On simply connected Godeaux surfaces

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Abstract. In this paper we provide a first step towards the classification of the numerical Godeaux surfaces in the still unknown open cases where $Tors(S) = 0$, or $Tors(S) = \mathbb{Z}/2\mathbb{Z}$.

Our method works in both cases, but in this paper, after some results which we establish in a greater generality, we mostly restrict ourselves to the case where the Torsion group is zero.

The bicanonical system yields, on a suitable blow up $\tilde{S}$ of the minimal model S, a fibration $f : \tilde{S} \to \mathbb{P}^1$ in curves of genus $2 \leq g \leq 4$, and the invariants of this fibration determine the equations of the image of S under a map which, in the general case, is the product of the tricanonical and of the bicanonical map.

This allows us to subdivide our surfaces into four classes, according to the behaviour of the bicanonical system. For each of these classes we have a complete description but the existence questions are not yet solved.

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Introduction

Algebraic surfaces with $p_g(S) = q(S) = 0$ were interesting ever since in the theory of algebraic surfaces, when it was first asked whether a surface with such invariants would be rational.

The first counterexample was given by Enriques, who constructed what are by now called the Enriques surfaces (cf. [E1], [E2], and [BPV] for further details and references); these have fundamental group $\mathbb{Z}/2\mathbb{Z}$, and are not of general type.

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In the 30's examples of surfaces of general type with $p_g(S) = q(S) = 0$ were constructed by Campedelli ([Cam], these have $K^2 = 2$), and by Godeaux ([G1], [G2], these have $K^2 = 1$).

Later on, Severi asked whether a simply connected surface with $p_g(S) = 0$ would be rational, and again this question had a negative answer by [Dol2], who constructed some simply connected elliptic surfaces, which are nowadays called Dolgachev surfaces.

The interest for surfaces of general type with $p_g(S) = q(S) = 0$, which are considered in Chapter VIII of Enriques' book [E2], with special regard to the question of the good properties of pluricanonical systems, was revived by Bombieri's article [Bo] which left some open problems about their pluricanonical systems.

After that, surfaces of general type with $p_g(S) = q(S) = 0$ and $K^2 = 1$ (for the minimal model) were called numerical Godeaux surfaces, while those with $p_g(S) = q(S) = 0, K^2 = 2$ were called numerical Campedelli surfaces.

In the 70's there were several papers devoted to these two classes of surfaces, especially to their tri-and quadri-canonical maps. Reid ([R2]) completely described the geometry of the numerical Godeaux surfaces with Torsion group of order ≥ 3, inverting the method of Godeaux who was constructing these surfaces as quotients by the free action of a cyclic group.

Finally, in the early 80's Barlow constructed ([R3], [Ba2]) a simply connected numerical Godeaux surface. The method here was a clever variant of Godeaux's method , in that the author used a non free action of a non cyclic group.

The Barlow surface was an interesting object both for applications to the differential topology of 4-manifolds ([Kot1], [OVdV]), and recently for problems on Einstein metrics ([CL]).

In fact, a classification of simply connected numerical Godeaux surfaces could produce new simply connected differentiable 4-manifolds with $b^+ = 1$. In this respect, we should point out that another beautiful example of such a surface was produced by Craighero and Gattazzo ([CG]), somehow in the line of thought introduced by Campedelli, i.e. as the minimal resolution of a normal singular surface (that the resulting surface is indeed simply connected, and has K_S ample was recently proved in [DW]). It is yet unclear whether the Barlow surface and the Craighero Gattazzo surface are diffeomorphic.

To conclude our historical motivation, we should point out that another source of interest for the numerical Godeaux surfaces stems from the conjecture of Bloch (cf. [Mu3], [BKL], [Blo]) that for a surface with $q = p_g = 0$ the Chow group of degree zero 0-cycles is trivial (this has been settled only in few very special cases, cf. [IM], [Ba3], [V]).

After all this, the reader might ask why the numerical Godeaux surfaces have not yet been classified. One reason is that the easy lines of the surfaces geography are the lines $K^2 = 2p_g - 4 + m$ with $m = 0, 1$, while here $m = 5$.

In fact, using unramified coverings which make m smaller, Reid was able to show that the numerical Godeaux surfaces with $|Tors(S)| \geq 3$ form three irreducible families, with fundamental group $\mathbb{Z}/n\mathbb{Z}, n = 3, 4, 5$.

The main purpose of the present paper is therefore to attack the classification of the numerical Godeaux surfaces with $Tors(S) = 0$, or $\mathbb{Z}/2\mathbb{Z}$ (also in the latter case special examples have been constructed, cf. e.g. [Ba1], [Wer1], [Wer2]).

Our method works in both cases, but in this paper, after some results which we establish in a greater generality, we restrict ourselves to the case where the Torsion group (i.e., the Abelianization of the fundamental group) of our surface is zero. We shall consider more amply also the latter case in a sequel to this article.

We would like now to explain what is the new method we are employing in order to classify the numerical Godeaux surfaces.

The first crucial property is that the bicanonical system yields, on a suitable blow up $\tilde{S}$ of the minimal model S, a fibration $f : \tilde{S} \to \mathbb{P}^1$ whose fibres are curves of genus g, where g can only be $= 2, 3, 4$.

To explain further our strategy, we have to subdivide our surfaces into several classes, according to the behaviour of the bicanonical system (cf. 1.1, 1.2).

Writing the bicanonical pencil $|2K_S|$ as $F + |M|$, where F is the fixed part, we see that $KF = 0$, $KM = 2$.

We have then four possibilities:

ia)	$M^2 = 4$	$F = 0$	M has genus 4	$\|M\|$ has 4 base points
ib)	$M^2 = 4$	$F = 0$	M has genus 3	$\|M\|$ has 1 double base point
ii)	$M^2 = 2$	$MF = 2$	$F^2 = -2$	M has genus 3
iii)	$M^2 = 0$	$MF = 4$	$F^2 = -4$	M has genus 2.

The second ingredient is to consider a product rational mapping $\varphi = \varphi_1 \times \varphi_2 : S \to \mathbb{P}^r \times \mathbb{P}^1$, where φ_2 is the bicanonical map φ_{2K} and φ_1 is

- Case ia): r=3 and φ_1 is the tricanonical morphism φ_{3K}
- Case ib): r =2 and φ_1 is given by the system $|3K_S - P|$, P being the (double) base point of $|2K|$
- Case ii): r=2 and φ_1 is given by the system $|3K_S - F|$
- Case iii): r=1 and φ_1 is given by the system $|3K_S - F|$.

The key idea is simple: namely, that φ_1 induces the complete canonical system on the fibres M of φ_2, and we use the fact that, in all the cases except the last, the general curve M is not hyperelliptic.

Therefore, in these cases φ yields a birational map, and indeed, on a blow up $\tilde{S}$ of S, we get a product morphism $\psi_1 \times f$, which fails to be an embedding exactly when we have a hyperelliptic fibre M of f.

In case iii), which will be treated in the sequel to this paper, we have that the blow up $\tilde{S}$ of S in 5 points is a double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on a curve Δ of bidegree (6,12). The fibration f given by the second projection has precisely 5 non 2-connected fibres and, of these, α are originated by a point of type (3,3) on Δ with horizontal tangency, whereas $5 - \alpha$ are originated by a horizontal fibre in $\mathbb{P}^1 \times \mathbb{P}^1$ contained in Δ, such that Δ has two ordinary quadruple points on it.

We have mentioned this last case because the case where $\alpha = 0$ was already treated in [R4], where Reid showed the existence of such a curve giving rise to a surface with $Tors(S) = \mathbb{Z}/5\mathbb{Z}$. Note however that we believe this case does not occur under the assumption Tors $(S) = 0$.

In the remaining cases, denoting by Y the image surface, our analysis shows that the blown up canonical model $\tilde{X}$ is precisely the normalization of Y, and the only singular curves of Y correspond to the hyperelliptic fibres of f.

The important conclusion is that the conductor ideal pulls back on $\tilde{S}$ to a divisor which is a sum of the hyperelliptic fibres, counted with a certan multiplicity.

In the cases ib) and ii) Y is a hypersurface, and, after we establish its divisor class, the main question about the existence and geometry of these classes in the moduli space is related to the problem of finding hypersurfaces with certain singularities yielding the right (sub)adjunction conditions.

Case 1a) is more interesting (in this case Y is a subvariety of codimension 2) because then the key idea comes into play.

The bicanonical curves in case 1a) (the most general case) are canonical curves of genus 4, whence the non hyperelliptic ones are proven to be complete intersection curves of type (2,3).

The situation globalizes, in the sense that Y is proven to be a divisor on a hypersurface $\mathcal{Q}$ of bidegree (2,7-2h) in $\mathbb{P}^3 \times \mathbb{P}^1$ (where $0 \leq h \leq 3$ is the number of hyperelliptic fibres in the bicanonical pencil, counted with multiplicity). Clearly, $\mathcal{Q}$ is swept out fibrewise by the quadrics containing the canonical curves, and the cubics also patch together fibrewise to yield a divisor Y.

To calculate the class of this divisor, we introduce a new technique, which is valid more generally for any genus 4 fibration whose fibres have ample canonical system and are 2-connected. The technique consists in defining a notion of multiplicity for the hyperelliptic fibres, and then in showing how this multiplicity determines the divisor classes of the relative quadric and of the relative cubic.

We can partly summarize here our main results as follows:

Theorem 0.1. *Assume that S is a numerical Godeaux surface with torsion $\{0\}$ and of type ia), i.e., s.t. the bicanonical pencil yields a genus 4 fibration f.*

Let $h = \sum_{C \ hyperelliptic} mult(C)$: then a priori $0 \leq h \leq 3$.

Moreover $\exists \mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$, s.t. $Y := \varphi(S)$ is a divisor in $|\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ whose singular curves are exactly the twisted cubic curves image of the (honestly) hyperelliptic bicanonical divisors. Moreover, if $\mathcal{C}$ is the conductor ideal, $h^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$.

Viceversa, assume that $0 \leq h \leq 3$ and that $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$ is an irreducible divisor, and that in turn $Y \in |\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ is an irreducible divisor whose normalization is a surface $\tilde{X}$ with rational double points as the only singularities. Suppose moreover that the conductor ideal $\mathcal{C}$ defines a divisor on $\tilde{X}$ equal to h fibres (counted with multiplicity). Assume moreover that the singular curves of Y are (irreducible) twisted cubics, and that $h^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$. Then Y is the tri-bicanonical model of a numerical Godeaux surface with torsion $\{0\}$ and of type ia).

A posteriori, the case with three distinct hyperelliptic fibres does not occur. Whereas, for the Barlow surface $h = 2$, while the Craighero Gattazzo surface has exactly two hyperelliptic fibres occurring with multiplicity 1.

The local moduli space of the Barlow and of the Craighero Gattazzo surface is smooth of the expected dimension = 8.

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A first description of the proposed approach was then exposed in seminar talks in Warwick and Oberwolfach in the summer of 1996.

Later on, the optimistic hope that there would exist such surfaces as complete intersections in $\mathbb{P}^3 \times \mathbb{P}^1$ having 3 hyperelliptic fibres turned out not to hold.

We are very grateful to F.-O. Schreyer who helped us with the Computer Algebra program Macaulay, and indeed his script allows us to show that the Barlow surface has 2 hyperelliptic fibres.

Notation

For the reader's convenience, we enclose here a list of the notation more often used, and of our abbreviations.

In this paper we denote by S the minimal model of a numerical Godeaux surface and (except in section 6) by X its canonical model.

We let moreover $Tors(S)$ be the torsion subgroup of the first homology group $H_1(S, \mathbb{Z})$ (equivalently, of $H^2(S, \mathbb{Z})$).

For a Gorenstein algebraic variety Z (e.g. S, X) we denote by K_Z a Cartier divisor associated to its dualizing sheaf ω_Z. The rational map associated to a divisor D is denoted by φ_D or $\varphi_{|D|}$; similarly the rational map associated to a line bundle $\mathcal{L}$ is denoted by $\varphi_{\mathcal{L}}$.

$\tilde{S}$ (resp. $\tilde{X}$) is a blow up of S (resp. X): except in case iii) (which is hardly treated here) it is the blow up in the base points of the movable part $|M|$ of $|2K_S|$ (resp. in the smooth base points of $|2K_X|$, we shall prove in lemma 1.1 that $|2K_X|$ has no fixed part). The induced morphism is denoted by β (resp. $\hat{\beta}$).

We have already defined $\varphi : S \dashrightarrow \mathbb{P}^r \times \mathbb{P}^1$ in the introduction; let us denote by $\hat{\varphi} : X \dashrightarrow \mathbb{P}^r \times \mathbb{P}^1$ the induced map on the canonical model, and set $Y := \varphi(S)$, $\Sigma := \varphi_{3K_S}(S)$.

Moreover, we denote by $\pi_1 : \mathbb{P}^r \times \mathbb{P}^1 \to \mathbb{P}^r$, $\pi_2 : \mathbb{P}^r \times \mathbb{P}^1 \to \mathbb{P}^1$ the natural projections.

This allows us to define the morphisms $g := \varphi \circ \beta : \tilde{S} \to Y$, $\hat{g} := \hat{\varphi} \circ \hat{\beta} : \tilde{X} \to Y$

Last, we denote by $f := \pi_2 \circ \varphi \circ \beta : \tilde{S} \to \mathbb{P}^1$ the fibration associated to the bicanonical pencil, and by $\hat{f} := \pi_2 \circ \hat{\varphi} \circ \hat{\beta} : \tilde{X} \to \mathbb{P}^1$ the analogous fibration on the canonical model X.

Quite often, given a Cartier divisor D on a scheme Z, by slight abuse of notation we denote also by D the associated invertible sheaf $\mathcal{O}_Z(D)$; and we often write, as shorthand notation, $H^0(D)$ instead of $H^0(\mathcal{O}_Z(D))$.

Moreover, $\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^1}(a,b)$ is also a quite understandable notation for $\pi_1^* \mathcal{O}_{\mathbb{P}^r}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(b)$.

1. The canonical ring

Let S be a numerical Godeaux surface, i.e. a minimal surface of general type with $K_S^2 = 1$, $p_g(S) = q(S) = 0$.

Recall that the canonical ring of S (cf. [Mu1]) is defined as the graded ring

$$R(S) := \oplus_{n \in \mathbb{N}} H^0(nK_S).$$

In our case of numerical Godeaux surfaces we have

$$h^0(K_S) = 0;\ \forall n \geq 2\ \ h^0(nK_S) = \binom{n}{2} + 1.$$

Let us look for a minimal system of generators of this ring (as a $\mathbb{C}$-algebra).

As usual we denote by 1 the identity of $R(S)$, given by the constant function equal to 1, moreover we fix a basis $\{x_0, x_1\}$ of $H^0(2K_S)$, and a basis $\{y_0, y_1, y_2, y_3\}$ of $H^0(3K_S)$.

We remark that x_0^2, x_0x_1, x_1^2 are independent in $H^0(4K_S)$, since $R(S)$ is an integral domain; whence, we can complete these elements to a basis $\{x_0^2, x_0x_1, x_1^2, v_0, v_1, v_2, v_3\}$ of $H^0(4K_S)$.

Let $X := Proj(R(S))$ be the canonical model of S, and let $\pi : S \to X$ be the natural map; X has an invertible dualizing sheaf and, as customary, we denote by K_X an associated Cartier divisor. Since $\pi^*(K_X) = K_S$, one has a natural isomorphism between the canonical rings $R(S)$ and $R(X)$.

Lemma 1.1. *The fixed part F of the bicanonical pencil $|2K_S|$ is supported on the fundamental cycles of S (normal crossing configurations of smooth rational curves with self-intersection -2).*

In particular $|2K_X|$ has no fixed part.

Proof.

We can write $|2K_S| = |M| + F$ where M is a linear pencil without fixed components; since K_S is nef and the only curves with $K_S C = 0$ are the finitely many smooth rational (-2) curves, building the so called fundamental cycles (cf. [Bo], [BPV]), we know that $K_S M > 0$, $K_S F \geq 0$.

Since $K_S M + K_S F = 2K_S^2 = 2$ we get $0 < K_S M \leq 2$, and clearly our purpose is to show that $K_S F = 0$, equivalently $K_S M = 2$.

Assume by contradiction that $K_S M = 1$. M being a pencil without fixed part, we have $M^2 \geq 0$, but $M^2 + K_S M = 0 \pmod 2$. It follows then that $M^2 = 1$, whence equality holds in the inequality given by algebraic index theorem.

Our conclusion is thus that M is numerically equivalent to K_S, and since $h^1(\mathcal{O}_S) = 0$ but $h^0(K_S) = 0 \neq h^0(M) = 2$, $M - K_S = \mu$ yields a non zero torsion element μ in Pic S.

An easy calculation $\chi(M) = \chi(K_S) = 1$, $h^0(M) = 2 \Rightarrow 1 \leq h^1(M) = h^1(K_S + \mu) = h^1(-\mu)$ shows that the covering of S induced by μ, yields an irregular covering of S. This is a contradiction, because the equality $K_S^2 = \chi(S)$ holds for S, hence for all its unramified coverings , whereas for minimal irregular surfaces Y we have the inequality $K_Y^2 \geq 2\chi(Y)$ (cf. [Bo]).

□

Remark 1.2. *We have seen that* $KF = 0$ $KM = 2$. *So, we have three possibilities for* F *and* M*, namely*

i) $M^2 = 4 \quad F = 0$

ii) $M^2 = 2 \quad MF = 2 \quad F^2 = -2$

iii) $M^2 = 0 \quad MF = 4 \quad F^2 = -4.$

In the second case F *is precisely a fundamental cycle, i.e., on the canonical model* X*, we get in the base point scheme a reduced singular point.*

Lemma 1.3. $H^0(2K) \otimes H^0(3K) \to H^0(5K)$ *is injective.*

Proof.

Otherwise we would have a relation $x_0 y = x_1 y'$ for suitable elements y, y' in $H^0(3K)$. By lemma 1.1, on X $Min(div(x_0), div(x_1)) = 0$; whence, $div(x_0) < div(y')$ and therefore the rational section y'/x_0 of $3K_X - 2K_X = K_X$is a regular section, contradicting $p_g(X) = 0$.

□

Corollary 1.4. *We can fix a basis of* $H^0(5K)$ *of the form*

$$\{x_i y_j, w_1, w_2, w_3\}.$$

Let us now consider the polynomial ring $A := \mathbb{C}[y_0, y_1, y_2, y_3]$, and let us look for a set of generators of $R(S)$ as $A-$module, ($R(S)$ is an $A-$algebra via the natural homomorphism $A \to R(S)$).

Define

$$R^{(0)} = \oplus_{n\geq 0} H^0(3nK_S),$$

$$R^{(1)} = \oplus_{n\geq 0} H^0((3n+1)K_S),$$

$$R^{(2)} = \oplus_{n\geq 0} H^0((3n+2)K_S),$$

Of course, there is a splitting (as $A-$modules) $R(S) = R^{(0)} + R^{(1)} + R^{(2)}$.

Theorem 1.5. *There are three resolutions*

$$0 \to A(-3)^7 \stackrel{\alpha}{\to} A \oplus A(-2)^6 \to R^{(0)} \to 0$$

$$0 \to A(-4) \oplus A(-2)^6 \stackrel{\alpha^t}{\to} A(-1)^7 \to R^{(1)} \to 0$$

$$0 \to A(-3)^2 \oplus A(-2)^3 \stackrel{\beta}{\to} A^2 \oplus A(-1)^3 \to R^{(2)} \to 0$$

where $\beta = \beta^t$.

Proof.
It is an easy exercise following the same argument of [Cat3].

□

Corollary 1.6. *$R(S) = R(X)$ is generated in degree ≤ 6 as an $A-$module.*

Remark 1.7. *In [Ci] is shown the weaker result that $R(S) = R(X)$ is generated in degree ≤ 6 as a ring.*

From now on, let us assume $\mathrm{Tors}(S) = 0$ or $\mathbb{Z}/2\mathbb{Z}$. In this last case, denote by μ the nonzero torsion element in Pic (S).

Under this assumption, we can prove

Proposition 1.8. *$R(S) = R(X)$ is generated as a ring in degree ≤ 5.*

Proof.
By corollary 1.6, we must only prove that every section of $H^0(6K_X)$ can be written as sum of products of sections of degree ≤ 5.

Take an effective divisor $C \in |2K_X|$, and let $C = div(c)$; then

$$H^0(6K_X) \supset W_C := cH^0(4K_X) + S^2(H^0(3K_X)).$$

We will prove that there exists some C s.t. this inclusion is an equality.

Since $H^1(4K) = 0$ we get the following exact sequence

$$0 \to H^0(\mathcal{O}_X(4K)) \stackrel{c\cdot}{\to} H^0(\mathcal{O}_X(6K)) \stackrel{\pi}{\to} H^0(\omega_C^2) \to 0.$$

By definition W_C contains Ker π; whence, it suffices to show that there exists C s.t. $\pi : S^2(H^0(3K_X)) \to H^0(\omega_C^2)$ is surjective; this is equivalent to the surjectivity of $\pi|_C : S^2(H^0(\omega_C)) \to H^0(\omega_C^2)$, that is verified for every C irreducible and non hyperelliptic by Noether's theorem.

It is clear that the general C is irreducible (since we have no fixed part, and $|M|$ is a linear pencil with $h^0(\mathcal{O}_S(M)) = 2$). That the general C is non-hyperelliptic follows from the forthcoming lemma 1.10.

□

Lemma 1.9. *Let C be a $3-$connected genus 4 Gorenstein curve, let ω be the dualizing sheaf of C. Assume that φ_ω embeds C. Then $\varphi_\omega(C)$ is a complete intersection of type $(2,3)$.*

Proof.
C has genus 4, so $h^0(C,\omega) = 4$, $h^0(C,\omega^2) = 9$, $h^0(C,\omega^3) = 15$. So the natural map $S^2(H^0(C,\omega)) \to H^0(C,\omega^2)$ has a non-trivial kernel.

Assume, by contradiction, that this kernel has dimension greater than 1.

Thus, there exist two distinct quadrics Q_1, Q_2 containing the degree 6 curve $\varphi_\omega(C)$. If Q_1, Q_2 have no common components, their intersection is a curve of degree 4, a contradiction.

Therefore there do exist linear forms L_0, L_1, L_2 such that $Q_1 = L_0L_1, Q_2 = L_0L_2$, and $\varphi_\omega(C) \subset L_0L_1 \cap L_0L_2 = L_0 \cup (L_1 \cap L_2)$.

But $\varphi_\omega(C)$ is non degenerate, so we can write $C = C_1 + C_2$, with $\varphi_\omega(C_1) \subset L_0$ of degree 5, $\varphi_\omega(C_2) = L_1 \cap L_2$ a line.

Now, recalling that C is assumed to be 3−connected, we can compute (note that C_1C_2 is defined as: $\deg_{C_1}(\omega_C)$ - $\deg_{C_1}(\omega_{C_1})$, cf. [CFHR])

$$4 = g(C) = g(C_1) + g(C_2) - 1 + C_1C_2 \geq 6 + 1 - 1 + 3 = 9,$$

hence we derive a contradiction.

Therefore there is only one quadric containing $\varphi_\omega(C)$, let us denote it by Q.

Now, by a dimension count, the map $S^3(H^0(C, \omega)) \to H^0(C, \omega^3)$ has a kernel of dimension at least 5. In particular we get at least one cubic surface G containing $\varphi_\omega(C)$ and not having Q as a component.

If G and Q have no common components, their intersection is a degree 6 curve containing $\varphi_\omega(C)$, so $\varphi_\omega(C) = Q \cap G$ and we are done.

Otherwise, there must exist linear forms L_0, L, and a quadratic form Q', such that $Q = L_0L$, $G = L_0Q'$, and $\varphi_\omega(C) \subset L_0 \cup (L \cap Q')$. Again, we can decompose C as $C_1 + C_2$, with $\varphi_\omega(C_1) \subset L_0$, $\varphi_\omega(C_2) \subset L \cap Q'$. If $\varphi_\omega(C_2) \neq L \cap Q'$, we have decomposed $\varphi_\omega(C)$ as the union of a plane quintic and of a line, and we have already excluded this case. Else, $\varphi_\omega(C_1)$ is a plane quartic, $\varphi_\omega(C_2)$ a plane conic, and again we get

$$4 = g(C) = g(C_1) + g(C_2) - 1 + C_1C_2 \geq 3 + 1 - 1 + 3 = 6,$$

a contradiction.

□

Let us recall (cf. [Cat2]) that a honestly hyperelliptic curve is a finite covering of degree 2 of $\mathbb{P}^1$.

Lemma 1.10. *Let $C \in |2K_X|$; one of the following holds:*

a) C is embedded by ω_C, $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is the complete intersection of a quadric and a cubic; moreover, if $\varphi_{\omega_C}(C)$ is reducible, it decomposes as the union of two plane cubics intersecting (with multiplicity) in three points;

b) C is honestly hyperelliptic, $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is a double twisted cubic curve;

c) $C = 2D$; in this case Tors(S)=$\mathbb{Z}/2\mathbb{Z}$, $D \in |K + \mu|$, and $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is a sextuple line.

Case a) is the general one.

Proof.

Let us consider first the case where C is not 3-connected. Then, by [CFHR], lemma 4.2 and its proof, π^*C is not 3−connected, and we have a decomposition $\pi^*(C) = D_1 + D_2$ with $D_1D_2 \leq 2$ and with $K_SD_i = 1$.

We get then $D_1^2 + D_2^2 = (2K_S)^2 - 2D_1D_2 \geq 0$, so we can assume D_1^2 non negative, whence positive because it must be odd; by the algebraic index theorem $D_1^2 = 1$, and $D_1 = K_S + \varepsilon$, $D_2 = K_S - \varepsilon$, $\varepsilon \in \mathrm{Tors}(S)$.

Since $H^0(K_S) = 0$, by our hypothesis on the torsion group follows that Tors(S) $= \mathbb{Z}/2\mathbb{Z}$, and that $\varepsilon = \mu = -\mu$.

Remark that $h^0(K+\mu) = 1$, whence $D_1 = D_2 \in |K_S + \mu|$.

Since $h^0(3K - (K+\mu)) = h^0(2K-\mu) = h^0(2K+\mu) = 2$, then $\varphi_{|3K|}(D_1)$ is a curve of degree $(K_S+\mu)3K_S = 3$ contained in a line, thus it is a triple line. This gives case c).

If C is 3−connected, by [CFHR], theorem 3.6, either $\omega_C = 3K_{|C}$ is very ample or C is honestly hyperelliptic. Note that if C is honestly hyperelliptic and reducible, then C consists of two smooth rational curves intersecting (with multiplicity) in 5 points. In this case $\varphi_{\omega_C}(C)$ is an irreducible non degenerate curve of degree 3 in $\mathbb{P}^3$, so its schematic image is a double structure on a twisted cubic curve.

Assume now that C is canonically embedded: by lemma 1.9 $\varphi_{\omega_C}(C)$ is a complete intersection of type $(2,3)$.

Finally, if C is reducible, $C = C_1 + C_2$, where C_1, C_2 are irreducible and $C_1 \neq C_2$ by the hypothesis of 3-connectedness (else, $C_1C_2 = 1$). Since $K_S\pi^*(C_i) = 1$, $\varphi_{\omega_{\pi^*(C)}}(C_i)$ is a plane curve of degree $3K_S\pi^*(C_i) = 3$, and we get thus two distinct irreducible plane cubics intersecting in three points.

Remark that case a) is the general one because (under our assumption about the torsion of S), $\varphi_{|3K_S|}$ is a birational morphism, as proved in [Cat1].

□

2. The tri-bicanonical map

As we recalled in the proof of lemma 1.10, by [Cat1], for a numerical Godeaux surface with torsion 0 or $\mathbb{Z}/2\mathbb{Z}$ the tricanonical system defines a birational morphism onto a surface $\Sigma \subset \mathbb{P}^3$ of degree 9.

We consider the rational map $\Phi : S \to \mathbb{P}^3 \times \mathbb{P}^1$ whose components are the tricanonical and the bicanonical maps. This is not a morphism in the base points of the movable part of the bicanonical system; its image Y is a birational model of our S that dominates Σ.

We know that the general bicanonical curve is a complete intersection of a quadric and a cubic; then, as mentioned in the introduction, our aim would be to construct two hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^1$ of bidegree respectively $(2,m)$ and $(3,m')$ such that their complete intersection is Y.

Let $\beta : \tilde{S} \to S$ be a minimal sequence of ordinary blow ups such that $\varphi \circ \beta$ is a morphism. Denote by π_1, π_2 the two respective projections of Y on $\mathbb{P}^3$ and $\mathbb{P}^1$, set $f := \pi_2 \circ \varphi \circ \beta : \tilde{S} \to \mathbb{P}^1$.

Lemma 2.1. *Let B be a smooth curve and $f : \tilde{S} \to B$ be a genus 4 fibration whose generic fibre is non hyperelliptic. Let F be a fibre of f, set $\omega = F + K_{\tilde{S}}$.*

Suppose moreover that the nonhyperelliptic fibres are 3-connected (equivalently, such that their canonical image is a complete intersection of type $(2,3)$).

Consider the homomorphisms of sheaves

$$S^n(f_*(\omega)) \xrightarrow{\sigma_n} f_*(\omega^n),$$

and set $\mathcal{L}_n = \ker \sigma_n$ *and* $\mathcal{T}_n = \operatorname{coker} \sigma_n$. *Then*

i) $\mathcal{T}_n$ *is a torsion sheaf supported on the image of the hyperelliptic fibers.*

ii) Let $p \in \mathbb{P}^1$ *be the image of a honestly hyperelliptic fibre (whose canonical image is a twisted cubic); then there is a positive integer s such that*

$$\forall k \geq 2 \ \ \textit{length}\ (\mathcal{T}_k, p) = s(3k-4).$$

Proof.

Let $\mathcal{M}_p$ be the maximal ideal sheaf of the point p in $\mathcal{O}_B$.

By Grauert's base change theorem (cf. e.g. [BPV]) $\forall p \in B$, $\frac{(f_*\omega^k)}{(\mathcal{M}_p f_*\omega^k)} \cong H^0(F_p, \omega^k)$; mod $\mathcal{M}_p$ the morphism σ_k acts on the stalks as $S^k(H^0(F_p,\omega)) \to H^0(F_p,\omega^k)$, whence it is surjective when F_p is non hyperelliptic.

By Nakayama's lemma, if p corresponds to a non hyperelliptic fibre, then $(\sigma_n)_p$ is surjective, and the first part of the teorem is proved.

Let us now assume that F_p is a honestly hyperelliptic fibre, so that F_p is a double cover of $\mathbb{P}^1$ branched, by Hurwitz formula, on a divisor of degree 10 . We can embed F_p in $\mathbb{P}(5,1,1)$ as the hypersurface defined by the equation $w^2 = P(t_0,t_1)$, where P is the homogeneus polynomial of degree 10 whose divisor is the branch divisor of the canonical map.

Following the same line of [ML] it is easy to prove that the canonical ring of F_p is generated in degree 2; this ring can be described as a subring of the ring $\mathcal{A} = \mathbb{C}[t_0,t_1,w]/ < w^2 = P(t_0,t_1) >$.

In fact, generators for $H^0(\omega)$ are $y_0 = t_0^3$, $y_1 = t_0^2 t_1$, $y_2 = t_0 t_1^2$, $y_3 = t_1^3$; the kernel of the map $S^2(H^0(F_p,\omega)) \to H^0(F_p,\omega^2)$ has dimension 3 (three independent quadrics through a twisted cubic), so the cokernel has dimension $9-10+3=2$, and we can see that it is generated by $v_0 = t_0 w$, $v_1 = t_1 w$.

It follows that, if we choose 3 degree 4 polynomials $P_{00}(y_i)$, $P_{01}(y_i)$, $P_{11}(y_i)$, s.t. in the ring $\mathcal{A}$ is $P_{00} = t_0^2 P$, $P_{01} = t_0 t_1 P$, $P_{11} = t_1^2 P$, we get the following 9 relations:

$$\begin{array}{lll} r_1 := y_1^2 - y_0 y_2 & r_2 := y_2^2 - y_1 y_3 & r_3 := y_0 y_3 - y_1 y_2 \\ r_4 := v_0 y_1 - v_1 y_0 & r_5 := v_0 y_2 - v_1 y_1 & r_6 := v_0 y_3 - v_1 y_2 \\ r_7 := v_0^2 - P_{00} & r_8 := v_0 v_1 - P_{01} & r_9 := v_1^2 - P_{11}. \end{array}$$

So we can describe the canonical ring R of F_p as a quotient of the graded ring $\mathbb{C}[y_0,y_1,y_2,y_3,v_0,v_1]/ < r_1,\dots,r_9 >$, where deg $y_i = 1$, deg $v_i = 2$. We have $H^0(F_p,\omega) = 4$, and $\forall k \geq 2$ $H^0(F_p,\omega^k) = 6k-3$ but on the other hand an easy calculation yields that the homogeneus part of degree k of our ring has at most the same dimension. Therefore follows that $R = \mathbb{C}[y_0,y_1,y_2,y_3,v_0,v_1]/ < r_1,\dots,r_9 >$.

Let us denote by R_k the homogeneus part of R of degree k. Remark that

$$(f_*\omega^k)_p \otimes_{\mathcal{O}_p} \mathbb{C} = R_k,$$

so, by flatness, $\oplus_k (f_*\omega^k)_p = \mathcal{O}_p[y_0,y_1,y_2,y_3,v_0,v_1]/ < \overline{r}_1,\dots,\overline{r}_9 >$, where the $\overline{r}_i$'s are lifts to $\mathcal{O}_p$ of the r_i's.

Moreover, every syzygy of R lifts to a syzygy of the $\mathcal{O}_p$-module $\oplus_k (f_*\omega^k)_p$.

Let t be a local parameter for $\mathcal{O}_p$, and write

$$\begin{pmatrix}\overline{r}_1\\ \overline{r}_2\\ \overline{r}_3\end{pmatrix} = \begin{pmatrix}\tilde{q}_1(y_j,t)\\ \tilde{q}_2(y_j,t)\\ \tilde{q}_3(y_j,t)\end{pmatrix} + \begin{pmatrix}\tilde{\alpha_{11}}(t) & \tilde{\alpha}_{12}(t)\\ \tilde{\alpha_{21}}(t) & \tilde{\alpha}_{22}(t)\\ \tilde{\alpha_{31}}(t) & \tilde{\alpha}_{32}(t)\end{pmatrix}\begin{pmatrix}v_0\\ v_1\end{pmatrix},$$

where $\tilde{q}_i(y_j,0) = r_i$, and $\tilde{\alpha}_{i,j}(0) = 0$. Let s be the minimum of the orders of vanishing of the $\tilde{\alpha}_{ij}(t)$'s; we can then find a new basis for the respective vectors spaces generated by v_0 and v_1, and by $\overline{r}_1,\overline{r}_2,\overline{r}_3$, and a new local parameter t, so that we can write our relations in the following simpler form:

$$\begin{pmatrix}\overline{r}_1\\ \overline{r}_2\\ \overline{r}_3\end{pmatrix} = \begin{pmatrix}q_1(y_j,t)\\ q_2(y_j,t)\\ q_3(y_j,t)\end{pmatrix} + \begin{pmatrix}t^s & t^{s+1}\alpha_{12}(t)\\ t^{s+1}\alpha_{21}(t) & t^s\alpha_{22}(t)\\ t^{s+1}\alpha_{31}(t) & t^s\alpha_{32}(t)\end{pmatrix}\begin{pmatrix}v_0\\ v_1\end{pmatrix}.$$

Clearly then the linear space of conics generated by the $q_i(y_j,0)$'s coincides with the space generated by the r_i's $(i = 1,2,3)$.

Lifting the syzygy $y_3r_1 + y_1r_2 + y_2r_3$, by degree reasons we get a syzygy of the form $L_3(y_j,t)\overline{r}_1 + L_1(y_j,t)\overline{r}_2 + L_2(y_j,t)\overline{r}_3 + f_4(t)\overline{r}_4 + f_5(t)\overline{r}_5 + f_6(t)\overline{r}_6$, where the $L_i(y_j,0)$'s are three independent linear forms, and $\overline{r}_4$, $\overline{r}_5$, $\overline{r}_6$ are lifts of r_4,r_5,r_6.

Working modulo the ideal generated by t^{s+1} and by the monomials of degree 3 in the (y_j)'s we get

$$t^s(L_3(y_j,0)v_0 + (\alpha_{22}(0)L_1(y_j,0) + \alpha_{23}(0)L_2(y_j,0))v_1) \in (\overline{r}_4,\overline{r}_5,\overline{r}_6)$$

But in fact, there are no constant coefficients syzygies among r_4,r_5,r_6, thus we conclude that

$$L_3(y_j,0)v_0 + (\alpha_{22}(0)L_1(y_j,0) + \alpha_{23}(0)L_2(y_j,0))v_1 \in (r_4,r_5,r_6)$$

which excludes the possibility that $\alpha_{22}(0) = \alpha_{23}(0) = 0$.

Therefore, choosing new bases for the respective $\mathcal{O}_p$-modules generated by v_0 and v_1, and by $\overline{r}_1,\overline{r}_2,\overline{r}_3$, we can write our relations in the following even simpler form:

$$\begin{pmatrix}\overline{r}_1\\ \overline{r}_2\\ \overline{r}_3\end{pmatrix} = \begin{pmatrix}q_1(y_j,t)\\ q_2(y_j,t)\\ q_3(y_j,t)\end{pmatrix} + \begin{pmatrix}t^s & 0\\ 0 & t^s\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_0\\ v_1\end{pmatrix}.$$

This allows us to compute, using the lifts of r_7,r_8,r_9 to eliminate the multiples of v_0^2, v_0v_1, v_1^2, and the lifts of r_4,r_5,r_6 to eliminate the multiples of v_0 as much as possible, that there exists a nonzero linear form $L_0(y_j)$ such that the set $\{t^iv_0L_0^{k-2}(y_j), t^iv_1q_l(y_j) | i < s\}$ is a basis for $(\mathcal{T}_k)_p$, when $\{q_l\}$ is a basis for the homogeneus degree $k-2$ part of $\mathbb{C}[y_0,y_1,y_2,y_3]/ < r_1,r_2,r_3 >$. But this is the projective coordinate ring of a twisted cubic, whose homogeneous part of degree d has dimension $3d+1$; whence the dimension of $(\mathcal{T}_k)_p$ equals $s(1+3(k-2)+1) = s(3k-4)$.

□

The integer s arising in lemma 2.1 can in fact be defined as follows:

Definition 2.2. *Let C be a honestly hyperelliptic curve of genus g , occurring as a fibre of a genus g fibration $f : \tilde{S} \to B$ where $\tilde{S}$ is smooth. Define the* **multiplicity** *of C (or mult(C)), as the multiplicity of C in the conductor ideal of f (recall that the conductor is a divisorial ideal).*

Proposition 2.3. *The integer s associated to a honestly hyperelliptic fibre C as in lemma 2.1 equals the multiplicity.*

Proof:

Let $p \in B$ be a point such that C is the fibre of p and U a sufficiently small affine open neighbourhood of p. Let $Y \subset \mathbb{P}^3 \times U$ be the image of the relative canonical map φ of f.

By abuse of notation let us still denote by $\tilde{S} = f^{-1}(U)$. The sheaf of double points Δ, supported on the image Γ of C is defined via the exact sequence

$$0 \to \mathcal{O}_Y \to \varphi_* \mathcal{O}_{\tilde{S}} \to \Delta \to 0.$$

Twisting the exact sequence by $\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(n, 0)$, and observing that $\varphi^* \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(1, 0) \cong \omega$, from the definition of $\mathcal{T}_n$ we get that

$$(\mathcal{T}_n)_p \cong H^0(\Gamma, \Delta(n)).$$

From lemma 2.1 we conclude that the length of Δ at the generic point of Γ equals s. Since (as we shall also see in lemma 2.6) at the general point of Γ we have a singularity consisting of two smooth branches, we conclude immediately that s equals the multiplicity of C in the conductor divisor.

□

The geometric meaning of the definition and proposition above is that s should be interpreted as the intersection multiplicity of the curve B with the hyperelliptic locus inside the moduli space of the curves of genus 4.

Remark 2.4. *The fibration f we had already defined is a genus 4 fibration if and only if (see remark 1.2) $F = 0$ and the map β is a sequence of blow-ups in smooth points, possibly infinitely near, of the generic bicanonical divisor; i.e., if and only if $F = 0$ and the base locus of $|2K_S|$ is not consisting of a single point where every bicanonical divisor has multiplicity 2.*

The above condition is of course an open condition; in fact we shall prove later that the Craighero Gattazzo surface enjoys such a property.

Theorem 2.5. *Assume that S is a numerical Godeaux surfaces with torsion $\{0\}$, s.t. f is a genus 4 fibration. Let $h = \sum_{C \ hyperelliptic} mult(C)$.*

Then $\exists \mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$, s.t. $Y := \varphi(S)$ is a divisor in $|\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$.

Proof.

By remark 2.4, the map β we had already defined at the beginning of this section, is a sequence of blow-ups of smooth points of the generic bicanonical divisor.

Let $E_i,\ i = 1, \dots, 4$ be the corresponding exceptional divisors of the first kind, and set $E = \sum E_i$. Then $K_{\tilde{S}} = \beta^*(K_S)+E$, and if F is a generic fibre of f (F is the strict transform of a generic bicanonical divisor on S by β), $F = \beta^*(2K_S) - E = 2K_{\tilde{S}} - 3E$ is a genus 4 curve.

The pull-back of the tricanonical system is given by $\omega := \beta^*(3K_S) = 3K_{\tilde{S}} - 3E = K_{\tilde{S}} + F$. In view of lemma 1.10 the hypotheses of lemma 2.1 are satisfied.

Consider now the exact sequences

$$0 \to \mathcal{L}_2 \to \mathcal{S}^2(f_*\omega) \xrightarrow{\sigma_2} f_*\omega^2 \to \mathcal{T}_2 \to 0; \tag{1}$$

$$0 \to \mathcal{L}_3 \to \mathcal{S}^3(f_*\omega) \xrightarrow{\sigma_3} f_*\omega^3 \to \mathcal{T}_3 \to 0. \tag{2}$$

$\forall F$, the map $H^0(\tilde{S}, \omega) \to H^0(F, \omega)$ is an isomorphism, therefore $f_*\omega \cong \mathcal{O}_{\mathbb{P}^1}^4$.

In particular $\mathcal{L}_2$ is a subsheaf of $\mathcal{S}^2(f_*\omega) \cong \mathcal{O}^{10}$, while $\mathcal{L}_3$ is a subsheaf of $\mathcal{S}^3(f_*\omega) \cong \mathcal{O}^{20}$.

Moreover, since $\mathcal{T}_k$ is a torsion sheaf, the rank of $\mathcal{L}_k$ equals the difference dim $S^k(H^0(F,\omega)) - h^0(F, \omega^k)$. Therefore rank $\mathcal{L}_2 = 1$, rank $\mathcal{L}_3 = 5$ and we can write $\mathcal{L}_2 \cong \mathcal{O}_{\mathbb{P}^1}(-m)$.

Since $\mathcal{L}_2$ is a subsheaf of $\mathcal{O}^{10}$, $m \geq 0$, and the injection $\mathcal{L}_2 \to \mathcal{O}^{10}$ defines a hypersurface $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, m)|$ that contains Y (in particular, then, $m \geq 1$).

Let us now compute the Euler characteristic of the exact sequence (1).

We get

$$\chi(\mathcal{L}_2) = 1 - m;$$

$$\chi(S^2(f_*\omega)) = \chi(\mathcal{O}^{10}) = 10;$$

$$R^1 f_*\omega^2 = 0 \Rightarrow \chi(f_*\omega^2) = \chi(\omega^2) = 16;$$

$$\chi(\mathcal{T}_2) = \text{length } (\mathcal{T}_2) = 2h;$$

so $1 - m + 16 = 10 + 2h$, i.e. $m = 7 - 2h$.

The splitting surjective homomorphism $S^2(f_*\omega) \otimes f_*\omega \to S^3(f_*\omega)$, induces a homomorphism $\mathcal{L}_2^4 \to \mathcal{L}_3$; it is easy to see that this is injective and that its cokernel is a subsheaf of the locally free sheaf $(f_*\omega^2)^4$.

So, $\mathcal{L}_3$ being a locally free sheaf of rank $20 - 15 = 5$, we can write the following exact sequence

$$0 \to \mathcal{O}(2h-7)^4 \to \mathcal{L}_3 \to \mathcal{O}(-m') \to 0, \tag{3}$$

But then

$$\chi(\mathcal{L}_3) = 8h - m' - 23;$$

$$\chi(S^3(f_*\omega)) = \chi(\mathcal{O}^{20}) = 20;$$

$$R^1 f_*\omega^3 = 0 \Rightarrow \chi(f_*\omega^3) = \chi(\omega^3) = 37;$$

$$\chi(\mathcal{T}_3) = \text{length } (\mathcal{T}_3) = 5h;$$

so by the exact sequence (2) we get $8h - m' - 23 + 37 = 20 + 5h$, i.e. $m' = 3h - 6$.

Remark now that the injection $\mathcal{O}_{\mathbb{P}^1}(6-3h) = \mathcal{L}_3/\mathcal{L}_2^4 \to S^3(f_*\omega)/\mathcal{L}_2 \cdot H^0(f_*\omega)$ defines a divisor $\mathcal{D} \in |\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ containing Y. $\mathcal{D}$ and Y have the same dimension; $\forall H_2 \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(0,1)|$, $H_2 \cap Y$ has degree 6. But if $H_2 \cap \mathcal{D}$ is a curve, it has the same degree and contains $H_2 \cap Y$, so they coincide. So, if $\mathcal{D} \neq Y$, $\exists H_2$ s.t. $H_2 \cap \mathcal{D}$ is a component of $H_2 \cap \mathcal{Q}$; if $\mathcal{Q} \cap H_2 = \mathcal{D} \cap H_2$ we get some torsion element in coker $\mathcal{L}_3 \to S^3(f_*\omega)$, that is a subsheaf of the locally free sheaf $f_*\omega^3$, a contradiction. Otherwise $\mathcal{Q} \cap H_2$ is the union of two planes, and $\mathcal{D} \cap H_2$ is exactly one of the two planes; but we have already excluded this case in the proof of lemma 1.9.

□

Now, let us understand the local behaviour of Y near the image of every honestly hyperelliptic curve occuring as fibre of $f : \tilde{S} \to \mathbb{P}^1$.

Proposition 2.6. *Let C be a honestly hyperelliptic genus 4 fibre of f, $\Gamma = \varphi(C)$. Assume that the multiplicity of C equals s: then, in the neighbourhood of a general point $p \in \Gamma$ there exist local coordinates (y_1, y_2, y_3, t), such that Y is defined by the equations $y_2 = y_1(y_1 - t^s) = 0$, Γ by $y_1 = y_2 = t = 0$, and the projection π_2 is (still) given by the coordinate t.*

Proof.

For the general $p \in \Gamma$ there exists a neighbourhood U of p in $\mathbb{P}^3 \times \mathbb{P}^1$ such that $\varphi^{-1}(U)$ has two smooth connected components, and φ identifies the two smooth holomorphic curves corresponding to C.

So, for a first suitable choice of local coordinates in the source and in the target we can assume that $\Gamma = \{y_1 = y_2 = t = 0\}$, the projection π_2 is given by the coordinate t, and the two branches of Y are parametrized as follows

$$(u_1, t_1) \to (0, 0, u_1, t_1)$$
$$(u_2, t_2) \to (t_2\phi_1(u_2, t_2), t_2\phi_2(u_2, t_2), u_2, t_2).$$

So, for a suitable local analytic coordinate change that fixes t, we get the simpler form

$$(u_1, t_1) \to (0, 0, u_1, t_1)$$
$$(u_2, t_2) \to (t_2^a, 0, u_2, t_2).$$

And Y is described by the equations $y_2 = y_1(y_1 - t^a) = 0$.

Finally, remarking that the conductor ideal is generated by y_1, t^a, we get $a = s$.

□

Corollary 2.7. *Assume that a fibre $F = \{t = 0\}$ appears in the conductor divisor with multiplicity s. Thus, if $Q(y_i, t)$ represents a divisor in $\mathbb{P}^3 \times \mathbb{P}^1$ s.t. $\varphi^* div(Q(y_i, t)) \geq 2sF$, then $t^s | Q(y_i, t) \pmod{\mathfrak{I}_Y}$.*

Proof.
By our assumption div $Q(y_i, t)$ pulls back to a divisor $\geq 2sF$.

Since we are interested in Q mod $\mathfrak{I}_Y$, this means, using the local coordinates introduced above, that we can look at Q modulo y_2, and writing $Q'(y_1, y_3, t) = Q(y_1, 0, y_3, t)$ we get as a first condition that

$$1)\ Q' \in (y_1, t^{2s}) \Leftrightarrow Q' = y_1 q' + t^{2s} g$$

and it suffices therefore to prove that $t^s | q'$.

The condition imposed by the second branch is that $t^{2s} | t^s q'(t^s, v, t) \Leftrightarrow q' \in (y_1, t^s)$. Thus, mod $\mathfrak{I}_Y$, $Q' \equiv y_1^2 a + y_1 t^s b + t^{2s} g$.

But $\mathfrak{I}_Y \ni y_1(y_1 - t^s)$ and thus $Q' \equiv t^s(y_1(a + b) + t^s g)$.

□

Consider now the case where $F = 0$, but f is not a genus 4 fibration. In this case we can consider the blow up $\beta' : \tilde{S}' \to S$ of S in the single base point P of $|2K_S|$. If E is the exceptional divisor of β', the strict transform of the bicanonical system is given by $\beta'^* 2K_S - 2E$.

Let $g' : \tilde{S}' \to \mathbb{P}^2 \times \mathbb{P}^1$ be the morphism obtained from $|\beta'^* 3K_S - E| \times |\beta'^* 2K_S - 2E|$, let $\pi_2' : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the second projection, set $f' = \pi_2' \circ g'$.

Recall, to understand the statement of the following lemma, that a curve C is said to be hyperelliptic if the canonical map is not birational.

Lemma 2.8. *Let $f' : \tilde{S} \to \mathbb{P}^1$ be a genus 3 fibration whose fibres are 2-connected and whose generic fibre is non hyperelliptic. Let F be the fibre of f', set $\omega = F + K_{\tilde{S}}$.*

Consider the homomorphisms of sheaves

$$S^n(f'_* \omega) \xrightarrow{\sigma_n} f'_*(\omega^n),$$

and denote by $\mathfrak{L}_n = \ker \sigma_n$ and $\mathfrak{T}_n =$ coker σ_n. Then

i) $\mathfrak{T}_n$ is a torsion sheaf supported on the image of the hyperelliptic fibres.

ii) Let $p \in \mathbb{P}^1$ be the image of some hyperelliptic fibre; then $\exists s > 0, s \in \mathbb{N}$, such that

$$\forall k \geq 2 \quad \textit{length } (\mathfrak{T}_k, p) = s(2k - 3).$$

Proof.
i) This point follows since if C is not hyperelliptic the canonical image of C is a plane quartic (in fact, the hypothesis of 2-connectedness ensures that the canonical system has no base points, see [CFHR], lemma 3.3.b).

ii) Recall that, by [ML], the canonical ring of a hyperelliptic fibre has the form

$$R = \mathbb{C}[x_1, x_2, x_3, y] / < r_1 := Q(x_i), r_2 := y^2 - F(x_i) >,$$

where deg $x_i = 1$, deg $y = 2$, deg Q=2, deg $F = 4$.

Acting as in lemma 2.1, if we choose a suitable local parameter t in $\mathcal{O}_p$, we can write a lift of r_1 as

$$\overline{r}_1 = \overline{Q}(x_i, t) + t^s y$$

for some $s > 0$, $\overline{Q}(x_i, 0) = Q(x_i)$.

This allows us to compute, using the lift of r_2 to eliminate the multiples of y^2, that the set $\{t^i q_j y | i < s\}$ is a basis for $\mathfrak{T}_k$ when the set $\{q_j\}$ is a basis for the homogeneus part of degree $k-2$ of the quotient ring $\mathbb{C}[x_1, x_2, x_3]/Q$.

□

Theorem 2.9. *For a numerical Godeaux surface with torsion $\{0\}$, and of type ib) (bicanonical system without fixed part possessing a double base point) f' is a genus 3 fibration, and g' yields fibrewise the canonical map of the fibres. Moreover, f' has exactly 7 hyperelliptic fibres (counted with multiplicity according to 2.8) and the image of g' is a divisor in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(4, 8)|$.*

We need the following

Lemma 2.10. *Under the above assumption, all the fibers are 2-connected and the generic fibre F is non hyperelliptic.*

Proof.

Let F be a fibre of f'. Since $EF = 2$ and E is irreducible, it follows that if we have a decomposition $F = A + B$, then $AE \geq 0$, and similarly $BE \geq 0$, whence

$$\begin{cases} AE = 0, BE = 2 \quad \text{or} \\ AE = BE = 1. \end{cases}$$

Assume that $AB = r$: then in the first case we get $A(B + 2E) = r$, in the second we get $(A + E)(B + E) = r + 1$. In both cases we obtain a decomposition $(A' + B') \in |2K_S|$ where $A'B' \leq (r + 1)$. We conclude that $r \geq 2$ because under our assumptions (see [Bo], lemma 2, page 181) $A'B' \geq 2$ and is equal to 2 only if, say, $A'K_S = 0$, what excludes $AE = 1$ (since otherwise there would be a fixed part of the bicanonical system).

Assume by contradiction that every F is hyperelliptic. We observe that the first component of g' restricts to every fibre F to the complete canonical system of F. Therefore we obtain that g' is $2:1$ so it defines a (birational) involution σ on $\tilde{S}$ that is the hyperelliptic involution on every fiber. Since σ acts biregularly on the minimal model S and clearly fixes P, σ also acts biregularly on $\tilde{S}$ leaving E invariant.

In particular on every F the hyperelliptic involution induces a involution on the corresponding bicanonical divisor in S, so every bicanonical divisor on S is hyperelliptic. This is a contradiction because the rational map induced by $|3K_S|$ is birational.

□

Proof of theorem 2.9.
Consider the exact sequences

$$0 \to \mathcal{L}_2 \to \mathcal{S}^2(f'_*\omega) \xrightarrow{\sigma_2} f'_*\omega^2 \to \mathcal{T}_2 \to 0; \tag{4}$$

$$0 \to \mathcal{L}_3 \to \mathcal{S}^3(f'_*\omega) \xrightarrow{\sigma_3} f'_*\omega^3 \to \mathcal{T}_3 \to 0; \tag{5}$$

$$0 \to \mathcal{L}_4 \to \mathcal{S}^4(f'_*\omega) \xrightarrow{\sigma_4} f'_*\omega^4 \to \mathcal{T}_4 \to 0. \tag{6}$$

Argueing as in theorem 2.5, we get that $\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are locally free sheaves of respective ranks 0, 0, 1, and that the $\mathcal{T}_j$ are torsion sheaves supported on the points corresponding to hyperelliptic fibers. By lemma 2.8 for every such point $p \in \mathbb{P}^1$ there is a multiplicity s_p s.t.

$$\text{length } (\mathcal{T}_2, p) = s_p,$$

$$\text{length } (\mathcal{T}_3, p) = 3s_p,$$

$$\text{length } (\mathcal{T}_4, p) = 5s_p.$$

Computing the Euler characteristics in the sequences (4), (5) and (6), we get $s = \sum s_p = 7$ and $\mathcal{L}_4 = \mathcal{O}_{\mathbb{P}^1}(-8)$.

So we can conclude that $g(\tilde{S}) \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4,8)|$.

□

Remark that, in the case of torsion $\mathbb{Z}/2\mathbb{Z}$, everything works almost identically, except that we have to calculate the hyperelliptic multiplicity corresponding to the non 1-connected (double) fibre: this will be done in the sequel to the present paper.

3. The genus 4 fibration cannot have three distinct hyperelliptic fibres.

The goal of this section is to prove the following

Proposition 3.1. *Let S a numerical Godeaux surface with torsion $\{0\}$ such that $|2K_S|$ has 4 base points possibly infinitely near (equivalently, s.t. f is a genus 4 fibration). Then the bicanonical pencil cannot contain three distinct honestly hyperelliptic fibres.*

We shall argue by contradiction and assume by theorem 2.5 that $h = 3$ and that Y is a divisor in $|\mathcal{O}_{\mathcal{Q}}(3,3)|$, with $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2,1)|$.

Remark 3.2. *Y is the complete intersection of $\mathcal{Q}$ with a hypersurface $\mathcal{G}$, where $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(3,3)|$.*

In fact, exact sequence 3 splits because $\mathrm{Ext}^1(\mathcal{O}(-3),\mathcal{O}(-1)^4) = H^1(\mathcal{O}(2)^4) = 0$; so exact sequence 2 induces a divisor $\mathcal{G}$ in $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(3,3)|$ that cuts out Y on $\mathcal{Q}$.

Lemma 3.3. *If the torsion group is $\{0\}$ and $h = 3$, then there exists a quadric Q'' in $\mathbb{P}^3$ containing each twisted cubic, image of a honestly hyperelliptic fibre. Moreover, the pull back of Q'' to $\tilde{S}$ yields an effective divisor which is greater than the adjoint (conductor) divisor.*

Proof.
Let X be the canonical model of S, and let $\hat{\beta} : \tilde{X} \to X$ be the blow up in the base points of $|2K_X|$ (they are smooth points of X by our hypothesis).

Let ψ be given by the relative canonical map of $\hat{f} : X \to \mathbb{P}^1$.

We have the following diagram

$$\begin{array}{ccccc} \tilde{X} & & \Sigma & \subset & \mathbb{P}^3 \\ \hat{\beta}\downarrow & \searrow^{\hat{g}} & \uparrow\pi_1 & & \\ X & \stackrel{\hat{\varphi}}{\dashrightarrow} & Y & \subset & \mathbb{P}^3\times\mathbb{P}^1 \\ & & \downarrow\pi_2 & & \\ & & \mathbb{P}^1 & & \end{array}$$

and recall that $\hat{g} = \hat{\varphi}\circ\hat{\beta}$ is a birational morphism factoring through a possible contraction $\epsilon : \tilde{X} \to \hat{X}$ of strings of (-2) curves (to rational double point singularities), and a finite birational map $\tilde{g} : \hat{X} \to Y$.

By [H], ex. III.6.10 and III.7.2, $\hat{g}_*(K_{\tilde{X}}) = \mathcal{H}om_{\mathcal{O}_Y}(\hat{g}_*\mathcal{O}_{\tilde{X}}, K_Y)$. Moreover $\omega_Y = \mathcal{O}_Y(2+3-4, 1+3-2) = \mathcal{O}_Y(1,2)$, whence $\hat{g}_*(K_{\tilde{X}}) = \mathcal{C}\otimes\mathcal{O}_Y(1,2)$, $\mathcal{C}$ being the conductor ideal of $\tilde{g}$.

The pull back to $\tilde{X}$ of the conductor ideal $\mathcal{C}$ is an invertible sheaf $\mathcal{O}_{\tilde{X}}(-D)$, D is here the adjunction divisor. We have $K_{\tilde{X}} + D = \hat{g}^*(\mathcal{O}_Y(1,2))$, so $D \equiv 3F$ (as we already know). More generally, since $\mathcal{C} = \hat{g}_*\mathcal{O}_{\tilde{X}}(-D)$, the n^{th} adjoint ideal $\hat{g}_*\mathcal{O}_{\tilde{X}}(-nD)$ equals $\mathcal{C}^n$.

Whence

$$h^0(S, nK_S) = h^0(\tilde{X}, nK_{\tilde{X}}) = h^0(Y, \hat{g}_*(nK_{\tilde{X}})) = h^0(Y, \mathcal{C}^n\mathcal{O}_Y(n,2n)).$$

In particular, a global section of $g_*(K_{\tilde{S}})$ is a global section of $\mathcal{O}_Y(1,2)$, whose divisor pulls back to an effective divisor containing the honestly hyperelliptic fibres with their multiplicity; in particular its divisor contains the special twisted cubics. Since no plane contains a twisted cubic curve, we recover the basic assumption $h^0(K_S) = 0$.

Moreover, letting E be the sum of the four (-1) divisors of the blow-up, since $|\varphi^*(\mathcal{O}_Y(0,1))| = |2K_S - E| = |2K_{\tilde{S}} - 3E| = |\varphi^*(\mathcal{O}_Y(2,4)) - 2D - 3E|$, there exists $Q' \in |\mathcal{O}_Y(2,3)|$ whose pull-back on $\tilde{S}$ is a divisor consisting of $3E$ plus the sum of the honestly hyperelliptic fibres counted each $2s$ times (s being their respective multiplicity).

By corollary 2.7 Q' belongs to the sheaf of ideals $(\mathcal{Q}, \mathcal{G}, P)$, where P is a polynomial of degree 3 on $\mathbb{P}^1$ such that its divisor pulls back to the adjunction divisor on $\tilde{S}$. Since $(\mathcal{Q}, \mathcal{G}, P)$ form a regular sequence, it follows easily that there exists a quadratic polynomial $Q''(y_i)$ such that $Q' = Q''P$.

Since the pull-back of Q' contains the adjunction divisor D doubly, while P pulls-back to D, follows that the pull back of div Q'' is at least D. □

Let us devote our analysis to the case of torsion $\{0\}$ and let us write down explicitly the equations of the two divisors whose complete intersection gives our image surface Y.

Let $Q_\lambda = \lambda_0 Q_0 + \lambda_1 Q_1$; $G_\lambda = \lambda_0^3 G_{000} + \lambda_0^2 \lambda_1 G_{100} + \lambda_0 \lambda_1^2 G_{011} + \lambda_1^3 G_{111}$.

We can assume that for $\lambda = (1,0)$, $(0,1)$ or $\mu = (\mu_0, \mu_1)$ (fixed), $Q_\lambda \cap G_\lambda$ is a double twisted cubic.

Lemma 3.4. *Q_0, Q_1, Q_μ are quadric cones of rank* 3.

Proof.

If one of these quadrics, say Q_0, were smooth then Q_0 would be isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1)$. Then the cubic G_{000} would cut on Q_0 a divisor in the linear system $(3,3)$, while we know that this intersection must be twice an irreducible twisted cubic curve (t.c.c. for short), a contradiction (observe that a t.c.c. lies in a linear system of type $(2,1)$ or $(1,2)$). Moreover, since the t.c.c. is irreducible and non degenerate, rank Q_0 = rank Q_1 = rank $Q_\mu = 3$.

□

By lemma 3.4, we know that Q_0, Q_1, Q_μ, are quadric cones. Let V_0, V_1, V_μ be their respective vertices, $\Gamma_0, \Gamma_1, \Gamma_\mu$ the corresponding twisted cubic curves. The tricanonical image Σ of S is the hypersurface of $\mathbb{P}^3$ defined by $\Sigma = \{Q_1^3 G_{000} - Q_1^2 Q_0 G_{100} + Q_1 Q_0^2 G_{011} - Q_0^3 G_{111} = 0\}$.

Lemma 3.5. *If $V_0 = V_1 \Rightarrow Q_0, Q_1, Q''$ have a common line L.*

Proof.

Let us consider the lines l_0, l_1, l_μ residual to the twisted cubics in the respective intersections of the three quadratic cones with the "adjoint" quadric Q". I.e., we have $Q_0 \cap Q'' = \Gamma_0 \cup l_0$, $Q_1 \cap Q'' = \Gamma_1 \cup l_1$, $Q_\mu \cap Q'' = \Gamma_\mu \cup l_\mu$.

Observe that, since $V_0 = V_1 = V_\mu$, then clearly $V_0 \in l_0 \cap l_1 \cap l_\mu$.

$Q'' \supset \Gamma_0 \Rightarrow$ rank $Q'' \geq 3$.

If Q'' is smooth, then $Q'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and every line in Q'' is contained in one of the two rulings. So, at least two of the above lines are in the same ruling, and since they intersect, they do coincide.

This line is in the base locus of the pencil Q_λ hence our assertion follows.

If Q'' is a quadric cone, denote by V'' its vertex. Every t.c.c. in a quadric cone passes trough the vertex, so $\forall i$ $V'' \in \Gamma_i \subset Q_i$; let $V = V_0 = V_1$, and observe that $V \neq V''$ (else the two quadric cones would intersect in 4 lines), whence the line $\overline{VV''}$ is contained in all these quadrics.

□

Lemma 3.6. $V_0 \neq V_1$

Proof.

Observe preliminarly that the previous lemma implies that the three twisted cubics $\Gamma_0, \Gamma_1, \Gamma_\mu$ are distinct (otherwise there would be a twisted cubic Γ contained in each Q_λ: but then $Q_0 = Q_1$, since they have the same vertex V and they are the join of V with Γ).

Observe now that Σ must be singular in our three twisted cubics: in fact Σ is the image of Y under the birational morphism given by the first projection, and Y is singular along the three twisted cubics.

Thus, $Q'' \cap \Sigma \geq 2\Gamma_0 + 2\Gamma_1 + 2\Gamma_\mu$.

Both Q'' and Σ are irreducible, so their intersection must be a curve of degree 18 and equality must hold.

But, by lemma 3.5, we have a line in $Q'' \cap Q_0 \cap Q_1$, which is a fortiori also in $Q'' \cap \Sigma$, whence a contradiction.

□

So, we can assume $V_0 \neq V_1$.

Lemma 3.7. $\forall \lambda \in \mathbb{P}^1$, *$Q_\lambda$ is a quadric cone and the line $\overline{V_O V_1}$ is contained in Q_λ.*

Proof.

Recall that $Q_0 \cap G_{000} = 2\Gamma_0$. But Q_0 is singular in V_0, so G_{000} must be smooth in V_0, thus also in a general point of Γ_0.

Observe that $V_0 \in \Gamma_0 \subset \text{Sing } \Sigma$, so, by inspecting the equation of Σ, we infer that $V_0 \in Q_1$. Similarly, $V_1 \in Q_0$, and the line $\overline{V_0V_1} \subset Q_\lambda \ \forall \lambda$.

Let us now fix coordinates s.t. $V_0 = (0,0,0,1)$, $V_1 = (0,0,1,0)$; $\overline{V_0V_1} = \{x_0 = x_1 = 0\}$.

Thus the matrix of the quadric Q_λ has the following form:

$$Q_\lambda = \begin{pmatrix} * & * & \lambda_0 * & \lambda_1 * \\ * & * & \lambda_0 * & \lambda_1 * \\ \lambda_0 * & \lambda_0 * & 0 & 0 \\ \lambda_1 * & \lambda_1 * & 0 & 0 \end{pmatrix},$$

whence the determinant of the matrix of $|Q_\lambda|$ equals the square p_2^2, of a homogeneous polynomial p_2 of degree 2 in the λ_i's; since we know that it has at least three distinct roots, we conclude that $p_2 = 0$, therefore Q_λ is a pencil of quadric cones.

□

So, after a suitable change of coordinates in $\mathbb{P}^1$ and in $\mathbb{P}^3$, we may assume that

$$Q_\lambda = \begin{pmatrix} 0 & 0 & -\frac{\lambda_0}{2} & -\frac{\lambda_1}{2} \\ 0 & \lambda_0+\lambda_1 & 0 & 0 \\ -\frac{\lambda_0}{2} & 0 & 0 & 0 \\ -\frac{\lambda_1}{2} & 0 & 0 & 0 \end{pmatrix},$$

i.e., $Q_0 = x_1^2 - x_0x_2$, $Q_1 = x_1^2 - x_0x_3$.

Remark that this choice imposes that it cannot be $\mu_0 = -\mu_1$, because otherwise we get a t.c.c. Γ_μ contained in a reducible quadric Q_μ.

End of the proof.

The vertices V_0, V_1 and V_μ of the quadric cones Q_0, Q_1 and Q_μ must be respectively contained in the twisted cubics Γ_0, Γ_1 and Γ_μ, therefore also in Q''. However, these three points lie on the same line $\overline{V_0V_1}$; in particular, we get a line intersecting a quadric in three distinct points. The conclusion is that $\overline{V_0V_1} \subset Q''$.

Recall that Σ must be singular in our three twisted cubics, and that by inspecting its equation, it follows easily that Σ is triple on the complete intersection of the two quadrics Q_0, Q_1, which contains the line $\overline{V_0V_1}$.

Let us write the complete intersection $Q_0 \cap Q_1$ as $\overline{V_0V_1} + T$, where T is thus a 1-cycle of degree 3.

Only two cases can occur:

- Γ_0, Γ_1 and Γ_μ are distinct
- $\Gamma_0 = \Gamma_1 = \Gamma_\mu = T$

In the first case, the schematic intersection $\Sigma \cap Q''$ has degree 18, however it contains Γ_0, Γ_1 and Γ_μ with multiplicity two and $\overline{V_0V_1}$ with multiplicity three: this is clearly a contradiction, since $18 + 3 = 21 > 18$.

In the second case, the irreducible twisted cubic T would intersect the line $\overline{V_0V_1}$ in the three distinct points V_0, V_1, V_μ, which is well known not to be possible.

□

4. The Barlow surface

Up to now there are only two known explicit constructions of numerical Godeaux surfaces with torsion $\{0\}$ (and indeed simply connected), respectively due to Barlow ([Ba2]), and Craighero and Gattazzo ([CG]): let us consider first Barlow's example .

For the Barlow surface, we can study the bicanonical and tricanonical system according to the manuscript [R3], where Reid describes the canonical ring of the Barlow surface as follows.

Let A the symmetric matrix

$$A = \begin{pmatrix} -2x_4 & x_2 - x_0 - x_4 & x_0 - x_1 - x_4 & x_3 - x_2 - x_4 & x_1 - x_3 - x_4 \\ & -2x_0 & x_3 - x_1 - x_0 & x_1 - x_2 - x_0 & x_4 - x_3 - x_0 \\ & & -2x_1 & x_4 - x_2 - x_1 & x_2 - x_3 - x_1 \\ & & & -2x_2 & x_0 - x_3 - x_2 \\ & & & & -2x_3 \end{pmatrix}.$$

Let A_{ij} the ij-th entry of A, B_{ij} the ij-th entry of the adjoint matrix B of A.

Let us consider the authomorphism β of $\mathbb{C}[x_0, \dots, x_4, y_0, \dots, y_4]$ that acts as $\beta(x_i) = x_{i+1}$, $\beta(y_i) = y_{i+1}$, and the automorphism α that acts as $\alpha(x_i) = x_{a(i)}$ ($a = (25)(34)$ in $\mathcal{S}_5$), $\alpha(y_i) = -y_{4-i}$, where all indices are to be taken in $\mathbb{Z}/5\mathbb{Z}$. They generate a subgroup G of the group of automorphisms of $\mathbb{C}[x_0, \dots, x_4, y_0, \dots, y_4]$. One can indeed check that $G \cong D_{10}$.

Let $R = \mathbb{C}[x_0, \dots, x_4, y_0, \dots, y_4]/I$, where the ideal I is generated by

$$\sum x_i = 0,$$

$$\forall 1 \leq i \leq 5, \quad \sum_1^5 A_{ij} y_{j-1} = 0,$$

$$\forall 1 \leq i, j \leq 5, \quad y_{i-1} y_{j-1} - B_{ij} = 0.$$

We consider the ring R as a graded ring via the following grading which makes I a homogeneus ideal: deg x_i=1, deg y_i=2.

One can check that the ideal I is G-invariant, whence G acts on R. Since the action acts only with isolated fixed points, it follows (cf. [R3]) that the canonical ring of the Barlow surface can be described as the ring of the invariants of R for the action of G.

In order to simplify the computations, one can choose as generators for G, β and $\alpha' = \beta\alpha$; $\alpha'(x_i) = x_{a'(i)}$, with $a' = (12)(35)$, and $\alpha'(y_i) = -y_{-i}$.

So we can easily compute that there are no nontrivial invariants in R_1, while the subspace of invariants in R_2 is generated by

$$\xi_0 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_0 + x_0x_1,$$

$$\xi_1 = x_1x_3 + x_2x_4 + x_3x_0 + x_4x_1 + x_0x_2,$$

$$\xi_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_0^2.$$

Moreover, the relation $\sum x_i = 0$ induces the relation

$$2\xi_0 + 2\xi_1 + \xi_2 = 0.$$

So we can take ξ_0, ξ_1 as generators of the bicanonical system.

The tricanonical system needs more computations.

We know that R_3 is generated by $x_ix_jx_k$ and x_iy_j; the invariants must have the same decomposition.

The subspace of invariants in the span of the monomials $x_i x_j x_k$ is generated by the invariants:

$$\eta_0 = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_0 + x_4x_0x_1 + x_0x_1x_2,$$

$$\eta_1 = x_1x_2x_4 + x_2x_3x_0 + x_3x_4x_1 + x_4x_0x_2 + x_0x_1x_3,$$

$$\eta_2 = x_1^2(x_2+x_0) + x_2^2(x_3+x_1) + x_3^2(x_4+x_2) + x_4^2(x_0+x_3) + x_0^2(x_1+x_4),$$

$$\eta_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_0^3,$$

$$\eta_4 = x_1^2(x_3+x_4) + x_2^2(x_4+x_0) + x_3^2(x_0+x_1) + x_4^2(x_1+x_2) + x_0^2(x_2+x_3).$$

The relation $\sum x_i = 0$ induces the three linear relations

$$\begin{cases} 2\eta_0 + \eta_1 + \eta_2 = 0 \\ \eta_0 + 2\eta_1 + \eta_4 = 0 \\ \eta_3 + \eta_2 + \eta_4 = 0. \end{cases}$$

Thus the above subspace is generated by two independent generators, say η_0, η_1.

Now we have to find two more independent generators for the subspace of invariants in the span of the monomials $\{x_i y_j\}$.

Here the β invariants are generated by $\zeta_j = \sum_i x_i y_{i+j}$, where the indices $0 \le j \le 4$ are again to be understood as elements of $\mathbb{Z}/5\mathbb{Z}$.

The ζ_j verify the trivial relation $\sum \zeta_j = 0$, and the sum of the five linear relations $\forall 1 \le i \le 5$ $\sum_1^5 A_{ij} y_{j-1} = 0$. An easy calculation shows that this sum yields exactly $(-6)\zeta_1$. Whence, we have only the other relation $\zeta_1 = 0$.

Another easy calculation shows that $\alpha'(\zeta_0) = -\zeta_2$, $\alpha'(\zeta_1) = -\zeta_1$, $\alpha'(\zeta_3) = -\zeta_4$, and we can easily conclude that a system of independent generators for the tricanonical system of the Barlow surface is given by $\eta_0, \eta_1, \zeta_0 - \zeta_2, \zeta_3 - \zeta_4$.

In order to understand how many hyperelliptic divisors (with multiplicity) there are in the bicanonical system of the Barlow surface, we have only to check what is the minimal m s.t. there exists a non trivial element in

$$(S^m(<\xi_0, \xi_1>) \otimes S^2(<\eta_0, \eta_1, \zeta_0 - \zeta_2, \zeta_3 - \zeta_4>)) \cap I.$$

We are indebted to F.-O. Schreyer who wrote a Macaulay script that verifies that this minimal number m is indeed equal to 3, and that the relation is given by the following polynomial

$$1728\xi_0^3\eta_0^2 + 1872\xi_0^2\xi_1\eta_0^2 - 1296\xi_0\xi_1^2\eta_0^2 - 1584\xi_1^3\eta_0^2 + 5472\xi_0^3\eta_0\eta_1 +$$

$$+5184\xi_0^2\xi_1\eta_0\eta_1 - 5184\xi_0\xi_1^2\eta_0\eta_1 - 5472\xi_1^3\eta_0\eta_1 + 1584\xi_0^3\eta_1^2 +$$

$$+1296\xi_0^2\xi_1\eta_1^2 - 1872\xi_0\xi_1^2\eta_1^2 - 1728\xi_1^3\eta_1^2 - 13\xi_0^3(\zeta_0 - \zeta_2)^2 +$$

$$-22\xi_0^2\xi_1(\zeta_0 - \zeta_2)^2 - 10\xi_0\xi_1^2(\zeta_0 - \zeta_2)^2 + \xi_1^3(\zeta_0 - \zeta_2)^2 +$$

$$+14\xi_0^3(\zeta_0-\zeta_2)(\zeta_3-\zeta_4)+24\xi_0^2\xi_1(\zeta_0-\zeta_2)(\zeta_3-\zeta_4)+$$

$$+24\xi_0\xi_1^2(\zeta_0-\zeta_2)(\zeta_3-\zeta_4)+14\xi_1^3(\zeta_0-\zeta_2)(\zeta_3-\zeta_4)-\xi_0^3(\zeta_3-\zeta_4)^2+$$

$$+10\xi_0^2\xi_1(\zeta_3-\zeta_4)^2+22\xi_0\xi_1^2(\zeta_3-\zeta_4)^2+13\xi_1^3(\zeta_3-\zeta_4)^2.$$

Afterwards we wrote a Macaulay2 script (available upon request) that obtains the same result in characteristic 0.

We can therefore summarize the main result of the foregoing section in the following

Theorem 4.1. *The bicanonical system of the Barlow surface has exactly* 4 *distinct base points and contains two hyperelliptic fibres (counted with multiplicity).*

The same result was obtained independently by [Lee].

5. The Craighero Gattazzo surface

Let us now compute what happens for the Craighero Gattazzo surface. As the Barlow surface, this is a numerical Godeaux surface with torsion $\{0\}$ (and indeed simply connected, as shown in [DW]).

The Craighero Gattazzo surface S is constructed in [CG] as the minimal resolution of the quintic $X \in \mathbb{P}^3$ defined by the equation F_5

$$F_5=(x+my+az)^2t^3+[a^2x^3+xy(bx+cy)+m^2y^3+(ex^2+fxy+cy^2)z+$$

$$+(bx+ey)z^2+z^3]t^2+[2ax^3y+ex^2y^2+2amxy^3+$$

$$+(2amx^3+fx^2y+fxy^2+2my^3)z+(cx^2+fxy+by^2)z^2+2(mx+ay)z^3]t+$$

$$+x^3y^2+a^2x^2y^3+xy(2mx^2+bxy+2ay^2)z+$$

$$+(m^2x^3+cx^2y+exy^2+y^3)z^2+(mx+ay)^2z^3=0.$$

where r is a root of the polynomial t^3+t^2-1 and where the various coefficients are defined as follows :

$$\begin{array}{ll} a=r^2 & b=-\frac{1}{7}(2r^2-13r-18) \\ c=\frac{1}{49}(73r^2+75r+92) & e=-\frac{1}{7}(r^2-24r-9) \\ f=\frac{1}{49}(181r^2+241r+163) & m=\frac{1}{7}(3r^2+5r+1). \end{array}$$

In [CG] are given different expressions for the coefficients a, e, b, m, f, c, expressed as rational functions of r; we have computed the equivalent expression as $\mathbb{Q}$-linear combinations of $1, r, r^2$ in order to simplify the calculations (we have done this both by hand and via a calculation using MAPLE).

This quintic surface X is invariant for the $\mathbb{Z}/4\mathbb{Z}$–action on $\mathbb{P}^3$ induced by the cyclical permutation of the coordinates $x \mapsto y \mapsto z \mapsto t$; the singular locus of X is the set of coordinate points $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

It is possible to show, as we shall do shortly, that in the neighbourhood of every singular point the singularity can be represented as a double cover of the plane branched on a curve with a singularity of type (3,3) (a triple point that has an infinitely near ordinary triple point). Therefore our singular points are simple elliptic (-1)-singularities (for which the exceptional curve in the minimal resolution is a smooth elliptic curve with self-intersection -1).

It follows that the adjoint divisor on the resolution is precisely the elliptic exceptional curve counted with multiplicity one, whence the bicanonical system of S is cut by the quadrics in $\mathbb{P}^3$ whose pull-back on S yields a divisor containing the exceptional locus twice, and the tricanonical system is cut by the cubics in $\mathbb{P}^3$ whose pull-back on S contains the exceptional locus with multiplicity three .

Craighero and Gattazzo compute explicitly both systems, but we found that their computation is different (and non-equivalent) to ours. It is possible that some misprint occurred, so let us sketch our calculation.

Let us look at the equation of X in a neighbourhood of $(0,0,0,1)$. Setting $w = (x + my + az)$ we can write the Taylor development of the equation F_5 in affine coordinates as follows:

$$w^2 + m^2y^3 + wf_2(w,y,z) + f_4(w,y,z) + f_5(w,y,z) = 0$$

with f_i homogeneus of degree i.

In local analytic coordinates (u,y,z), where $u = w+1/2f_2(w,y,z)$, the equation takes the form

$$u^2 + m^2y^3 + g_4(u,y,z) + ... = 0.$$

Whence $y = 0$ is the equation (in the plane of coordinates (y,z)) of the direction of the tangent cone of the branching locus, and therefore the pull-back on S of the divisor $div(y)$ is easily shown to contain the exceptional curve E at least twice.

Of course the multiplicity in the exceptional curve of w and z is at least one. But, since w^2 belongs to the cube of the maximal ideal, it follows that $div(w) \geq 2E$.

Again, writing

$$f_2(w,y,z) = \alpha z^2 + wF_1(w,y,z) + yG_1(w,y,z)$$

$$f_4(w,y,z) = \beta z^4 + wF_3(w,y,z) + yG_3(w,y,z)$$

we are able to rewrite our equation in a slightly different way as follows:

$$w^2 + m^2y^3 + [\alpha wz^2 + w^2F_1(w,y,z) + wyG_1(w,y,z)]+$$

$$+[\beta z^4 + wF_3(w,y,z) + yG_3(w,y,z)] + f_5(w,y,z) = 0.$$

From the above remarks follows that the function

$$w^2 + \alpha wz^2 + \beta z^4$$

has a divisor which is greater than $5E$.

But a tedious calculation shows that $w^2+\alpha wz^2+\beta z^4 = (w-\frac{1}{7}(6r^2+3r-5)z^2)^2$. Whence the multiplicity of $w - \frac{1}{7}(6r^2+3r-5)z^2$ is at least 3.

It is in fact obvious that $p_g(S) = 0$; moreover it is also clear that $|2K_S|$ contains the divisors corresponding to the quadrics $Q_0 = xz$ and $Q_1 = yt$: on the other hand these two quadrics generate a fixed part free pencil on X, therefore the corresponding pencil in $|2K_S|$ has no rational curve in its fixed part; whence S is minimal. Since $K_S^2 = 1$, it follows that the bigenus $P_2(S) = 2$, hence the bicanonical system is precisely the above pencil.

We can proceed further by using the $\mathbb{Z}/4\mathbb{Z}$–invariance of F_5, since then $|3K_S|$ is generated by the $\mathbb{Z}/4\mathbb{Z}$ orbit of the cubic $C_0 = a(x+my+az)t^2+txy+a^3yzt+\frac{1}{7}(6r^2+3r-5)xzt$.

If σ is the generator of the $\mathbb{Z}/4\mathbb{Z}$ action such that $\sigma(x) = y$, let us set $C_1 = \sigma(C_0)$, $C_2 = \sigma^2(C_0)$, $C_3 = \sigma^3(C_0)$.

In this way, also for the Craighero Gattazzo surface, we can calculate using the computer algebra program Macaulay2 what is the minimal number m such that the kernel of the map $S^m(H^0(2K)) \otimes S^2(H^0(3K)) \to H^0((2m+6)K)$ is not trivial; and again the answer we get is $m = 3$.

At the moment we cannot yet determine whether there do exist numerical Godeaux surfaces with bigger values of $m = 5$ or 7; we hope to address this question in a sequel to this paper.

The explicit equation of this polynomial is

$$-(3r^2+5r+1)Q_0^3C_0C_2+$$

$$+Q_0^2Q_1[-7r(C_0^2+C_2^2)-14(r+1)(C_0C_1+C_2C_3)-7(r+1)(C_1^2+C_3^2)+$$

$$+(r^2+4r-9)C_0C_2-7(r^2+r+1)(C_1C_2+C_0C_3)-(11r^2+16r+6)C_1C_3]+$$

$$+Q_0Q_1^2[7(r+1)(C_0^2+C_2^2)+7(r^2+r+1)(C_0C_1+C_2C_3)+7r(C_1^2+C_3^2)+$$

$$+(11r^2+16r+6)C_0C_2+14(r+1)(C_1C_2+C_0C_3)-(r^2+4r-9)C_1C_3]+$$

$$+(3r^2+5r+1)Q_1^3C_1C_3$$

We can combine the results of our calculations above with the previous results of Craighero and Gattazzo ([CG]) and Dolgachev and Werner ([DW]),

Theorem 5.1. *The Craighero Gattazzo surface is a simply connected numerical Godeaux surface with ample canonical bundle. The bicanonical system has exactly 4 distinct base points and contains exactly two hyperelliptic fibres with multiplicity 1.*

Proof.

We need only to verify the last two assertions.

Recall that by [DW], S does not contain (-2)–curves; so, by lemma 1.1, $2K_S$ has no fixed part.

Restricting to the line $x = y = 0$ the equation F_5, we get the polynomial $az^2t^3 + z^3t^2$. So the smooth point of X of coordinates $(0, 0, -a, 1)$ is a base point of the bicanonical system of $2K_S$; since its orbit by the $\mathbb{Z}/4\mathbb{Z}$ action consists of four distinct points, we have gotten 4 distinct base points. These build up the whole base locus because $(2K_S)^2 = 4$.

We have shown before that the minimal m such that the kernel of the map $S^m(H^0(2K)) \otimes S^2(H^0(3K)) \to H^0((2m+6)K)$ is not trivial, is 3. This allows us to conclude, by theorem 2.5, that there are two hyperelliptic bicanonical divisors (counted with multiplicity).

But the $\mathbb{Z}/4\mathbb{Z}$ action on X induces a $\mathbb{Z}/2\mathbb{Z}$ action on the bicanonical system (since the bicanonical sections are invariant by σ^2). So, if there were only one hyperelliptic bicanonical divisor (with multiplicity two), it would be cut by a $\sigma-$invariant quadric in the pencil generated by Q_0 and Q_1, i.e. by Q_0+Q_1 or Q_0-Q_1.

But we have written down explicitly the tricanonical system, so we can explicitly write the tricanonical images of these two divisors. We can prove that neither of them is hyperelliptic, because otherwise we would find three quadrics containing the image of one of them, whereas we have checked with the program Macaulay2 that in both cases there is only one such a quadric.

□

6. The local moduli space of the Craighero Gattazzo surface

It is known that the local moduli space of the Barlow surface is smooth of dimension 8 (cf. [CL], and also [Lee]). The main scope of this section is to prove that the same holds for the Craighero Gattazzo surface:

Theorem 6.1. *The local moduli space of the Craighero Gattazzo surface is smooth of dimension* 8.

Let X be the quintic constructed by Craighero and Gattazzo, and $\pi : S \to X$ its minimal resolution.

By Kodaira and Spencer's first main result in deformation theory (cf. [KS], also [KM]) our claim will be stablished if we show that $h^1(\Theta_S) = 8$, $h^2(\Theta_S) = 0$.

In fact, $h^0(\Theta_S) = 0$, since S is of general type, and moreover $h^1(\Theta_S) - h^2(\Theta_S) = -\chi(\Theta_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 = 8$. Therefore, it suffices to prove that $h^1(\Theta_S) = 8$.

Applying to the standard exact sequence

$$0 \to \mathcal{O}_X(-5) \to \Omega^1_{\mathbb{P}|X} \to \Omega^1_X \to 0$$

the functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{O}_X)$, we get the standard long exact sequence

$$H^0(\Theta_{\mathbb{P}^3|X}) \to H^0(\mathcal{O}_X(5)) \to$$

$$\to Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^1(\Theta_{\mathbb{P}^3|X}) \to H^1(\mathcal{O}_X(5)) \to$$

$$\to Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^2(\Theta_{\mathbb{P}^3|X}).$$

However, taking the restriction to X of the Euler exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^4 \to \Theta_{\mathbb{P}^3|X} \to 0$$

we can easily compute that $H^1(\Theta_{\mathbb{P}^3|X}) = H^2(\Theta_{\mathbb{P}^3|X}) = 0$.

Therefore, keeping also in mind that $H^1(\mathcal{O}_X(5)) = 0$, we find that the map $H^0(\mathcal{O}_X(5)) \xrightarrow{f} Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is surjective and that $Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$.

In turn, applying the Ext spectral sequence, we obtain the following exact sequence:

$$0 \to H^1(\Theta_X) \to Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \xrightarrow{g} H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \to H^2(\Theta_X) \to 0.$$

We are now going to show the vanishing of $H^1(\Theta_X)$.

Let us denote by p the natural projection $H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{O}_X(5))$, and consider the map $g \circ f \circ p : H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$.

The $\mathbb{Z}/4\mathbb{Z}$ action on X allows us to choose a basis in $H^0(\mathcal{O}_{\mathbb{P}^3}(5))$, say $v_1, \dots, v_{56}$, s.t., if σ is the generator of the action given in the previous section,

$$\sigma(v_j) = \begin{cases} v_j & \text{if } 1 \le v_j \le 14 \\ iv_j & \text{if } 15 \le v_j \le 28 \\ -v_j & \text{if } 29 \le v_j \le 42 \\ -iv_j & \text{if } 43 \le v_j \le 56 \end{cases}$$

(notice in fact that σ acts freely on the set of monomials of degree 5).

We observe that $H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$, as a representation of $\mathbb{Z}/4\mathbb{Z}$, is isomorphic to the direct sum of the quotients of $\mathcal{O}_X$ by the jacobian ideal in the 4 singular points of X, and these addenda are permuted by σ, since the 4 singular points are a orbit for σ.

Thus the map $H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$ is given via a matrix of the following form:

$$\begin{pmatrix} A & B & C & D \\ A & iB & -C & -iD \\ A & -B & C & -D \\ A & -iB & -C & iD \end{pmatrix}$$

where every block is a matrix of size 10×14. We observe immediately that the above matrix has the same rank of the matrix

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix}.$$

We have explicitly checked with the program Macaulay2 that the matrices A, B, C, D have maximal rank, so that g is a surjective map; since

$$\dim\ Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \dim\ H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) = 40,$$

it follows that g is an isomorphism and therefore $H^1(\Theta_X) = H^2(\Theta_X) = 0$.

By [BW] $\pi_*(\Theta_S) = \Theta_X$. So, by the Leray spectral sequence we get $H^1(\Theta_S) \cong H^0(R^1\pi_*\Theta_S)$, and the last vector space equals, by the theorem on formal functions ([H]) to

$$\lim_{\leftarrow} H^1(\Theta_{S|nD}),$$

where D is the exceptional locus of π.

Since D consists of the sum of the four elliptic curves $D_1, \ldots, D_4$, corresponding to the 4 singular points of X, we can conclude that

$$h^1(\Theta_S) = 4\ \dim\ \lim_{\leftarrow} H^1(\Theta_{S|nC}),$$

where C is a smooth elliptic curve with $C^2 = -1,\ K_S C = 1$.

So we are left with proving the following lemma:

Lemma 6.2. *Let S a smooth surface containing a smooth elliptic curve with normal bundle of degree -1. Then*

$$dim\ \lim_{\leftarrow} H^1(\Theta_{|nC}) = 2.$$

Proof.

Since a simple elliptic singularity is analytically isomorphic to the blow down of the 0-section in the normal bundle to the exceptional curve (cf. [R1], [Lau]), we can assume, w.l.o.g., that S the total space of a line bundle over C of degree -1, that is, $\mathcal{O}_C(-p)$ for some $p \in C$.

By the exact sequence

$$0 \to \Theta_C \to \Theta_{S|C} \to \mathcal{O}_C(-p) \to 0,$$

where C is a smooth elliptic curve (thus $\Theta_C = \mathcal{O}_C$), we get $h^1(\Theta_{S|C}) = 2$.

Tensoring this exact sequence by $\mathcal{O}_C(mp)$, we obtain, $\forall m > 0$, $h^1(\Theta_{S|C}(mp)) = h^1(\mathcal{O}_C((m-1)p))$, whence we get 0 if $m \geq 2$, 1 for $m = 1$.

Applying this result to the exact sequence

$$0 \to \Theta_{S|C}(-(n-1)C) \to \Theta_{S|nC} \to \Theta_{S|(n-1)C} \to 0$$

we get that for $n \geq 3$, the restriction map $H^1(\Theta_{S|nC}) \to H^1(\Theta_{S|(n-1)C})$ is an isomorphism, therefore

$$\lim_{\leftarrow} H^1(\Theta_{|nC}) \cong H^1(\Theta_{|2C})$$

and $2 \leq h^1(\Theta_{S|2C}) \leq 3$.

Let us now consider the canonical projection $q : S \to C$; for every line bundle L on S, $h^0(R^1 q_* L) = 0$, so $h^1(q_* L) = h^1(L)$. Moreover, $q_*(\mathcal{O}_S) \cong \bigoplus_{n \geq 0} \mathcal{O}_C(np)$.

Consider the exact sequence

$$(\#) \qquad 0 \to q^*\mathcal{O}_C(-p) \to \Theta_S \to q^*\Theta_C (\cong \mathcal{O}_S) \to 0.$$

Tensoring this sequence by $\mathcal{O}_S(-2C) \cong q^*\mathcal{O}_C(2p)$, since

$$H^i(q^*\Theta_C \otimes \mathcal{O}_S(-2C)) = H^i(q_*\mathcal{O}_S \otimes \mathcal{O}_C(2p)) =$$

$$= H^i((\bigoplus_{n\geq 0} \mathcal{O}_C(np)) \otimes \mathcal{O}_C(2p)) = \bigoplus_{n\geq 2} H^i(\mathcal{O}_C(np)),$$

we get $h^1(\Theta_S)(-2C) = h^2(\Theta_S)(-2C) = 0$, so $h^1(\Theta_{S|2C}) = h^1(\Theta_S)$.

Again by (#), since

$$h^1(q^*\Theta_C) = h^1(q_*\mathcal{O}_S) = \sum_{n\geq 0} h^1(\mathcal{O}_C(np)) = 1$$

$$h^1(q^*\mathcal{O}_C(-p)) = \sum_{n\geq -1} h^1(\mathcal{O}_C(np)) = 2$$

remembering that we have shown that $2 \leq h^1(\Theta_{S|2C}) = h^1(\Theta_S) \leq 3$, we see that we have to prove that the projection map

$$H^0(\Theta_S) \to H^0(q^*\Theta_C)$$

is not surjective.

We claim that we can write $S = (\mathbb{C}^* \times \mathbb{C})/\sim$, where $\sim$ is the equivalence relation generated by $(z,w) \sim (\mu^2 z, \mu w z)$. $C \subset S$ is defined by the equation $w = 0$, so that $C \cong \mathbb{C}^* / < z \sim \mu^2 z >$.

In fact we can assume the point p to be the origin of the elliptic curve C, and we observe that every elliptic curve occurs as a quotient of $\mathbb{C}^*$ as above. Since the functional equation of the Riemann theta function is then $f(\mu^2 z) = \mu^{-1} z^{-1} f(z)$, we obtain the desired assertion.

We shall prove now that the global holomorphic never vanishing section of $q^*\Theta_C$ defined by $z\frac{\partial}{\partial z}$ is not a projection of a global section of Θ_S.

In fact, a global holomorphic vector field on S can be written as $a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w}$ with a, b global holomorphic functions on $\mathbb{C}^* \times \mathbb{C}$ satisfying the following functional equations: $\forall z, w \in \mathbb{C}^* \times \mathbb{C}$

$$a(z,w) = \mu^2 a(\frac{z}{\mu^2}, \mu\frac{w}{z})$$

$$b(z,w) = \mu w a(\frac{z}{\mu^2}, \mu\frac{w}{z}) + \mu z b(\frac{z}{\mu^2}, \mu\frac{w}{z}).$$

If there were a global holomorphic vector field on S whose projection on $q^*\Theta_C$ is $z\frac{\partial}{\partial z}$, then there would be a global holomorphic function b in $\mathbb{C}^* \times \mathbb{C}$ s.t.

$$b(z,w) = \mu^{-1} wz + \mu z b(\frac{z}{\mu^2}, \mu\frac{w}{z}).$$

Let us write b as a power series

$$b(z,w) = \sum_{n\in\mathbb{Z}, i\in\mathbb{N}} b_{ni} z^n w^i.$$

Then our condition can be written as :

$$\mu^{-1}wz = b(z,w) - \mu z b(\frac{z}{\mu^2}, \mu\frac{w}{z}) = \sum_{n,i} b_{ni}(z^n w^i - \mu z(\frac{z}{\mu^2})^n(\mu\frac{w}{z})^i) =$$

$$= \sum_{n,i} b_{ni}(z^n w^i - \mu^{i+1-2n} z^{n+1-i} w^i);$$

looking at the coefficient of wz we get $\mu^{-1} = b_{11}(1-\mu^0) = 0$, a contradiction.

□

7. End of the proof of the main theorem

In this section we summarize some of the previous results, in order to prove theorem 0.1. The first two assertions are already proven in theorem 2.5.

That the curves in Y which are images of the hyperelliptic bicanonical divisors are irreducible twisted cubic curves was proved in lemma 1.10, part b); the nature of the singularity along these curves was explained in proposition 2.6.

Remark that $\omega_Y = \mathcal{O}_Y(2+3-4, 7-2h+3h-6-2) = \mathcal{O}_Y(1, h-1)$. Moreover, recall that $g^*\mathcal{O}_Y(0,1)$ gives the movable part of the bicanonical system, and that $g_*\omega_{\tilde{S}}^2 = \mathcal{C}^2\omega_Y^2$. So we have a non trivial section Q' in $H^0(\mathcal{C}^2\mathcal{O}_Y(2, 2h-3))$, which, by proposition 2.6, induces a non trivial section Q'' in $H^0(\mathcal{C}\mathcal{O}_Y(2, h-3))$.

Let us denote by H_1 the class of a divisor in $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,0)|$ and by H_2 the class of a divisor in $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(0,1)|$.

The divisor associated to the non trivial section Q' of $H^0(\mathcal{C}^2\mathcal{O}_Y(2, 2h-3))$ gives a curve in $\mathbb{P}^3\times\mathbb{P}^1$ of class $(2, 7-2h)(3, 3h-6)(2, h-3) = 12H_1^3 + 12hH_1^2H_2$. It must contain doubly the h singular twisted cubic curves, so we can consider the residual curve E'', of class $12H_1^3$.

The bicanonical system of $\tilde{S}$ has $2(K_{\tilde{S}} - \beta^*K_S)$ as fixed part, a divisor which is easily shown to contain (with multiplicity) 12 (-1)-curves that are not contracted by g. So their image in Y is precisely E''.

Viceversa, let $Y \subset \mathbb{P}^3\times\mathbb{P}^1$ be as described. Let us consider the normalization $\varepsilon : \tilde{X} \to Y$, and a minimal resolution of singularities $\delta : \tilde{S} \to \tilde{X}$; we have $\varepsilon_*\omega_{\tilde{X}} \cong \mathcal{C}\omega_Y$. By our assumptions, $\omega_Y = \mathcal{O}_Y(1, h-1)$.

First, we claim that $p_g(\tilde{X}) = 0$.

In fact, an easy computation shows that the restriction maps

$$H^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1, h-1)) \to H^0(\mathcal{O}_Q(1, h-1)) \to H^0(\mathcal{O}_Y(1, h-1))$$

are isomorphisms.

So, if $h=0$, $p_g(\tilde{X}) = h^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,-1)) = 0$, while, if $h>0$, a non trivial section of $H^0(\varepsilon_*\omega_{\tilde{X}})$ induces a non trivial section of $H^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,h-1))$ containing some of the singular twisted cubic curves; since a plane in $\mathbb{P}^3$ cannot contain a twisted cubic curve, we derive a contradiction.

Let us now denote by Q'' a non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(2,h-3))$; let moreover F' be a non trivial section (unique up to scalar multiplication) in $H^0(\mathcal{O}_Y(0,h))$ whose pull back in $\tilde{X}$ gives the conductor divisor. Let us set $Q' = F'Q'' \in H^0(\mathcal{C}^2\mathcal{O}_Y(2,2h-3))$, $\overline{Q}' = F'Q' \in H^0(\mathcal{C}^3\mathcal{O}_Y(2,3h-3))$.

The sections Q' and $\overline{Q}'$ define two injective homomorphisms of sheaves

$$\mathcal{O}_Y(0,1) \to \mathcal{C}^2\mathcal{O}_Y(2,2h-2) \cong \varepsilon_*\omega^2_{\tilde{X}}$$

$$\mathcal{O}_Y(1,0) \to \mathcal{C}^3\mathcal{O}_Y(3,3h-3) \cong \varepsilon_*\omega^3_{\tilde{X}}.$$

In particular we can conclude that the morphisms $\pi_2\circ\varepsilon : \tilde{X}\to\mathbb{P}^1$ and $\pi_1\circ\varepsilon : \tilde{X}\to\mathbb{P}^3$ are induced by some subsystem of the bicanonical, respectively of the tricanonical system. It follows that $\tilde{X}$ is of general type.

Since $\tilde{X}$ has only R.D.P.'s as singularities, $\tilde{S}$ is a surface of general type with geometric genus $p_g=0$; in particular $q=0$ and $\chi=1$.

Let us denote by S the minimal model of $\tilde{S}$; then $K_S^2\geq 1$. In order to prove that S is a numerical Godeaux surface, we need only to prove that $K_S^2=1$.

Observe that the divisor associated to Q'' gives a curve in $\mathbb{P}^3\times\mathbb{P}^1$ of class $12H_1^3 + 6hH_1^2H_2$.

The assumption $Q''\in H^0(\mathcal{C}\mathcal{O}_Y(2,h-3))$ ensures that such a divisor contains h fibres; so we can consider the residual curve E'' of class $12H_1^3$ (thus consisting with multiplicity of exactly 12 fibres of the projection over $\mathbb{P}^3$). Let us denote by E' and by E the respective divisors in $\tilde{X}$ and $\tilde{S}$ given by the difference between the pull back of $div(Q'')$ and the h fibres corresponding to the conductor divisor.

We have

$$(2K^2_{\tilde{S}}) = ((\varepsilon\circ\delta)^*\mathcal{O}_Y(0,1)+E)^2 = 24+E^2$$

$$(3K^2_{\tilde{S}}) = ((\varepsilon\circ\delta)^*\mathcal{O}_Y(1,0)+E)^2 = 9+E^2.$$

In particular $K^2_{\tilde{S}} = (9+E^2-24-E^2)/5 = -3$.

The morphism $\beta:\tilde{S}\to S$ is a sequence of n blow ups. Since $\tilde{S}$ is of general type and $K^2_{\tilde{S}}=-3$, it follows that $n = K^2_{\tilde{S}} - K^2_S \geq 4$.

An easy computation shows that, if we denote by $\overline{E}$ the difference $K_{\tilde{S}} - \beta^*K_S$, $\overline{E}$ contains, with multiplicity, at least n $(-1)-$curves. Remark that the morphism $\tilde{S}\to Y$ is composition of a finite map $(\tilde{X}\to Y)$ and of the minimal resolution of the singularities of $\tilde{X}$. By hypotheses, $\tilde{X}$ has only R.D.P., so the only curves contracted are $(-2)-$curves, and our $(-1)-$curves cannot be contracted to Y.

Now we only need to remark that the fixed part of $3K_{\tilde{S}}$ contains $3\overline{E}$, whence at least $3n$ (-1)-curves; and the corresponding divisor maps on Y to E'', which has 12 components.

Since $n \geq 4$, $3\overline{E}$ is exactly the fixed part of $3K_{\tilde{S}}$; in particular $n = 4$, $K_S^2 = 1$ and S is a numerical Godeaux surface.

Thus $3\overline{E}$ is the fixed part of both $2K_{\tilde{S}}$ and $3K_{\tilde{S}}$; the rational map $S \dashrightarrow Y$ is the tri-bicanonical morphism, $3K_S$ has no base points, whence (as shown in [Cat1], [Mi1]) the torsion group of S is either 0 or $\mathbb{Z}/2\mathbb{Z}$. But if the torsion were $\mathbb{Z}/2\mathbb{Z}$, by lemma 1.10, part c), in the singular locus we would obtain a fibre consisting of a line with multiplicity 6, a contradiction.

Since the bicanonical system yields a genus 4 fibration, we are in case 1a).

We proved that the case with three distinct hyperelliptic fibres cannot occur in proposition 3.1; the computations of the number of hyperelliptic fibres for the Barlow and the Craighero Gattazzo surface are given in sections 4 and 5.

Finally, the local moduli space of the Barlow surface is computed in [CL], whereas the assertion concerning the local moduli space of the Craighero Gattazzo surface is the contents of theorem 6.1.

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