times C''_2) \to \hat{X}$, in order to derive some interesting consequences.

Case ++: assume here that $h_1, h_2 \geq 2$.

Then, by the extension of the Siu-Beauville theorem given in [Cat03b], each surjection $\mathcal{N} \to \Pi_{h_j}$ is given by a surjective holomorphic map with connected fibres $\varphi_j \colon \hat{X} \to C_j''''$, thus we get a surjective holomorphic map $\varphi \colon \hat{X} \to (C_1'''' \times C_2'''')$, where C_i'''' has genus h_i, and φ_* induces the above isomorphism $\mathcal{N} \cong \Pi_{h_1} \times \Pi_{h_2}$.

The first geometrical consequence is that, since the holomorphic map $f \colon (C_1'' \times C_2'') \to (C_1'''' \times C_2'''')$ is by the rigidity lemma (see [Cat00]) necessarily of product type (w.l.o.g., $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$), the foliations on $\hat{X}$ induced by the fibrations of X onto C_1/G and C_2/G are the same as the ones induced by the maps of $\hat{X}$ onto C_1'''' and C_2''''.

Moreover, G' acts on $\hat{X}$, hence also on its Albanese variety, which is the product $\mathrm{Alb}(C_1'''') \times \mathrm{Alb}(C_2'''')$. Thus G' acts also on each C_i'''', in particular C_i''''/G' maps onto C_i/G, and in view of the connectedness of the general fibre we get that $C_i''''/G' \cong C_i/G$.

Since the unramified G'-cover $\hat{X}$ and $\varphi \colon \hat{X} \to (C_1'''' \times C_2'''')$ are dictated by the fundamental groups, a natural way to study the deformations of X is to study the deformations of the morphism φ which are G'-equivariant.

Case +0: assume here that $h_1 \geq 2$, $h_2 = 1$.

Then we have $\varphi_1 \colon \hat{X} \to C_1''''$ such that φ_{1*} induces the above epimorphism $\mathcal{N} \cong \Pi_{h_1}$.

The morphism $\Phi_1 \colon \hat{X} \to \mathrm{Alb}(C_1'''') := J$ factors , by the universal property of the Albanese map, through the Albanese map $\alpha \colon \hat{X} \to A := \mathrm{Alb}(\hat{X})$, hence A is isogenous to a product of J with an elliptic curve.

Since however $H_1(\hat{X}, \mathbb{Z}) = H_1(C_1'''', \mathbb{Z}) \times \mathbb{Z}^2$, we conclude that

$$\mathrm{Alb}(\hat{X}) = J \times C_2'''',$$

where C_2'''' is an elliptic curve.

Hence we get again a morphism $\varphi \colon \hat{X} \to (C_1'''' \times C_2'''')$ such that φ_* induces the above isomorphism $\mathcal{N} \cong \Pi_{h_1} \times \Pi_{h_2}$, and we can argue exactly as before.

Case 00: assume here that $h_1 = h_2 = 1$.

Then the Albanese variety $A := \mathrm{Alb}(\hat{X})$ is a surface, and we get a finite morphism $f \colon (C_1'' \times C_2'') \to A$.

Consider the induced injective homomorphism

$$f^* \colon H^0(\Omega^1_A) \to H^0(\Omega^1_{C_1'' \times C_2''}) = H^0(\Omega^1_{C_1''}) \oplus H^0(\Omega^1_{C_2''}).$$

This is compatible with wedge product, and since the image of $H^0(\Omega^1_A)$ inside each $H^0(\Omega^1_{C_j''})$ is an isotropic subspace, it follows that $H^0(\Omega^1_A)$ splits as $\mathbb{C}\omega_1 \oplus \mathbb{C}\omega_2$ where, for $i \neq j$, $f^*(\omega_i)$ maps to zero in $H^0(\Omega^1_{C_j''})$.

We conclude that A is isogenous to a product of elliptic curves, and again looking at $H_1(\hat{X}, \mathbb{Z})$ we derive that A is a product of elliptic curves, thereby showing that also in this case we get a morphism $\varphi\colon \hat{X} \to (C_1'''' \times C_2'''')$ to which we can apply the previous analysis.

We do not apply these considerations here in order to study the deformations of the surfaces X that we construct.

The situation can become rather involved, since φ is not necessarily Galois. We defer to [BC09a] for an interesting and completely worked out example, where φ has degree 2.

In the case where $h_1 \neq 0$, $h_2 = 0$, we only have $\varphi_1\colon \hat{X} \to C_1''''$ at our disposal, and the analysis seems much more difficult to handle.

5. The classification of canonical product-quotient surfaces with $p_g = q = 0$, i.e., where $X = (C_1 \times C_2)/G$ has rational double points

In this section we will give a complete classification of the surfaces S occurring as the minimal resolution of the singularities of a surface $X := (C_1 \times C_2)/G$, where G is a finite group with an unmixed action on a product of smooth projective curves $C_1 \times C_2$ of respective genera $g_1, g_2 \geq 2$, and such that

i) X has only *rational double points* as singularities,

ii) $p_g(S) = q(S) = 0$.

We denote by $\rho\colon S \to X$ the minimal resolution of the singularities.

Remark 5.1. 1) Note that the assumption $g_i \geq 2$ is equivalent to S being of general type. In fact, $K_S = \rho^* K_X$ is nef, and $K_S^2 = K_X^2 = \frac{8(g_1-1)(g_2-1)}{|G|} > 0$, whence S is a minimal surface of general type.
2) The analogous situations, with the assumption ii) replaced by $p_g = q = 1$, resp. $p_g = q = 2$, have been classified by F. Polizzi in [Pol07], resp. by M. Penegini in [Pen09].
3) Recall that a cyclic quotient singularity is a rational double point if and only if it is of type A_n, for $n \in \mathbb{N}$.

5.1. The singularities of X.

We shall show in the sequel that, under the above hypotheses, X has only ordinary double points (i.e., singularities of type A_1). More precisely, we shall derive the following improvement of [Pol07, Prop. 4.1].

Proposition 5.2. *Assume that $X := C_1 \times C_2/G$ has only rational double points as singularities and that $\chi(\mathcal{O}_X) = 1$.*
Then X has $t := 8 - K_X^2$ ordinary double points as singularities. Moreover, $t \in \{0, 2, 4, 6\}$.

We use the following lemma (cf. [Pol09, Prop. 2.8]).

Lemma 5.3. *Let C_1, C_2 be two compact Riemann surfaces, and let G be a finite group with an unmixed action on $C_1 \times C_2$. Consider the*

surface $X := C_1 \times C_2/G$, and its minimal resolution of the singularities $\rho\colon S \to X$.
Let F be a reduced fibre of the natural map $p_1\colon X \to C_1/G$ as a Weil divisor, and let $\tilde{F} := \rho_*^{-1}F$ be its strict transform.
Assume that each singular point p of X on F is of type $A_{n(p)}$.
Then

$$-\tilde{F}^2 = \sum_{p\in F}\left(1 - \frac{1}{n(p)+1}\right).$$

Proof. Set $q := p_1(F)$. Then $\rho^* p_1^*(q) = k\tilde{F} + \sum_{E\in Exc(\rho)} m_E E$ for some $k, m_E \in \mathbb{N}$.

By assumption, for each $p \in F \cap \mathrm{Sing}(X)$, the exceptional divisor of ρ mapping to p is union of $n(p)$ rational curves, say $E_1, \dots, E_{n(p)}$, with $E_i^2 = -2$, $E_iE_{i+1} = 1$, and $E_i \cap E_j = \emptyset$ if $|i-j| > 1$. By Serrano's theorem (cf. [Serra96, (2.1)]), we can assume $\tilde{F}E_i = 0$ for each $i < n(p)$ and $\tilde{F}E_{n(p)} = 1$.

Since $E_i \cdot \rho^* p_1^*(q) = 0$, it follows that $m_{E_i} = \frac{ik}{n(p)+1}$. Therefore

$$-\tilde{F}^2 = \frac{1}{k}\tilde{F}(\rho^*p_1^*(q) - k\tilde{F}) =$$
$$= \frac{1}{k}\tilde{F}\left(\sum_{E\in Exc(\rho)} m_E E\right) = \frac{1}{k}\left(\sum_{p\in F}\frac{n(p)k}{n(p)+1}\right).$$

□

It follows

Corollary 5.4. *Assume that $X := C_1 \times C_2/G$ has only rational double points as singularities and that $\chi(\mathcal{O}_X) = 1$.*
Then

i) *either X has $t := 8 - K_X^2 \in \{0, 2, 4, 6\}$ ordinary double points as singularities,*
ii) *or X has three singular points, one of type A_1 and two of type A_3.*

Proof. By lemma 5.3, considering all the fibres through the singular points, $\sum_{p\in Sing X}\left(1 - \frac{1}{n(p)+1}\right) \in \mathbb{N}$, or, equivalently

$$\sum_{p\in Sing X}\frac{1}{n(p)+1} \in \mathbb{N} \tag{16}$$

By [Pol07, prop. 4.1], either

i) X has only nodes, or
ii) X has two A_3 singularities, and at most two nodes.

Then formula (16) shows that in case i) the number of nodes is even whereas in case ii) it is odd.

Note that (cf. 2.7)

$$K_S^2 = K_X^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|}, \quad e(S) = \frac{K_S^2 + 3t}{2}.$$

By Noether's formula

$$1 = \chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + e(S)\right) = \frac{3K_S^2 + 3t}{12 \cdot 2} = \frac{K_S^2 + t}{8},$$

and it follows that $K_S^2 = 8 - t$. Since $K_S^2 > 0$, $t \leq 6$.

□

Remark 5.5. 1) In a forthcoming paper ([BP10]) it is shown that case ii) of corollary 5.4 does not occur.

2) The case $t = 0$ (i.e., G acts freely on $C_1 \times C_2$) has been completely classified in [BCG08]. Therefore in what follows we shall assume $t > 0$.

5.2. **The signatures.** We know that X determines the following data

- a finite group G,
- two orbifold surface groups $\mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \ldots, m_r)$, $\mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \ldots, n_s)$,
- two appropriate homomorphisms $\varphi_i \colon \mathbb{T}_i \to G$, such that the stabilizers fulfil certain conditions ensuring that X has only $2, 4$ or 6 nodes respectively.

The assumption $q = 0$ is equivalent (cf. [Serra96, Proposition 2.2]) to C_i/G is rational, i.e., $g_1' = g_2' = 0$, whence $\mathbb{T}_i$ is indeed a polygonal group for $i = 1, 2$.

We will use the combinatorial restriction forced by the assumption $\chi(S) = 1$, and the conditions on the singularities of X in order to determine the possible signatures $T_1 = (m_1, \ldots, m_r)$, $T_2 = (n_1, \ldots, n_s)$ of the respective polygonal groups.
We associate to an r-tuple T_1 the following numbers:

$$\Theta_1 := -2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right), \qquad \alpha_1 := \frac{K_S^2}{4\Theta_1} \tag{17}$$

and similarly we associate Θ_2 and α_2 to T_2.

The geometric meaning of α_i is given by Hurwitz' formula.

Lemma 5.6. *$\alpha_1 = g_2 - 1$, $\alpha_2 = g_1 - 1$, and $|G| = \frac{8\alpha_1\alpha_2}{K_S^2}$.*

Proof. Hurwitz' formula applied to p_i gives

$$2(g_i - 1) = |G|\Theta_i.$$

Therefore by (6)and (17), we obtain

$$\alpha_1 = \frac{K_S^2}{4\Theta_1} = \frac{2(g_1-1)(g_2-1)}{|G|\Theta_1} = g_2 - 1,$$

and similarly $\alpha_2 = g_1 - 1$. In particular, $|G| = \frac{8\alpha_1\alpha_2}{K_S^2}$.

□

Remark 5.7. Note that the above lemma implies that, $\forall i,\ \alpha_i \in \mathbb{N}$. As we shall see, this is a strong restriction on the set of possible signatures.

Lemma 5.8. *Let $T_1 = (m_1, \dots, m_r)$ be the signature of one of the polygonal groups yielding a surface with t nodes. Then each m_i divides $2\alpha_1$. Moreover, there are at most $\frac{t}{2}$ indices $i \in \{1, \dots, r\}$ such that m_i does not divide α_1.*

Proof. Consider $p_1 \colon X \to C_1' := C_1/G$. Fix m_i in T_1 and let $h_i = \varphi_1(c_i)$ where the $c_i \in \mathbb{T}_1$ are generators as in (4). Then h_i determines a point $q_i \in C_1/G$, image of the points stabilized by h_i. Now, $p_1^*(q_i) = m_i W_i$ for some irreducible Weil divisor W_i. It follows that the intersection number of K_X with a fibre F, which equals on one side $2g_2 - 2 = 2\alpha_1$, can be written as $m_i W_i \cdot K_X$. We conclude since K_X is Cartier, whence $W_i \cdot K_X$ is an integer (cf. [Ful84]).

If $C_2 \to C_2/h_i \cong W_i$ is étale, then, by Hurwitz' formula, m_i divides α_1. Else h_i has order $= 2d$ and its d-th power stabilizes some points of C_2, and therefore W_i passes through some nodes of X. By lemma (5.3) (for $F = W_i$), since $\tilde{F}^2 \in \mathbb{Z}$ there are at least two nodes on W_i, whence this can happen for at most $\frac{t}{2}$ indices i.

□

Given $t \in \{2, 4, 6\}$ we want now to determine all r-tuples $T := (m_1, \dots, m_r)$ of positive integers which fulfill the numerical conditions we just found *i.e.*,

i) $\alpha := \frac{8-t}{4\Theta(m_1,\dots,m_r)}$ (cf. (17)) is a strictly positive integer;
ii) each m_i divides 2α;
iii) at most $\frac{t}{2}$ of the m_i's do not divide α.

In other words we are looking for a priori possible signatures of polygonal groups giving rise to standard isotrivial fibrations with $p_g = q = 0$, and t nodes.
In order to write a MAGMA script to list all the possible candidates for the signatures, we still have to give effective bounds for the numbers r and m_i. The following lemma will do this.

Lemma 5.9. *Let $t \in \{2, 4, 6\}$ and assume that $(m_1, \dots, m_r)$ is a sequence of integers $m_i \geq 2$ fulfilling conditions i), and ii) above. Then we have:*
*1) $r \leq 7$ (*i.e.*, p_1 (resp. p_2) has at most 7 branch points);*
2) for all i *we have* $m_i \leq 3(10 - t)$.

Proof. 1) If $r \geq 8$, then $\Theta \geq -2 + \frac{8}{2} = 2$. Then $1 \leq \alpha = \frac{8-t}{4\Theta} \leq \frac{6}{4 \cdot 2}$, a contradiction.
2) We can assume $m_1 \geq m_i$ for all i. We want to show that $m_1 \leq 3(10 - t)$.

Note that, since α (and therefore also Θ) is positive, $r \geq 3$ and if $r = 3$ at most one m_i can be equal to 2. Therefore it holds

$$\Theta + \frac{1}{m_1} = -1 + \sum_{i=2}^{r}(1 - \frac{1}{m_i}) \geq \frac{1}{6},$$

which is equivalent to $6(\Theta m_1 + 1) \geq m_1$. Since, by condition ii), $m_1 \leq 2\alpha$, we conclude

$$m_1 \leq 6(2\alpha\Theta + 1) = 6\left(\frac{8-t}{2} + 1\right)$$

□

The MAGMA script **ListOfTypes**, which can be found in the appendix as all other scripts, lists all signatures fulfilling conditions i), ii), iii): see table 3.

Table 3. The signatures which respect conditions i), ii) and iii)

K^2	T	α	K^2	T	α	K^2	T	α
2	(2,3,7)	21	4	(2,3,7)	42	6	(2,3,7)	63
2	(2,3,8)	12	4	(2,3,8)	24	6	(2,3,8)	36
2	(2,3,9)	9	4	(2,3,9)	18	6	(2,3,9)	27
2	(2,3,12)	6	4	(2,3,10)	15	6	(2,3,12)	18
2	(2,4,5)	10	4	(2,3,12)	12	6	(2,3,15)	15
2	(2,4,6)	6	4	(2,3,18)	9	6	(2,3,24)	12
2	(2,4,8)	4	4	(2,4,5)	20	6	(2,4,5)	30
2	(2,5,5)	5	4	(2,4,6)	12	6	(2,4,6)	18
2	(2,6,6)	3	4	(2,4,8)	8	6	(2,4,7)	14
2	(3,3,4)	6	4	(2,4,12)	6	6	(2,4,8)	12
2	(3,3,6)	3	4	(2,5,5)	10	6	(2,4,10)	10
2	(4,4,4)	2	4	(2,5,10)	5	6	(2,4,16)	8
2	(2,2,2,3)	3	4	(2,6,6)	6	6	(2,5,5)	15
2	(2,2,2,4)	2	4	(2,8,8)	4	6	(2,6,12)	6
			4	(3,3,4)	12	6	(2,7,7)	7
			4	(3,3,6)	6	6	(3,3,4)	18
			4	(3,4,4)	6	6	(3,3,6)	9
			4	(3,6,6)	3	6	(3,3,12)	6
			4	(4,4,4)	4	6	(3,4,6)	6
			4	(2,2,2,3)	6	6	(4,4,8)	4
			4	(2,2,2,4)	4	6	(2,2,2,4)	6
			4	(2,2,3,3)	3	6	(2,2,2,8)	4
			4	(2,2,4,4)	2	6	(2,3,3,3)	3
			4	(2,2,2,2,2)	2	6	(2,2,2,2,4)	2

5.3. **The possible groups** G**.** For each surface $X = (C_1 \times C_2)/G$ with $p_g = q = 0$ and nodes as above we have two associated signatures T_1 and T_2, such that G is a quotient of the corresponding polygonal groups.

Moreover, as shown by lemma 5.6, the order of the group depends only on T_1, T_2 and $K^2 := K_X^2 = 8 - t$. Having found, for each possible K^2, a finite list of possible signatures, we have then only finitely many groups to consider.

We can then write a script which computes, for each K^2, all possible triples (T_1, T_2, G), where G is a group of order $|G| = \frac{8\alpha_1\alpha_2}{K^2}$, which is a quotient of both polygonal groups of respective signatures T_1, T_2. This is done by the script **ListGroups**.

Note that our code skips those pairs of signatures giving rise to groups of order 1024 or bigger than 2000, since these cases are not covered by the MAGMA SmallGroup database of finite groups.

Moreover, the code skips the case $|G| = 1152$, since there are more than 10^6 groups of this order and this causes extreme computational complexity.

We shall see now, that the cases skipped by the MAGMA script actually do not occur.

$|G| = 1024$: For each pair of signatures T_1, T_2 in the same column of the table 3, $\frac{8\alpha_1\alpha_2}{K^2} \neq 1024$. So this case does not occur.

$|G| = 1152$: Looking again at table 3, this can happen only in one case:

- $K^2 = 4$, $T_1 = T_2 = (2, 3, 8)$.

Since the abelianization of $\mathbb{T}(2, 3, 8)$ is $\mathbb{Z}_2$, the abelianization of G can have order at most 2. The MAGMA smallgroup database shows that there are exactly 44 groups of order 1152 with abelianization of order at most 2. It can now easily be checked that none of these groups is a quotient of $\mathbb{T}(0; 2, 3, 8)$.

$|G| > 2000$: This can happen in 6 cases, all with $T_1 = (2, 3, 7)$.

More precisely we have:

- $K^2 = 4$, $T_2 = (2, 3, 7)$ or $(2, 3, 8)$, or
- $K^2 = 6$, and T_2 one of the following: $(2, 3, 7)$, $(2, 3, 8)$, $(2, 3, 9)$, $(2, 4, 5)$.

Since each quotient of $\mathbb{T}(0; 2, 3, 7)$ is a perfect group we can look at the MAGMA database of perfect groups, which is complete up to order 50000. Table 3 gives that $|G|$, in the six respective cases, is one of the numbers 3528, 2016, 5292, 3024, 2268, and 2520. But there is only one perfect group having order equal to one of these numbers, namely, $\mathfrak{A}_7$; a quick verification shows that $\mathfrak{A}_7$ is not a quotient of $\mathbb{T}(0; 2, 3, 7)$.

5.4. **The classification.** We have now, for each value $K^2 = 2, 4, 6$, a finite list of triples (T_1, T_2, G) such that T_1 and T_2 fulfill the numerical conditions i), ii), iii), G has the order given by lemma 5.6 and is a quotient of both polygonal groups $\mathbb{T}_1$ and $\mathbb{T}_2$ associated to the signatures T_1 and T_2.

To each of those triples correspond many families of surfaces, one for each pair of appropriate homomorphisms $\mathbb{T}_1 \to G$, $\mathbb{T}_2 \to G$.

We construct these families varying (here $C_1' = C_2' = \mathbb{P}^1$) the branch point sets $\mathcal{B}_1 := \{p_1, \dots p_r\} \subset \mathbb{P}^1$, $\mathcal{B}_2 := \{q_1, \dots q_s\} \subset \mathbb{P}^1$, and choosing respective standard bases for the fundamental groups $\pi_1(\mathbb{P}^1 \setminus \mathcal{B}_i)$. Then we use Riemann's existence theorem to construct the two curves C_1 and C_2 with an action of G of respective signatures T_1 and T_2, we finally consider $X = (C_1 \times C_2)/G$.

Observe that, since $C_i' = \mathbb{P}^1$, any appropriate homomorphism of $\mathbb{T}_i$ to G is completely determined by a so called *spherical system of generators* of G of signature T_i, i.e., in the case of $\mathbb{T}_1$, a sequence of elements $h_1, \dots, h_r$ which generate G, satisfy $h_1 \cdot \dots \cdot h_r = 1$, and fulfill moreover $ord(h_i) = m_i$.
It turns out that most of these surfaces have too many or too bad singularities, and we only take into consideration the surfaces whose singular locus consists of exactly $t = 8 - K^2$ nodes.
However, different pairs of appropriate homomorphisms can yield the same family of surfaces, due to different choices of a standard basis: this is taken into account by declaring that two pairs of appropriate homomorphisms are equivalent iff they are in the same orbit of the natural action of $\mathrm{Aut}(G)$ (simultaneously on both elements of the pair) and of the respective braid groups (the second equivalence relation is generated by the so-called *Hurwitz moves*, cf. [BC04]).

We determine our equivalence classes by using two MAGMA scripts (and many subroutines).
ExistingNodalSurfaces gives as output, for each K^2, all possible triples (T_1, T_2, G) which yield at least one nodal surface with the right number of nodes. It gives 7 possible triples (T_1, T_2, G) in the case $K^2 = 2$, 11 possible triples in the case $K^2 = 4$ and 6 possible triples in the case $K^2 = 6$.

These results are listed in table 2 and, more precisely, in the first 6 columns, showing respectively K^2, T_1, T_2, g_1, g_2 and the group G.

The corresponding families of surfaces have dimension equal to $r + s - 6 = \#T_1 + \#T_2 - 6$.

To find all the irreducible families corresponding to a given triple we write the script **FindAllComponents** which determines all possible equivalence classes for pairs of spherical systems of generators, producing one representative for each class.

Running **FindAllComponents** on the 24 triples given by **ExistingNodalSurfaces**, we have found one family in 21 cases and two families in the remaining three cases. The number of families found is written in the seventh column of the table 2. Finally, we compute a presentation of the fundamental group of each of the constructed surfaces. To do this we start from the given presentation of the direct product of the two given polygonal groups. Using the Reidemeister - Schreier routine, we compute a presentation of its finite index subgroup $\mathbb{H}$, and divide out the appropriate torsion elements in $\mathbb{H}$ to obtain a presentation of π_1. Since we know by theorem 4.1, that π_1 is surface times surface by finite, we go through all the normal subgroups of finite index until such a group appears.

Let us point out that, in the three cases in which there are two families, they both have the same fundamental group. Therefore we listed the fundamental groups in the ninth column of the table 2 without distinguishing the two cases.
In the eight column we have listed $H_1(X,\mathbb{Z})$, which is always finite.

Remark 5.10. The description of the groups in the table 2 requires some explanation.
As in the table of notation, $\mathbb{Z}_d := \mathbb{Z}/d = \mathbb{Z}/d\mathbb{Z}$ is the cyclic group of order d, $\mathfrak{S}_n$ is the symmetric group in n letters, $\mathfrak{A}_n$ is the alternating group.
$PSL(2,7)$ is the group of 2×2 matrices over $\mathbb{F}_7$ with determinant 1 modulo the subgroup generated by $-Id$.
$D_{p,q,r} = \langle x, y | x^p, y^q, xyx^{-1}y^{-r}\rangle$, and $D_n = D_{2,n,-1}$ is the usual dihedral group of order $2n$.

$G(32,2)$ is the second group of order 32 in the MAGMA database, a 2-group of exponent 4 whose elements of order 2 form the center, giving a central extension $1 \to \mathbb{Z}_2^3 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$, which does not split.

Finally, we have semidirect products $H \rtimes \mathbb{Z}_r$; to specify them, we have to indicate the image of the generator of $\mathbb{Z}_r$ in $Aut(H)$.

For $H = \mathbb{Z}^2$ either r is even, and then the image of the generator of $\mathbb{Z}_r$ in $Aut(H)$ is $-Id$, or $r = 3$ and the image of the generator of $\mathbb{Z}_3$ is the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Else H is finite and $r = 2$; for $H = \mathbb{Z}_3^2$ the image of the generator of H is $-Id$; for $H = \mathbb{Z}_2^4$ it is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Except for the last two cases, all the groups appearing as fundamental groups are polycyclic by finite. The groups in line 10 and 16 of table 2 have the same commutator quotient and also the same commutator subgroup, but they are not isomorphic since, for example, the one in line 16 has no normal subgroups of index 4 with free abelianization. The groups in the lines 8 and 14 are isomorphic to each other, as the ones in the lines 10 and 13, as well as the groups in the lines 17 and 18 (they are isomorphic to the fundamental groups of the Keum Naie surfaces, [BC09a]).

We observe here that the isomorphism problem for polycyclic by finite groups is decidable ([Se90]).

5.5. **Bloch's conjecture.** Another important feature of surfaces with $p_g = 0$ is the following

Conjecture 5.11. *Let S be a smooth surface with $p_g = 0$ and let $A_0(S) = \oplus_{i=0}^{\infty} A_0^i(S)$ be the group of rational equivalence classes of zero cycles on S. Then the kernel $T(S)$ of the natural morphism $A_0^0(S) \to \mathrm{Alb}(S)$ is trivial.*

The conjecture has been proven for $\kappa(S) < 2$ by Bloch, Kas and Liebermann (cf. [BKL76]). If instead S is of generaltype, then $q(S) = 0$, whence Bloch's conjecture asserts for those surfaces that $A_0(S) \cong \mathbb{Z}$.

Inspite of the efforts of many authors, there are only few cases of surfaces of general type for which Bloch's conjecture has been verified (cf. e.g. [InMi79], [Bar85b], [Keu88], [Voi92]).

Recently S. Kimura introduced the following notion of *finite dimensionality* of motives ([Kim05]).

Definition 5.12. Let M be a motive.

Then M is *evenly finite dimensional* if there is a natural number $n \geq 1$ such that $\wedge^n M = 0$.

M is *oddly finite dimensional* if there is a natural number $n \geq 1$ such that $\mathrm{Sym}^n M = 0$.

And, finally, M is *finite dimensional* if $M = M^+ \oplus M^-$, where M^+ is evenly finite dimensional and M^- is oddly finite dimensional.

Using this notation, he proves the following

Theorem 5.13. *1) The motive of a smooth projective curve is finite dimensional (*[Kim05]*, cor. 4.4.).*

2) The product of finite dimensional motives is finite dimensional (loc. cit., cor. 5.11.).

3) Let $f\colon M \to N$ be a surjective morphism of motives, and assume that M is finite dimensional. Then N is finite dimensional (loc. cit., prop. 6.9.).

4) Let S be a surface with $p_g = 0$ and suppose that the Chow motive of S is finite dimensional. Then $T(S) = 0$ (loc.cit., cor. 7.7.).

Using the above results we obtain

Theorem 5.14. *Let S be the minimal model of a product-quotient surface with $p_g = 0$.*

Then Bloch's conjecture holds for S, namely, $A_0(S) \cong \mathbb{Z}$.

Proof. Let S be a product-quotient surface.

Then S is the minimal model of $X = (C_1 \times C_2)/G$. Since X has rational singularities $T(X) = T(S)$.

By thm. 5.13, 2), 3) we have that the motive of X is finite dimensional, whence, by 4), $T(S) = T(X) = 0$.

Since S is of general type we have also $q(S) = 0$, hence $A_0^0(S) = T(S) = 0$.

□

Corollary 5.15. *All the surfaces in table 2 and all the surfaces in* [BC04], [BCG08] *satisfy Bloch's conjecture.*

6. The surfaces

This section is an expanded version of table 2. In the sequel we will follow the scheme below:

- G: here we write the Galois group G (most of the times as permutation group);
- T_i: here we specify the respective types of the pair of spherical generators of the group G;
- S_1: here we list the first set of spherical generators;
- S_2: here we list the second set of spherical generators;
- H_1: the first homology group of the surface;
- π_1: the fundamental group of the surface;
- $\mathbb{H}$: the generators of the preimage $\mathbb{H}$ of the diagonal of $G \times G$; and their images in the fundamental group $\mathbb{H}/\operatorname{Tors}(\mathbb{H})$.

6.1. $K^2 = 2$, Galois group $PSL(2,7)$:

- G: $\langle (34)(56), (123)(457)\rangle < \mathfrak{S}_7$
- T_i: $(2,3,7)$ and $(4,4,4)$
- S_1: (13)(26), (127)(345), (1762354)
- S_2: (1632)(47), (1524)(36), (1743)(25)
- H_1: $\mathbb{Z}_2^2$
- π_1: $\mathbb{Z}_2^2$
- $\mathbb{H}$: $\mathbb{H}_1 := c_1c_3^2c_2d_2$, $\mathbb{H}_2 := c_2c_3^{-1}c_1c_3d_3^{-1}$
  $\mathbb{H}_1 \mapsto (\overline{1},\overline{0})$, $\mathbb{H}_2 \mapsto (\overline{0},\overline{1})$

6.2. $K^2 = 2$, Galois group $PSL(2,7)$:

- G: $\langle (34)(56), (123)(457)\rangle < \mathfrak{S}_7$
- T_i: $(2,3,7)$ and $(4,4,4)$
- S_1: (13)(26),(127)(345),(1762354)
- S_2: (16)(2537),(1734)(26),(1452)(67)
- H_1: $\mathbb{Z}_2^2$
- π_1: $\mathbb{Z}_2^2$
- $\mathbb{H}$: $\mathbb{H}_1 := c_2c_1d_3^{-1}d_2d_1^{-2}$, $\mathbb{H}_2 := c_3c_2c_1d_1d_3^{-1}d_1^{-1}$
  $\mathbb{H}_1 \mapsto (\overline{1},\overline{0})$, $\mathbb{H}_2 \mapsto (\overline{0},\overline{1})$

6.3. $K^2 = 2$, **Galois group** $\mathfrak{S}_5$**:**

G: $\mathfrak{S}_5$
T_i: $(2,4,5)$ and $(2,6,6)$
S_1: (25),(1435),(12534)
S_2: (13)(25),(142)(35),(15)(243)
H_1 : $\mathbb{Z}_3$
π_1 : $\mathbb{Z}_3$
$\mathbb{H}$: $\mathbb{H}_1 := c_1c_2^{-1}c_3c_2^{-1}d_2$, $\mathbb{H}_2 := c_3^2c_2^{-1}c_3^{-1}d_1d_2^{-1}$, $\mathbb{H}_3 := c_3c_2c_3^{-1}c_1d_3^{-2}$
$\mathbb{H}_1 \mapsto \overline{1}$ and $\mathbb{H}_2$, $\mathbb{H}_3 \mapsto \overline{2}$

6.4. $K^2 = 2$, **Galois group** $\mathfrak{A}_5$**:**

G: $\mathfrak{A}_5$
T_i: $(2,2,2,3)$ and $(2,5,5)$
S_1: (14)(35), (15)(24), (13)(24), (154)
S_2: (23)(45), (15342), (13425)
H_1: $\mathbb{Z}_5$
π_1: $\mathbb{Z}_5$
$\mathbb{H}$: $\mathbb{H}_1 := c_1c_3d_2^{-1}$, $\mathbb{H}_2 := c_2c_3d_2^{-1}d_3$
$\mathbb{H}_1 \mapsto \overline{1}$, $\mathbb{H}_2 \mapsto \overline{2}$

6.5. $K^2 = 2$, **Galois group** $\mathfrak{S}_4 \times \mathbb{Z}_2$**:**

G: $\langle(12),(13),(14),(56)\rangle < \mathfrak{S}_6$
T_i: $(2,2,2,4)$ and $(2,4,6)$
S_1: (12)(56), (34), (14), (1342)(56)
S_2: (13)(56), (1342), (124)(56)
H_1: $\mathbb{Z}_2^2$
π_1: $\mathbb{Z}_2^2$
$\mathbb{H}$: $\mathbb{H}_1 := c_1c_3d_3$, $\mathbb{H}_2 := c_1c_4^{-1}d_1d_3d_2$, $\mathbb{H}_3 := c_3d_2d_1d_2^{-1}d_3$
$\mathbb{H}_1$, $\mathbb{H}_3 \mapsto (\overline{1},\overline{0})$, $\mathbb{H}_2 \mapsto (\overline{0},\overline{1})$

6.6. $K^2 = 2$, **Galois group** $\mathfrak{S}_3 \times \mathfrak{S}_3$**:**

G: $\langle(12),(13),(45),(46)\rangle < \mathfrak{S}_6$
T_i: $(2,2,2,3)$ and $(2,6,6)$
S_1: (13),(46),(23)(56),(132)(465)
S_2: (23)(46),(123)(45),(12)(456)
H_1: $\mathbb{Z}_3$
π_1: $\mathbb{Z}_3$
$\mathbb{H}$: $\mathbb{H}_1 := c_2c_1d_2^{-1}d_3$, $\mathbb{H}_2 := c_3c_2d_1d_2^{-3}$, $\mathbb{H}_3 := c_4c_1d_2^2d_1d_2^{-1}$
$\mathbb{H}_1$, $\mathbb{H}_3 \mapsto \overline{1}$ and $\mathbb{H}_2 \mapsto \overline{0}$.

6.7. $K^2 = 2$, **Galois group** $\mathbb{Z}_4^2$**:**

G: $\mathbb{Z}_4^2$
T_i: $(4,4,4)$ and $(4,4,4)$
S_1: $(\overline{1},\overline{3}),(\overline{1},\overline{0}),(\overline{2},\overline{1})$
S_2: $(\overline{3},\overline{2}),(\overline{0},\overline{1}),(\overline{1},\overline{1})$
H_1: $\mathbb{Z}_2^3$

π_1: $\mathbb{Z}_2^3$
$\mathbb{H}$: $\mathbb{H}_1 := c_1 d_2 d_1^{-1}$, $\mathbb{H}_2 := c_2 c_1^{-1} d_2$, $\mathbb{H}_3 := c_1 d_1^{-1} d_2$
$\mathbb{H}_1 \mapsto (\overline{1}, \overline{0}, \overline{0})$, $\mathbb{H}_2 \mapsto (\overline{0}, \overline{1}, \overline{0})$, $\mathbb{H}_3 \mapsto (\overline{0}, \overline{0}, \overline{1})$

6.8. $K^2 = 2$**, Galois group** $D_4 \times \mathbb{Z}_2$**:**

G: $\langle (1234), (14)(23), (56) \rangle < \mathfrak{S}_6$
T_i: $(2, 2, 2, 4)$ and $(2, 2, 2, 4)$
S_1: (24),(56),(12)(34),(1432)(56)
S_2: (12)(34),(13)(56),(13)(24)(56),(1234)
H_1: $\mathbb{Z}_4 \times \mathbb{Z}_2$
π_1: $\mathbb{Z}_4 \times \mathbb{Z}_2$
$\mathbb{H}$: $\mathbb{H}_1 := c_1 c_3 d_4$, $\mathbb{H}_2 := c_1 d_1 d_4^{-1}$, $\mathbb{H}_3 := c_4 d_3 d_4$, $H_4 := c_1 c_4^{-1} c_1 d_3 d_4$
$\mathbb{H}_1 \mapsto (\overline{1}, \overline{0})$, $\mathbb{H}_2 \mapsto (\overline{3}, \overline{0})$, $\mathbb{H}_3$, $\mathbb{H}_4 \mapsto (\overline{1}, \overline{1})$

6.9. $K^2 = 4$**, Galois group** $\mathfrak{S}_5$**:**

G : $\mathfrak{S}_5$
T_i: $(2, 4, 5)$ and $(3, 6, 6)$
S_1: (25), (1435), (12534)
S_2: (132), (135)(24), (15)(243)
H_1: $\mathbb{Z}_2^3$
π_1: $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}_3$, $\varphi(\overline{1}) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
$\mathbb{H}$: $\mathbb{H}_1 := c_3^2 c_2^{-1} d_2$, $\mathbb{H}_2 := c_3^{-1} c_2 c_3^{-1} d_3^{-1}$
$\mathbb{H}_1 \mapsto \overline{1}$, $\mathbb{H}_2 \mapsto (-1, 0) \cdot \overline{2}$

6.10. $K^2 = 4$**, Galois group** $\mathfrak{A}_5$**:**

G: $\mathfrak{A}_5$
T_i: $(2, 2, 3, 3)$ and $(2, 5, 5)$
S_1: (15)(34), (12)(35), (354), (125)
S_2: (23)(45), (15342), (13425)
H_1: $\mathbb{Z}_{15}$
π_1: $\mathbb{Z}_{15}$
$\mathbb{H}$: $\mathbb{H}_1 := c_4 d_2 d_3^{-1}$, $\mathbb{H}_2 := c_3^{-1} d_2^2 d_3^{-1}$, $\mathbb{H}_3 := c_3 c_1 d_2^{-1} d_3$
$\mathbb{H}_1 \mapsto \overline{1}$, $\mathbb{H}_2 \mapsto \overline{4}$, $\mathbb{H}_3 \mapsto \overline{14}$

6.11. $K^2 = 4$**, Galois group** $\mathfrak{S}_4 \times \mathbb{Z}_2$**:**

G: $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$
T_i: $(2, 2, 4, 4)$ and $(2, 4, 6)$
S_1: (24), (24), (1324)(56), (1423)(56)
S_2: (13)(56), (1342), (124)(56)
H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_4$
π_1: $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}_4$, $\varphi(\overline{1}) = -Id$
$\mathbb{H}$: $\mathbb{H}_1 := c_1 c_4^{-1} c_1 d_1 d_3^2$, $\mathbb{H}_2 := c_3 c_1 d_3^{-1} d_1 d_2^{-1} d_3$, $\mathbb{H}_3 := c_1 c_3^{-1} c_1 d_3^{-2} d_1$
$\mathbb{H}_1 \mapsto \overline{1}$, $\mathbb{H}_2 \mapsto (1, 0)$, $\mathbb{H}_3 \mapsto (0, 1) \cdot \overline{1}$

6.12. $K^2 = 4$**, Galois group** $\mathfrak{S}_4 \times \mathbb{Z}_2$**:**

G: $\langle (12),(13),(14),(56)\rangle < \mathfrak{S}_6$
T_i: $(2,2,2,2,2)$ and $(2,4,6)$
S_1: (12)(34)(56), (34)(56), (13), (23), (13)
S_2: (13)(56), (1342), (124)(56)
H_1: $\mathbb{Z}_2^3$
π_1: $\pi_1 = \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}_2$, $\varphi(\overline{1}) = -Id$
$\mathbb{H}$: $\mathbb{H}_1 := c_3 d_3^2 d_2^{-1}$, $\mathbb{H}_2 := c_1c_4c_2d_3^2$, $\mathbb{H}_3 := c_1c_4d_1d_3^2$, $\mathbb{H}_4 := c_2c_3c_1d_3^2$
$\mathbb{H}_1 \mapsto \overline{1}$, $\mathbb{H}_2$, $\mathbb{H}_3 \mapsto (1,0)$, $\mathbb{H}_4 \mapsto (0,1)$.

6.13. $K^2 = 4$**, Galois group** $\mathbb{Z}_2^4 \rtimes_\varphi \mathbb{Z}_2$**:** $\varphi(\overline{1}) = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix} \oplus \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$

G: $\langle (12)(34),(14)(23),(56)(78),(58)(67),(13)(57)\rangle < \mathfrak{S}_8$
T_i: $(2,2,2,4)$, and $(2,2,2,4)$
S_1: (24)(68), (12)(34)(56)(78), (12)(34), (24)(5876)
S_2: (12)(34)(57)(68), (56)(78), (24)(68), (1234)(5876)
H_1: $\mathbb{Z}_4^2$
π_1: $\langle x_1, x_2, y | x_i^4, y^2, [x_i, y], x_1^{-1}x_2^{-1}x_1x_2y\rangle$ of order 32
$\mathbb{H}$: $\mathbb{H}_1 := c_4c_3d_4$, $\mathbb{H}_2 := c_4d_1d_4^{-1}$, $\mathbb{H}_3 := c_1c_4^{-1}d_1d_3d_4$
$\mathbb{H}_1 \mapsto x_1$, $\mathbb{H}_2 \mapsto x_2$, $\mathbb{H}_3 \mapsto x_2 y$

6.14. $K^2 = 4$**, Galois group** $\mathfrak{S}_4$**:**

G: $\mathfrak{S}_4$
T_i: $(2,2,2,2,2)$ and $(3,4,4)$
S_1: (13), (14), (12)(34), (12), (14)
S_2: (132), (1432), (1342)
H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_4$
π_1: $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}_4$, $\varphi(\overline{1}) = -Id$
$\mathbb{H}$: $\mathbb{H}_1 := c_5c_2$, $\mathbb{H}_2 := c_2c_3d_3$, $\mathbb{H}_3 := c_1c_2d_2d_3^{-1}$, $\mathbb{H}_4 := c_2c_1c_2d_1^{-1}d_2$
$\mathbb{H}_1 \mapsto (1,0)$, $\mathbb{H}_2 \mapsto (0,-1)\cdot\overline{1}$, $\mathbb{H}_3 \mapsto (0,-1)\cdot\overline{2}$, and $\mathbb{H}_4 \mapsto \overline{1}$

6.15. $K^2 = 4$**, Galois group** $\mathfrak{S}_3 \times \mathbb{Z}_3$**:**

G: $\langle (12),(13),(456)\rangle < \mathfrak{S}_6$
T_i: $(2,2,3,3)$ and $(3,6,6)$
S_1: (13), (13), (123)(456), (132)(465)
S_2: (132), (23)(465), (12)(456)
H_1: $\mathbb{Z}_3^2$
π_1: $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}_3$, $\varphi(\overline{1}) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
$\mathbb{H}$: $\mathbb{H}_1 := c_2c_1$, $\mathbb{H}_2 := c_3c_1d_3$, $\mathbb{H}_3 := c_4c_1d_2$, $\mathbb{H}_4 := c_4c_1d_3^{-1}d_1$
$\mathbb{H}_1 \mapsto Id$, $\mathbb{H}_2 \mapsto \overline{1}$, $\mathbb{H}_3 \mapsto (1,0)\cdot\overline{1}$, $\mathbb{H}_4 \mapsto (-1,0)$

6.16. $K^2 = 4$**, Galois group** $\mathbb{Z}_3^2 \rtimes_\varphi \mathbb{Z}_2$**:** $\varphi(\overline{1}) = -Id$

G $\langle (123),(456),(12)(45)\rangle < \mathfrak{S}_6$
T_i: $(2,2,3,3)$ and $(2,2,3,3)$
S_1: (23)(56), (23)(45), (123)(465), (132)

S_2: (23)(56), (12)(46), (132)(465), (465)
H_1: $\mathbb{Z}_3^3$
π_1: $\mathbb{Z}_3^3$
$\mathbb{H}$: $\mathbb{H}_1 := c_1c_2d_4^{-1}$, $\mathbb{H}_2 := c_2d_1d_4^{-1}$, $\mathbb{H}_3 := c_3c_2c_1d_3^{-1}$, $\mathbb{H}_4 := c_3c_4^{-1}d_3$
$\mathbb{H}_1$, $\mathbb{H}_2 \mapsto (\overline{1}, \overline{0}, \overline{0})$, $\mathbb{H}_3 \mapsto (\overline{0}, \overline{1}, \overline{0})$, $H_4 \mapsto (\overline{0}, \overline{0}, \overline{1})$

6.17. $K^2 = 4$, **Galois group** $D_4 \times \mathbb{Z}_2$**:**

G: $\langle (1234), (12)(34), (56) \rangle < \mathfrak{S}_6$
T_i: $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 4)$
S_1: (56), (24), (12)(34), (12)(34)(56), (24)
S_2: (13)(56), (14)(23)(56), (13)(24)(56), (1234)(56)
H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_4$
π_1: $\langle x_1, x_2, x_3 | x_3^2, x_2^4, x_1^2x_2^2, (x_1^{-1}x_3x_2)^2, (x_2^{-1}x_1^{-1}x_3)^2, x_3x_2x_1x_3x_2^{-1}x_1^{-1} \rangle$
$\mathbb{H}$: $\mathbb{H}_1 := c_1c_3c_4$, $\mathbb{H}_2 := c_2d_1d_3$, $\mathbb{H}_3 := c_3c_2c_1d_4^{-1}$, $\mathbb{H}_4 := c_3d_1d_4^{-1}$
$\mathbb{H} \to \pi_1$ maps $H_1 \mapsto x_1x_3^{-1}x_2x_3^{-1}$, $H_2 \mapsto x_1$, $H_3 \mapsto x_2$, $H_4 \mapsto x_3$

The normal subgroups of π_1 having minimal index among the normal subgroups with free abelianization are 4. All of them are isomorphic to $\mathbb{Z}^2$ and have index 8.

The corresponding quotient of π_1 is either D_4 (in two cases) or $\mathbb{Z}_2 \times \mathbb{Z}_4$ (in two cases).

This shows two things.

First, it shows in particular that this fundamental group $\pi_1 = \pi_1(S)$ is different from all fundamental groups of surfaces with $p_g = q = 0$ and $K^2 = 4$ we knew before.

Second, and more important, it shows that, unlike the case of surfaces isogenous to a higher product (cf. [Cat00],Prop. 3.13) there is no unicity for a minimal realization.

6.18. $K^2 = 4$, **Galois group** $\mathbb{Z}_4 \times \mathbb{Z}_2$**.**

G: $\mathbb{Z}_4 \times \mathbb{Z}_2$
T_i: $(2, 2, 4, 4)$ and $(2, 2, 4, 4)$
S_1: $(\overline{2}, \overline{1})$, $(\overline{2}, \overline{1})$, $(\overline{3}, \overline{1})$, $(\overline{1}, \overline{1})$
S_2: $(\overline{0}, \overline{1})$, $(\overline{0}, \overline{1})$, $(\overline{3}, \overline{0})$, $(\overline{1}, \overline{0})$
H_1: $\mathbb{Z}_2^3 \times \mathbb{Z}_4$
π_1: $\langle x_1, x_2, x_3, x_4 | (x_3x_4)^2, (x_3^{-1}x_4)^2, x_2x_4^{-1}x_2x_4, [x_1, x_2], [x_2, x_3],$
$[x_1, x_4], x_1^{-1}x_3x_1^{-1}x_3^{-1}, x_4^{-1}x_3^{-1}x_1x_3x_4x_1, x_2x_4^{-1}x_3^{-1}x_4^{-1}x_2^{-1}x_3^{-1},$
$x_1^{-1}x_4x_1^{-1}x_3^{-1}x_4x_2x_1^{-2}x_3^{-1}x_2^{-1} \rangle$
$\mathbb{H}$: $\mathbb{H}_1 := c_2c_1$, $\mathbb{H}_2 := d_2d_1$, $\mathbb{H}_3 := c_3d_1d_4^{-1}$, $\mathbb{H}_4 := c_3c_1d_4$, $\mathbb{H}_5 :=$
$c_1d_4^2d_1$
$\forall i \leq 4 :$ $\mathbb{H}_i \mapsto x_i$, $\mathbb{H}_5 \mapsto x_3x_4$

There is an exact sequence

$$\begin{array}{ccccc} 1 \to & \mathbb{Z}^4 & \to & \pi_1 & \to \mathbb{Z}_2^2 \to 1 \\ & e_1 & \mapsto & x_1 & \\ & e_2 & \mapsto & x_2 & \\ & e_3 & \mapsto & x_3^2 & \\ & e_4 & \mapsto & x_4^2 & \end{array}.$$

The image of $\mathbb{Z}^4$ in π_1 is the only normal subgroup of index ≤ 4 with free abelianization.

6.19. $K^2 = 4$**, Galois group** $\mathbb{Z}_2^3$**.**

G: $\mathbb{Z}_2^3$
T_i: $(2,2,2,2,2)$ and $(2,2,2,2,2)$
S_1: $(\overline{0},\overline{0},\overline{1})$, $(\overline{0},\overline{1},\overline{1})$, $(\overline{0},\overline{0},\overline{1})$, $(\overline{1},\overline{1},\overline{1})$, $(\overline{1},\overline{0},\overline{0})$
S_2: $(\overline{1},\overline{0},\overline{0})$, $(\overline{1},\overline{0},\overline{1})$, $(\overline{0},\overline{1},\overline{0})$, $(\overline{1},\overline{1},\overline{0})$, $(\overline{1},\overline{0},\overline{1})$
H_1: $\mathbb{Z}_2^3 \times \mathbb{Z}_4$
π_1: $\langle x_1, x_2, x_3, x_4 | x_2^2, (x_2x_4)^2, (x_2x_1^2)^2, (x_2x_3^2)^2, [x_1, x_3], x_4x_3^{-1}x_4x_3, x_1^{-1}x_4x_1x_4, x_2x_1x_3x_2x_3x_1, x_3^{-1}x_1x_2x_1x_3^{-1}x_2\rangle$
$\mathbb{H}$: $\mathbb{H}_1 := c_1c_2d_3$, $\mathbb{H}_2 := c_1c_5d_2$, $\mathbb{H}_3 := c_1d_1d_2$, $\mathbb{H}_4 := c_2c_3d_3$, $\mathbb{H}_5 := c_5d_3d_4$
$\mathbb{H}_1 \mapsto x_1$, $\mathbb{H}_2 \mapsto x_2$, $\mathbb{H}_3 \mapsto x_2^{-1}$, $\mathbb{H}_4 \mapsto x_3$, $\mathbb{H}_5 \mapsto x_4$

There is an exact sequence

$$\begin{array}{ccccc} 1 \to & \mathbb{Z}^4 & \to & \pi_1 & \to \mathbb{Z}_2^2 \to 1 \\ & e_1 & \mapsto & x_3x_1 & \\ & e_2 & \mapsto & x_4 & \\ & e_3 & \mapsto & x_1^2 & \\ & e_4 & \mapsto & (x_2x_1)^2 & \end{array}.$$

The image of $\mathbb{Z}^4$ in π_1 is the only normal subgroup of index ≤ 4 with free abelianization.

This group is isomorphic to the one in 6.18.

6.20. $K^2 = 6$**, Galois group** $\mathfrak{A}_6$**:**

G: $\mathfrak{A}_6$
T_i: $(2,5,5)$ and $(3,3,4)$
S_1: $(16)(34)$, (25436), (16452)
S_2: (156), $(146)(235)$, $(1532)(46)$
H_1: $\mathbb{Z}_{15}$
π_1: $\mathfrak{A}_4 \times \mathbb{Z}_5$
$\mathbb{H}$: $\mathbb{H}_1 := c_2c_1d_2d_3^{-1}$, $\mathbb{H}_2 := c_2c_3^{-1}c_1c_2^{-2}d_3^{-1}d_1d_3^{-1}$
$\mathbb{H}_1 \mapsto (234) \cdot \overline{1}$, $\mathbb{H}_2 \mapsto (123)$

6.21. $K^2 = 6$**, Galois group** $\mathfrak{A}_6$**:**

G: $\mathfrak{A}_6$
T_i: $(2,5,5)$ and $(3,3,4)$
S_1: $(16)(34)$, (25436), (16452)

S_2: (123)(456), (125), (1465)(23)
H_1: $\mathbb{Z}_{15}$
π_1: $\mathfrak{A}_4 \times \mathbb{Z}_5$
$\mathbb{H}$: $\mathbb{H}_1 := c_2^{-2} d_1 d_2^{-1}$, $\mathbb{H}_2 := c_2 c_3^{-1} c_1 d_1 d_3^{-2}$
$\mathbb{H}_1 \mapsto (234) \cdot \overline{1}$, $\mathbb{H}_2 \mapsto (123) \cdot \overline{4}$

6.22. $K^2 = 6$**, Galois group** $\mathfrak{S}_5 \times \mathbb{Z}_2$**:**
G: $\langle (12), (13), (14), (15), (67) \rangle < \mathfrak{S}_7$
T_i: $(2, 4, 6)$ and $(2, 4, 10)$
S_1: (13)(45)(67), (1524)(67), (153)(24)
S_2: (25)(67), (1432), (15234)(67)
H_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$
π_1: $\mathfrak{S}_3 \times D_{4,5,-1}$, recall that $D_{4,5,-1} = \langle x, y | x^4, y^5, xyx^{-1}y \rangle$
$\mathbb{H}$: $\mathbb{H}_1 := c_2 c_1 d_2 d_1 d_2^{-1} d_3^{-1}$, $\mathbb{H}_2 := c_2 c_3^{-2} d_2^{-1} d_3$, $\mathbb{H}_3 := c_2^2 c_1 c_2^{-1} d_2^2 d_3^{-1} d_1$
$\mathbb{H}_1 \mapsto (123)xy$, $\mathbb{H}_2 \mapsto (12)x$, $\mathbb{H}_3 \mapsto (132)xy^3$

6.23. $K^2 = 6$**, Galois group** $PSL(2, 7)$**:**
G: $\langle (34)(56), (123)(457) \rangle < \mathfrak{S}_7$
T_i: $(2, 7, 7)$ and $(3, 3, 4)$
S_1: (27)(46), (1235674), (1653742)
S_2: (157)(234), (145)(367), (2476)(35)
H_1: $\mathbb{Z}_{21}$
π_1: $\mathfrak{A}_4 \times \mathbb{Z}_7$
$\mathbb{H}$: $\mathbb{H}_1 := c_2^3 d_3^{-1} d_1$, $\mathbb{H}_2 := c_3^3 d_1 d_3^{-1}$, $\mathbb{H}_3 := c_3^2 c_2^{-1} d_1 d_3$
$H_1 \mapsto (123) \cdot \overline{1}$, H_2, $H_3 \mapsto (134) \cdot \overline{6}$

6.24. $K^2 = 6$**, Galois group** $PSL(2, 7)$**:**
G: $\langle (34)(56), (123)(457) \rangle < \mathfrak{S}_7$
T_i: $(2, 7, 7)$ and $(3, 3, 4)$
S_1: (27)(46), (1235674), (1653742)
S_2: (127)(364), (157)(234), (1263)(57)
H_1: $\mathbb{Z}_{21}$
π_1: $\mathfrak{A}_4 \times \mathbb{Z}_7$
$\mathbb{H}$: $\mathbb{H}_1 := c_3 c_2^{-1} d_2 d_1^{-1}$, $\mathbb{H}_2 := c_1 c_3^2 d_1^{-1} d_3$, $\mathbb{H}_3 := c_2^2 c_1 d_1 d_3 d_1^{-1}$
$\mathbb{H}_1 \mapsto (134) \cdot \overline{1}$, $\mathbb{H}_2 \mapsto (132) \cdot \overline{1}$, $\mathbb{H}_3 \mapsto (12)(34) \cdot \overline{6}$

6.25. $K^2 = 6$**, Galois group** $\mathfrak{A}_5$**:**
G: $\mathfrak{A}_5$
T_i: $(2, 3, 3, 3)$ and $(2, 5, 5)$
S_1: (13)(24), (123), (235), (254)
S_2: (23)(45),(15342), (13425)
H_1: $\mathbb{Z}_3 \times \mathbb{Z}_{15}$
π_1: $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}_{15}$, $\varphi(\overline{1}) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
$\mathbb{H}$: $\mathbb{H}_1 = c_1 d_1 d_3 d_2 d_3^{-1}$, $\mathbb{H}_2 = c_1 c_4^{-1} c_2^{-1} d_2 d_3^{-1}$, $\mathbb{H}_3 = c_1 c_2^{-1} c_1 d_1 d_3^2$
$\mathbb{H}_1 \mapsto \overline{12}$, $\mathbb{H}_2 \mapsto (1, 0) \cdot \overline{14}$, $\mathbb{H}_3 \mapsto \overline{1}$

6.26. $K^2 = 6$**, Galois group** $\mathfrak{S}_4 \times \mathbb{Z}_2$**:**

G: $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$
T_i: $(2, 2, 2, 2, 4)$ and $(2, 4, 6)$
S_1: (23)(56), (12)(34)(56), (13)(56), (13)(56), (1342)
S_2: (13), (1324)(56), (142)(56)
H_1: $\mathbb{Z}_2^3 \times \mathbb{Z}_4$
π_1: $\langle x_1, x_2, x_3, x_4 | x_4^4, x_4^{-2}x_3^2, x_3^{-1}x_2^{-1}x_3x_2^{-1},$
$x_1^{-1}x_3x_4^{-1}x_1^{-1}x_3^{-1}x_4, x_2x_3x_1^{-1}x_2x_3^{-1}x_1^{-1}, x_1x_4^{-1}x_3^{-1}x_1x_4x_3 \rangle$
$\mathbb{H}$: $\mathbb{H}_1 = c_4c_3$, $\mathbb{H}_2 = c_3c_2c_1d_3$, $\mathbb{H}_3 = c_5d_2d_3^{-1}$, $\mathbb{H}_4 = c_1d_2^{-1}d_3^2$
$\mathbb{H}_i \mapsto x_i$

There is an exact sequence

$$\begin{array}{ccccc} 1 \to & \Pi_2 & \to & \pi_1 & \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1 \\ & \alpha_1 & \mapsto & x_3x_4x_2^{-1}x_4^{-1}x_3^{-1} & \\ & \beta_1 & \mapsto & x_1[x_4^{-1}, x_3^{-1}] & \\ & \alpha_2 & \mapsto & x_1 & \\ & \beta_2 & \mapsto & x_2^{-1} & \end{array}.$$

The image of Π_2 in π_1 is of minimal index among the normal subgroup of π_1 with free abelianization.

6.27. $K^2 = 6$**, Galois group** $D_4 \times \mathbb{Z}_2$**:**

G: $\langle (1234), (12)(34), (56) \rangle < \mathfrak{S}_6$
T_i: $(2, 2, 2, 2, 4)$ and $(2, 2, 2, 4)$
S_1: (56), (56), (12)(34)(56), (13)(56), (1432)
S_2: (24), (14)(23), (13)(24)(56), (1432)(56)
H_1: $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
π_1: $\langle x_1, x_2, x_3, x_4 | x_4^2, (x_1^{-1}x_4)^2, (x_3^2x_4)^2, (x_4x_2^2)^2,$
$[x_1, x_2^2], [x_1, x_3^2], [x_2, x_3^2], [x_3, x_2^2], x_1^{-1}x_2x_3^{-1}x_1^{-1}x_2^{-1}x_3 \rangle$
$\mathbb{H}$: $\mathbb{H}_1 = c_2c_1$, $\mathbb{H}_2 = c_3d_1d_4^{-1}$, $\mathbb{H}_3 = c_4d_1d_3$, $\mathbb{H}_4 = c_1d_4^2d_3$
$\mathbb{H}_i \mapsto x_i$

There is an exact sequence

$$\begin{array}{ccccc} 1 \to & \mathbb{Z}^2 \times \Pi_2 & \to & \pi_1 & \to \mathbb{Z}_2^2 \to 1 \\ & ((1,0), id) & \mapsto & x_2^2 & \\ & ((0,1), id) & \mapsto & x_3^2 & \\ & ((0,0), \alpha_1) & \mapsto & x_1x_2 & \\ & ((0,0), \beta_1) & \mapsto & x_3^{-1}x_1^{-1} & \\ & ((0,0), \alpha_2) & \mapsto & x_4x_2x_4x_1^{-1} & \\ & ((0,0), \beta_2) & \mapsto & x_1x_4^{-1}x_3^{-1}x_4^{-1} & \end{array}.$$

The image of $\mathbb{Z}^2 \times \Pi_2$ in π_1 is the only normal subgroup of index ≤ 4 with free abelianization.

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Appendix: MAGMA scripts

```
/* The first script, ListOfTypes, produces, for each value of K^2=8-t,
a list containing all signatures fulfilling the conditions i), ii) and iii)
in subsection 5.2. We represent the signatures by sequences of 7
positive integers [m1,..,m7], adding some "1" if the length of the
sequence is shorter. E.g., the signature (2,3,7) is represented by the
sequence [1,1,1,1,2,3,7]. Note that by lemma 5.9 the length of the
signatures is bounden by 7.

The script takes all nondecreasing sequences of 7 positive integers
between 1 and  3(K^2+2) (cf. lemma 5.9) and discards those
which do not fulfill all three conditions.
The script uses two auxiliary functions Theta and Alpha computing the
```

```
respective numbers (see (17) in subsection 5.2). */

Theta:=function(sig)
  a:=5;
  for m in sig do a:=a-1/m; end for;
  return a;
end function;

Alpha:=func<sig,Ksquare | Ksquare/(4*Theta(sig))>;

ListOfTypes:=function(Ksquare: MaximalCoefficient:=3*(Ksquare+2))
     list:=[* *];
     for m1 in [ 1..3*(Ksquare+2)] do
     for m2 in [m1..3*(Ksquare+2)] do
     for m3 in [m2..3*(Ksquare+2)] do
     for m4 in [m3..3*(Ksquare+2)] do
     for m5 in [m4..3*(Ksquare+2)] do
     for m6 in [m5..3*(Ksquare+2)] do
     for m7 in [m6..3*(Ksquare+2)] do
       sig:=[Integers() | m1,m2,m3,m4,m5,m6,m7 ];
       if Theta(sig) ne 0 then A:=Alpha(sig,Ksquare);
       if A in IntegerRing() and A ge 1 then
       if forall{m : m in sig | 2*A/m in IntegerRing()} then bads:=0;
          for m in sig do
            if A/m notin IntegerRing() then bads +:=1;
          end if; end for;
       if bads le 4-Ksquare/2 then Append(~list,sig);
       end if; end if; end if; end if;
     end for; end for; end for; end for; end for; end for; end for;
return list;
end function;

/* The second important script is ListGroups, which returns for each
K^2, the list of all triples [G,T1,T2] where T1 and T2 are two
signatures in ListOfTypes(K^2) and G is a group of the prescribed
order 8*Alpha(T1)*Alpha(T2)/K^2 having sets of spherical generators of
both signatures. Note that the script skips the cases: |G| = 1024,
1152, or bigger than 2000, cases which we have excluded by hand (cf.
subsection(5.3)).

It uses two subscripts:
  ElsOfOrd which returns the set of elements of a group of a given
order,
  ExistSphericalGenerators, a boolean function which answers if a
group has a system of generators of a prescribed signature */

ElsOfOrd:=function(G,order)
     Els:={ };
```

```
     for g in G do if Order(g) eq order then Include(~Els, g);
     end if; end for;
     return Els; end function;

ExistSphericalGenerators:=function(G,sig)
     test:=false;
     for x1 in ElsOfOrd(G,sig[1]) do
     for x2 in ElsOfOrd(G,sig[2]) do
     for x3 in ElsOfOrd(G,sig[3]) do
     for x4 in ElsOfOrd(G,sig[4]) do
     for x5 in ElsOfOrd(G,sig[5]) do
     for x6 in ElsOfOrd(G,sig[6]) do
       if Order(x1*x2*x3*x4*x5*x6) eq sig[7]
          and #sub<G|x1,x2,x3,x4,x5,x6> eq #G
          then test:=true; break x1;
       end if;
     end for; end for; end for; end for; end for; end for;
return test;
end function;

ListGroups:=function(Ksquare)
     list:=[* *]; L:=ListOfTypes(Ksquare); L1:=L;
       for T1 in L do
       for T2 in L1 do
         ord:=8*Alpha(T1,Ksquare)*Alpha(T2,Ksquare)/Ksquare;
       if ord in IntegerRing() and ord le 200 and ord notin {1024, 1152} then
       for G in SmallGroups(IntegerRing()!ord: Warning := false) do
       if ExistSphericalGenerators(G,T1) then
       if ExistSphericalGenerators(G,T2) then
         Append(~list,[* G,T1,T2 *]);
       end if; end if; end for; end if; end for;
       L1:=Reverse(Prune(Reverse(L1)));
       end for;
return list;
end function;

/* Each triple [G,T1,T2] corresponds to many surfaces, one for each
pair of spherical generators of G of the prescribed signatures, but
still these surfaces can be too singular.

The script ExistingNodalSurfaces returns all triples in the output of
ListGroups such that at least one of the surfaces has exactly the
prescribed number 8-K^2 of nodes as singularities.
It uses three more scripts.

FSGUpToConjugation returns a list of spherical generators of the group
of given type, and more precisely, one for each conjugacy
class. It divides the set of spherical generators into two sets,
```

```
Heaven and Hell, according to the rule "if a conjugate of the set I'm
considering is in Heaven, go to Hell, else to Heaven", and returns Heaven.

CheckSingsEl is a Boolean function answering if the surface S given by
two sets of generators of G has the right number of nodes and no other
singularities.
It uses the following: given a pair of spherical generators, the singular
points of the resulting surface S come from pairs g,h of elements, one
for each set, such that there are n,m with g^n nontrivial and
conjugated to h^m. If the order of g^n is 2 then S has some nodes,
else S has worse singularities, a contradiction.

Checksings is a Boolean function answering if there is apair of spherical
generators of G of signatures sig1 and sig2 giving a surface with the
right number of nodes.
To save time it checks only one set of spherical generators for each
conjugacy class, using FSGUpToConjugation. In fact, the isomorphism
class of the surface obtained by a pair of spherical generators does
not change if we act on one of them by an inner automorphism. */


FSGUpToConjugation:=function(G,sig)
    Heaven:={@ @}; Hell:={@ @};
     for x1 in ElsOfOrd(G,sig[1]) do
     for x2 in ElsOfOrd(G,sig[2]) do
     for x3 in ElsOfOrd(G,sig[3]) do
     for x4 in ElsOfOrd(G,sig[4]) do
     for x5 in ElsOfOrd(G,sig[5]) do
     for x6 in ElsOfOrd(G,sig[6]) do x7:=x1*x2*x3*x4*x5*x6;
       if Order(x7) eq sig[7] then
       if #sub<G|x1,x2,x3,x4,x5,x6> eq #G then
       if [x1,x2,x3,x4,x5,x6,x7^-1] notin Hell then
        Include(~Heaven,[x1,x2,x3,x4,x5,x6,x7^-1]);
         for g in G do
         Include(~Hell, [x1^g,x2^g,x3^g,x4^g,x5^g,x6^g,(x7^-1)^g]);
          end for;
         end if; end if; end if;
       end for; end for; end for; end for; end for; end for;
return Heaven;
end function;

CheckSingsEl:=function(G,seq1,seq2,Ksquare)
     Answer:=true; Nodes:=0;
       for g1 in seq1 do
       for g2 in seq2 do
       for d1 in [1..Order(g1)-1] do
       for d2 in [1..Order(g2)-1] do
       if IsConjugate(G,g1^d1,g2^d2) then
```

```
       if Order(g1^d1) ge 3 then Answer:=false; break g1;
       elif Order(g1^d1) eq 2 then
         Nodes +:=Order(G)/(2*d1*d2*#Conjugates(G,g1^d1));
       if Nodes gt 8-Ksquare then Answer:=false; break g1;
       end if; end if; end if; end for; end for; end for; end for;
    return Answer and Nodes eq 8-Ksquare;
end function;

CheckSings:=function(G,sig1,sig2,Ksquare)
     test:=false;
       for gens1 in FSGUpToConjugation(G,sig1) do
       for gens2 in FSGUpToConjugation(G,sig2) do
       if CheckSingsEl(G,gens1,gens2,Ksquare) then test:=true; break gens1;
       end if; end for; end for; return test;
end function;

ExistingNodalSurfaces:=function(Ksquare)
     M:=[* *];
     for triple in ListGroups(Ksquare) do
        G:=triple[1]; T1:=triple[2]; T2:=triple[3];
     if CheckSings(G,T1,T2,Ksquare) then Append(~M, triple);
     end if; end for; return M; end function;

/* ExistingNodalSurfaces produces a list of triples [G,T1,T2] more
precisely 7 for K^2=2, 11 for K^2=4 and 6 for K^2=6, one for each row
of table 2. To each of these triples correspond at least a surface
with p_g=q=0.

Recall that two of these surfaces are isomorphic if the two pairs
of  spherical generators are equivalent for the equivalence
relation generated by Hurwitz moves on each set and by the
automorphism group of G (acting simultaneously on the pair).
We want one surface for each equivalence class. This is
done by FindAllComponents, which needs 4 new scripts.

AutGr  describes the automorphism group of G as set.

HurwitzMove runs an Hurwitz move on a sequence of elements of a
group.

HurwitzOrbit computes the orbit of a sequence of elements of
the group under Hurwitz moves, and then returns (to save memory)
the subset of the sequences such that the  corresponding sequence of integers,
given by the orders of the group elements, is non
decreasing.

SphGens gives all sets of spherical generators of a group of
prescribed signature of length 5. Note that we have reduced the
```

```
length of the signature from 7 to 5, because the output of
ListOfTypes (see table 3) shows that the bound of r in lemma 5.9 can
be sharpened to 5.

Finally FindAllComponents produces (fixing the group and the
signatures) one pair of spherical generators for each isomorphism
class.
Running FindAllComponents on all 7+11+6 triples obtained by
ExixtingNodalSurfaces (remembering that we have to shorten the
signatures removing the first two 1) we always find one surface
except in three cases, giving two surfaces.
*/

AutGr:=function(G)
     Aut:=AutomorphismGroup(G); A:={ Aut!1 };
     repeat
       for g1 in Generators(Aut) do
       for g2 in A do
         Include (~A,g1*g2);
       end for; end for;
     until  #A eq #Aut; return A; end function;

HurwitzMove:=function(seq,idx)
     return Insert(Remove(seq,idx),idx+1,seq[idx]^seq[idx+1]);
end function;

HurwitzOrbit:=function(seq)
     orb:={ }; Purgatory:={ seq };
     repeat
       ExtractRep(~Purgatory,~gens); Include(~orb, gens);
         for k in [1..#seq-1] do hurgens:=HurwitzMove(gens,k);
           if hurgens notin orb then Include(~Purgatory, hurgens);
         end if; end for;
     until IsEmpty(Purgatory);
     orbcut:={  };
     for gens in orb do test:=true;
       for k in [1..#seq-1] do
         if Order(gens[k]) gt Order(gens[k+1]) then test:=false; break k;
         end if;
       end for;
       if test then Include(~orbcut, gens);
       end if;
     end for; return orbcut; end function;

SphGens:=function(G,sig)
     Gens:={ };
     for x1 in ElsOfOrd(G,sig[1]) do
     for x2 in ElsOfOrd(G,sig[2]) do
```

```
     for x3 in ElsOfOrd(G,sig[3]) do
     for x4 in ElsOfOrd(G,sig[4]) do
     if Order(x1*x2*x3*x4) eq sig[5] then
     if sub<G|x1,x2,x3,x4> eq G then
       Include(~Gens, [x1,x2,x3,x4,(x1*x2*x3*x4)^-1]);
     end if; end if; end for; end for; end for; end for;
return Gens; end function;

FindAllComponents:=function(G,sig1,sig2,Ksquare)
     Comps:={@ @}; Heaven:={ }; Hell:={ }; Aut:=AutGr(G);
     NumberOfCands:=#SphGens(G,sig1)*#SphGens(G,sig2);
     for gen1 in SphGens(G,sig1) do
     for gen2 in SphGens(G,sig2) do
       if gen1 cat gen2 notin Hell then
         Include(~Heaven, [gen1,gen2]);
         orb1:=HurwitzOrbit(gen1); orb2:=HurwitzOrbit(gen2);
         for g1 in orb1 do for g2 in orb2 do for phi in Aut do
            Include(~Hell, phi(g1 cat g2));
            if #Hell eq NumberOfCands then break gen1;
            end if;
         end for; end for; end for;
       end if;
     end for; end for;
     for gens in Heaven do
     if CheckSingsEl(G,gens[1],gens[2],Ksquare) then
       Include(~Comps, gens);
     end if; end for; return Comps; end function;

/* Finally we have to compute the fundamental group of the resulting
surfaces, which is done by the script Pi1. It uses the script
Polygroup which, given a sequence of 5 spherical generators of a group,
produces the corresponding Polygonal Group P and the surjective
morphism P->G.

Pi1 uses the two sequences seq1, seq2 of 5 spherical generators of G
to construct the surjection f from the product of the respective
polygonal groups T1 x T2 to G x G, defines the subgroup H (=preimage
of the diagonal in GxG), and takes the quotient by a sequence of
generators of Tors(H), which are obtained as follows: consider each pair
of elements (g1,g2), where gi is an element of seqi, such that g1^a is
nontrivial and conjugate to g2^b. Note that they give rise to a node, so
they both have order 2). Let h be a fixed element of G such that
g1^a*h=h*g2^b.
We let c vary between the elements in G commuting with g1^a.
Let t1,t2 be the natural preimages of g1,g2 in the
respective polygonal groups T1 and T2, t a preimage of h^-1*c in T2.
Then t1^a*t^-1*t2^b*t has order 2 and belongs to H.
These elements generate Tors(H) (see prop. 4.3). */
```

```
PolyGroup:=function(seq)
  F:=FreeGroup(#seq);
  P:=quo<F | F.1^Order(seq[1]), F.2^Order(seq[2]), F.3^Order(seq[3]),
            F.4^Order(seq[4]), F.5^Order(seq[5]), F.1*F.2*F.3*F.4*F.5>;
  return P, hom<P->Parent(seq[1])|seq>;
end function;

Pi1:=function(seq1,seq2)
     T1:=PolyGroup(seq1); T2,f2:=PolyGroup(seq2); G:=Parent(seq1[1]);
     T1xT2:=DirectProduct(T1,T2);
     inT2:=hom< T2->T1xT2 | [T1xT2.6, T1xT2.7, T1xT2.8, T1xT2.9, T1xT2.10]>;
     GxG,inG:=DirectProduct(G,G); m:=NumberOfGenerators(G); L:=[ ];
     for i in [1..m] do Append(~L,GxG.i*GxG.(i+m)); end for;
     Diag:=hom<G->GxG|L>(G);
     f:=hom<T1xT2->GxG|
                     inG[1](seq1[1]),inG[1](seq1[2]),inG[1](seq1[3]),
                                  inG[1](seq1[4]),inG[1](seq1[5]),
                     inG[2](seq2[1]),inG[2](seq2[2]),inG[2](seq2[3]),
                                  inG[2](seq2[4]),inG[2](seq2[5])>;
     H:=Rewrite(T1xT2,Diag@@f); TorsH:=[ ];
     for i in [1..5] do if IsEven(Order(seq1[i])) then
     for j in [1..5] do if IsEven(Order(seq2[j])) then
     a:=IntegerRing()!(Order(seq1[i])/2); b:=IntegerRing()!(Order(seq2[j])/2);
     test,h:= IsConjugate(G,seq1[i]^a,seq2[j]^b);
        if test then for c in Centralizer(G,seq1[i]^a) do
          Append(~TorsH, T1xT2.i^a * ((T1xT2.(j+5)^b)^(inT2((h^-1*c) @@ f2))));
        end for; end if;
       end if; end for; end if; end for;
     return Simplify(quo<H | TorsH>);
end function;

/* This following script does the same computation as the previous one,
but instead of returning the fundamental group as astract group it
returns T1xT2, H as subgroup of T1xT2 and a list of generators of
Tors(H) */

Pi1Detailed:=function(seq1,seq2)
     T1:=PolyGroup(seq1); T2,f2:=PolyGroup(seq2); G:=Parent(seq1[1]);
     T1xT2:=DirectProduct(T1,T2);
     inT2:=hom< T2->T1xT2 | [T1xT2.6, T1xT2.7, T1xT2.8, T1xT2.9, T1xT2.10]>;
     GxG,inG:=DirectProduct(G,G); m:=NumberOfGenerators(G); L:=[ ];
     for i in [1..m] do Append(~L,GxG.i*GxG.(i+m)); end for;
     Diag:=hom<G->GxG|L>(G);
     f:=hom<T1xT2->GxG|
                     inG[1](seq1[1]),inG[1](seq1[2]),inG[1](seq1[3]),
                                  inG[1](seq1[4]),inG[1](seq1[5]),
```

```
                    inG[2](seq2[1]),inG[2](seq2[2]),inG[2](seq2[3]),
                                      inG[2](seq2[4]),inG[2](seq2[5])>;
     H:=Rewrite(T1xT2,Diag@@f); TorsH:=[ ];
     for i in [1..5] do if IsEven(Order(seq1[i])) then
     for j in [1..5] do if IsEven(Order(seq2[j])) then
     a:=IntegerRing()!(Order(seq1[i])/2); b:=IntegerRing()!(Order(seq2[j])/2);
     test,h:= IsConjugate(G,seq1[i]^a,seq2[j]^b);
        if test then for c in Centralizer(G,seq1[i]^a) do
          Append(~TorsH, T1xT2.i^a * ((T1xT2.(j+5)^b)^(inT2((h^-1*c) @@ f2))));
        end for; end if;
       end if; end for; end if; end for;
     return T1xT2,H, TorsH;
end function;
```