APPENDIX B: SKIPPED CASES

Our code skips some signatures giving rise to groups of large order, either not covered by the MAGMA SmallGroup database, or causing extreme computational complexity. The program returns the list of the skipped cases, which have to be studied separately. The program returns the skipped cases in a different list. For the cases $p_g = q \neq 0$, this list is empty, while for $p_g = q = 0$, this list is collected in Table 1. We prove that no one of these cases occur.

K_S^2	Sing$X$	type	$\lvert G^0\rvert$
1	$2 \times C_{8,1} + C_{4,1}$	2,3,8	6336
1	$3 \times C_{4,1} + C_{4,3}$	2,3,8	2304
1	$C_{8,1} + C_{4,1} + C_{8,5}$	2,3,8	4032
1	$4 \times C_{4,1} + C_{2,1}$	2,3,8	2880
1	$2 \times C_{8,3} + C_{4,1} + C_{2,1}$	2,3,8	2304
1	$2 \times C_{2,1} + C_{8,3} + C_{8,1}$	2,3,8	3744
2	$2 \times C_{8,3} + C_{4,1}$	2,3,8	2880
2	$C_{8,3} + C_{8,1} + C_{2,1}$	2,3,8	4320
2	$4 \times C_{4,1}$	2,4,5	2400
2	$4 \times C_{4,1}$	2,3,8	3456
2	$C_{8,3} + C_{8,5} + C_{2,1}$	2,3,8	2016
2	$2 \times C_{4,1} + 3 \times C_{2,1}$	2,3,8	2304
2	$2 \times C_{4,1} + C_{3,1} + C_{3,2}$	2,3,8	2496
3	$2 \times C_{4,1} + 2 \times C_{2,1}$	2,3,8	2880
3	$C_{8,3} + C_{8,1}$	2,3,8	4896
3	$2 \times C_{4,1} + C_{5,3}$	2,4,5	2160
3	$C_{8,3} + C_{8,5}$	2,3,8	2592
3	$C_{4,3} + C_{4,1} + C_{2,1}$	2,3,8	2304
4	$C_{4,3} + C_{4,1}$	2,3,8	2880
4	$4 \times C_{2,1}$	2,3,8	2304
4	$C_{3,1} + C_{3,2} + C_{2,1}$	2,3,8	2496
4	$2 \times C_{4,1} + C_{2,1}$	2,4,5	2400
4	$2 \times C_{4,1} + C_{2,1}$	2,3,8	3456
5	$C_{5,2} + C_{2,1}$	2,4,5	2160
5	$3 \times C_{2,1}$	2,3,8	2880
5	$C_{3,1} + C_{3,2}$	2,3,8	3072
5	$2 \times C_{4,1}$	2,4,5	2800
5	$2 \times C_{4,1}$	2,3,8	4032
6	$2 \times C_{2,1}$	2,4,5	2400
6	$2 \times C_{2,1}$	2,3,8	3456
6	$2 \times C_{5,3}$	2,4,5	2560
7	$C_{2,1}$	2,3,9	2268
7	$C_{2,1}$	2,4,5	2800
7	$C_{2,1}$	2,3,8	4032
8	$\emptyset$	2,3,9	2592
8	$\emptyset$	2,4,5	3200
8	$\emptyset$	2,3,8	4608

TABLE 1. The skipped cases for $p_g = q = 0$ and $K^2 > 0$

Definition 0.1. Let H be a finite group and let $m_1, \dots, m_r > 1$ be integers. A *spherical system of generators* for H of signature $[m_1, \dots, m_r]$ is a generating vector for H of signature $(0; m_1, \dots, m_r)$.

Remark 0.2. In the following we will sometimes need the number of perfect groups of a given order; we compute it by the MAGMA function:

`NumberOfGroups(PerfectGroupDatabase(),order);`

while the other functions that we use are in the MAGMA script we developed.

1. Non generation results

Lemma 1.1. *No group of order 2016, 2304, 2496, 2592, 2880, 3456 or 3744 has a spherical system of generators of type* $[2, 3, 8]$.

Proof. Assume that G^0 is a group of order 2016 (2304, 2496, 2592, 2880, 3456, 3744 resp.) admitting a surjective homomorphism $\mathbb{T}(2, 3, 8) \to G^0$.

Since $\mathbb{T}(2, 3, 8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2016 (2304, 2496, 2592, 2880, 3456, 3744 resp.), the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1008 (1152, 1248, 1296, 1440, 1728, 1872 resp.) and it is a quotient of $[\mathbb{T}(2, 3, 8), \mathbb{T}(2, 3, 8)] \cong \mathbb{T}(3, 3, 4)$. The following MAGMA computations

```
>  Test([3,3,4], 1008,0);
{}
>
> TestBAD([3,3,4], 1152,0);
{}
>
>  Test([3,3,4], 1248,0);
{}
>
>  Test([3,3,4], 1296,0);
{}
>
> Test([3,3,4], 1440,0);
{}
>
> Test([3,3,4], 1728,0);
{}
>
>  Test([3,3,4], 1872,0);
{}
>
```

show that there are no groups of order 1008 (1152, 1248, 1296, 1440, 1728, 1827 resp.) with a spherical system of generators of type $[3, 3, 4]$, a contradiction. □

Remark 1.2. Note that if (a, b, c) is a spherical system of generators of type $[2, 3, 8]$ of G^0, then (b, cbc^{-1}, c^2) is a spherical system of generators of type $[3, 3, 4]$ of $G^{0\prime}$.
Analogous considerations hold in all the following proofs.

Lemma 1.3. *No group of order 4608 or 6336 has a spherical system of generators of type* $[2, 3, 8]$.

Proof. Assume that G^0 is a group of order 4608 (6336 resp.) admitting a surjective homomorphism $\mathbb{T}(2, 3, 8) \to G^0$.

Since $\mathbb{T}(2, 3, 8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4608 (6336 resp.), the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2304 (3168 resp.) and it is a quotient of $[\mathbb{T}(2, 3, 8), \mathbb{T}(2, 3, 8)] \cong \mathbb{T}(3, 3, 4)$.

Since $\mathbb{T}(3,3,4)^{\text{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2304 (3168 resp.), the commutator subgroup $G^{0''} = [G^{0'}, G^{0'}]$ of $G^{0'}$ has order 768 (1056 resp.) and it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$. The following MAGMA computations

```
> TestBAD([4,4,4], 768,0);
{}
>
> Test([4,4,4], 1056,0);
{}
>
```

show that there are no groups of order 768 (1056 resp.) with a spherical system of generators of type $[4,4,4]$, a contradiction. □

Lemma 1.4. *No group of order 2400, 2800 or 3200 has a spherical system of generators of type* $[2,4,5]$.

Proof. Assume that G^0 is a group of order 2400 (2800, 3200 resp.) admitting a surjective homomorphism $\mathbb{T}(2,4,5) \to G^0$.

Since $\mathbb{T}(2,4,5)^{\text{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2400 (2800, 3200 resp.), the commutator subgroup $G^{0'} = [G^0, G^0]$ of G^0 has order 1200 (1400, 1600 resp.) and it is a quotient of $[\mathbb{T}(2,4,5), \mathbb{T}(2,4,5)] \cong \mathbb{T}(2,5,5)$. The following MAGMA computations

```
> Test([2,5,5], 1200,0);
{}
>
> Test([2,5,5], 1400,0);
{}
>
> Test([2,5,5], 1600,0);
{}
```

show that there are no groups of order 1200 (1400, 1600 resp.) with a spherical system of generators of type $[2,5,5]$, a contradiction. □

Lemma 1.5. *No group of order 2268 has a spherical system of generators of type* $[2,3,9]$.

Proof. Assume that G^0 is a group of order 2268 admitting a surjective homomorphism $\mathbb{T}(2,3,9) \to G^0$.

Since $\mathbb{T}(2,3,9)^{\text{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2268, the commutator subgroup $G^{0'} = [G^0, G^0]$ of G^0 has order 756 and is a quotient of $[\mathbb{T}(2,3,9), \mathbb{T}(2,3,9] \cong \mathbb{T}(2,2,2,3)$. The following MAGMA computation

```
> Test([2,2,2,3], 756,0);
{}
>
```

shows that there are no groups of order 756 with a spherical system of generators of type $[2,2,2,3]$, a contradiction. □

Lemma 1.6. *No group of order 2160 has a spherical system of generators of type* $[2,4,5]$.

Proof. Assume that G^0 is a group of order 2160 admitting a surjective homomorphism $\mathbb{T}(2,4,5) \to G^0$. It holds $\mathbb{T}(2,4,5)^{\mathrm{ab}} \cong \mathbb{Z}_2$.
There is only one perfect group of order 2160, we denote it by H. $H = 6.\mathcal{A}_6$ has the following MAGMA representation:

```
>  F<w>:=GF(9);
>
>  x:=CambridgeMatrix(1,F,6,[
>  "010000",
>  "200000",
>  "000100",
>  "002000",
>  "000001",
>  "000020"]);
>
>  y:=CambridgeMatrix(1,F,6,[
>  "300000",
>  "550000",
>  "007000",
>  "126600",
>  "000030",
>  "240155"]);
>  H<x,y>:=MatrixGroup<6,F|x,y>;
>
> #H;
2160
> IsPerfect(H);
true
```

The following MAGMA computation

```
> ExVectGens(H,[2,4,5],0);
false
```

shows that H does not have a spherical system of generators of type $[2,4,5]$
If G^0 is a group of order 2160 with a spherical system of generators of type $[2,4,5]$, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1080 and it is a quotient of $[\mathbb{T}(2,4,5), \mathbb{T}(2,4,5)] \cong \mathbb{T}(2,5,5)$. The following MAGMA computation

```
> Test([2,5,5], 1080,0);
{}
>
```

shows that there are no groups of order 1080 with a spherical system of generators of type $[2,5,5]$, a contradiction. $\square$

Lemma 1.7. *No group of order 4320 has spherical system of generators of type* $[2,3,8]$.

Proof. Assume that G^0 is a group of order 4320 admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4320, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2160 and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8)] \cong \mathbb{T}(3,3,4)$.

Now $\mathbb{T}(3,3,4)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and there is only one perfect group of order 2160: the $H = 6.\mathcal{A}_6$ in the proof of Lemma 1.6. The following MAGMA computation

```
> ExVectGens(H,[3,3,4],0);
false
```

shows that H does not have a spherical system of generators of type $[3,3,4]$

If G^0 is a group of order 2160 with a spherical system of generators of type $[3,3,4]$, the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 720 and it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$. The following MAGMA computation

```
> Test([4,4,4], 720,0);
{ 584, 585, 763, 765, 766, 773, 776 }
>
```

shows that only the groups $G(720, j)$ with $j \in \{584, 585, 763, 765, 766, 773, 776\}$ have a spherical system of generators of type $[4,4,4]$.

Assume that $G^{0\prime}$ has a spherical system of generators of type $(3,3,4)$. Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & \mathbb{T}(3,3,4) \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{T}(3,3,4)^{ab} = \mathbb{Z}_3 \arrow[d, equal] & \\
1 \arrow[r] & G^{0\prime\prime} \arrow[r] & G^{0\prime} \arrow[r, "f"] & \mathbb{Z}_3 \arrow[r] & 1
\end{tikzcd}$$

where $q(c_i) = d_i$. Let

$$\mathbb{T}(3,3,4) = \langle c_1, c_2, c_3 \mid c_1^3, c_2^3, c_3^4, c_1c_2c_3 \rangle$$

$$\begin{aligned}
\mathbb{T}(3,3,4)^{ab} &= \langle d_1, d_2, d_3 \mid d_1^3, d_2^3, d_3^4, d_1d_2d_3, [d_i, d_j]_{1 \le i,j \le 3} \rangle \\
&= (\mathbb{Z}_3 d_1 \times \mathbb{Z}_3 d_2 \times \mathbb{Z}_4 d_3)/\langle (1,1,1) \rangle
\end{aligned}$$

since $[d_1] = (1,0,0) \notin \langle (1,1,1) \rangle$, then $[d_1] \neq [0]$; so we have $q(c_1) \neq [0]$, and $f(p(c_1)) = f(g_1) \neq 0$. We have found an element of $G^{0\prime}$ of order 3 that does not belong to $G^{0\prime\prime}$, this means that the following exact sequence

$$1 \longrightarrow G^{0\prime\prime} \longrightarrow G^{0\prime} \xrightarrow{f} \mathbb{Z}_3 \longrightarrow 1$$

splits with map

$$\begin{aligned}
\alpha : \mathbb{Z}_3 &\longrightarrow G^{0\prime} \\
d_1 &\longmapsto g_1
\end{aligned}$$

and so $G^{0\prime} \cong G^{0\prime\prime} \rtimes \mathbb{Z}_3$.

The next claim, that we do not prove, is a standard result about semi-direct products.

Claim. Let L be a finite group and let K be a cyclic group of order p. Let $\varphi_1, \varphi_2 \colon K \to \mathrm{Aut}(L)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugated. Then $L \rtimes_{\varphi_1} K \cong L \rtimes_{\varphi_2} K$.

This means that, in order to build up the group $G^{0\prime}$, we have only to look at the conjugacy classes of elements of order 3 in $\mathrm{Aut}(G^{0\prime\prime})$ and at $\mathrm{Id}(\mathrm{Aut}(G^{0\prime\prime}))$. The function `ConjugCl(A,n)`returns a representative of each conjugacy class of elements of A of order n.

The following MAGMA script

```
>  v:={ 584, 585, 763, 765, 766, 773, 776 };
>  for j in  v do
for> H2:=SmallGroup(720, j);
for> Aut2:=AutGr(H2);
for> A2:=AutomorphismGroup(H2);
for> R2:=ConjugCl(Aut2,3);
for> C3:=CyclicGroup(3);
for> R2[1+#R2]:=Id(A2);
for> f2:=[]; for i in [1..#R2] do
for|for> f2[i]:=hom<C3->A2|R2[i]>;end for;
for> h1:=[]; for i in [1..#R2] do
for|for> h1[i]:=SemidirectProduct(H2,C3,f2[i]);
for|for> j, i, ExVectGens(h1[i],[3,3,4],0); end for; end for;
584 1 false
584 2 false
585 1 false
585 2 false
773 1 false
773 2 false
763 1 false
763 2 false
765 1 false
765 2 false
776 1 false
776 2 false
766 1 false
766 2 false
>
```

shows that no group isomorphic to $G^{0\prime} = G^{0\prime\prime} \rtimes \mathbb{Z}_3$ has a spherical system of generators of type $[3, 3, 4]$. □

2. Non existence results

Remark 2.1. Let $X = (C \times C)/G^0$ be a mixed q.e. surface given by a set spherical system of generators $(h_1, \ldots, h_r)$ of $G^0 \subseteq G$; in order to compute the basket of singularities we have to compare $(h_1, \ldots, h_r)$ with its conjugate by $\tau' \in G \setminus G^0$. Note that $(\tau' h_1 \tau'^{-1}, \ldots, \tau' h_r \tau'^{-1})$ is a spherical system of generators of $G^0 \subseteq G$ of the same type.
Hence, if a group has a set of spherical generators of the required type, we check if this group has a pair of set of spherical generators that give the right

singularities (on Y). If this is not the case surely a set of spherical generators and its conjugated by τ' in G cannot give the required singularities.

Lemma 2.2. *No group of order 4032 has a pair of spherical system of generators of type* $[2,3,8]$ *which give the expected singularities on* Y*, i.e. either* $\{2\times C_{8,1}, 2\times C_{4,1}, 2\times C_{8,5}\}$ *or* $\{4\times C_{4,1}\}$ *or* $\{2\times C_{2,1}\}$*.*

Proof. Assume that G^0 is a group of order 4032 admitting a surjective homomorphism $\mathbb{T}(2,3,8)\to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}}\cong\mathbb{Z}_2$ and since there are no perfect groups of order 4032, the commutator subgroup $G^{0\prime}=[G^0,G^0]$ of G^0 has order 2016 and it is a quotient of $[\mathbb{T}(2,3,8),\mathbb{T}(2,3,8)]\cong\mathbb{T}(3,3,4)$.

Since $\mathbb{T}(3,3,4)^{\mathrm{ab}}\cong\mathbb{Z}_3$ and since there are no perfect groups of order 2016, the commutator subgroup $G^{0\prime\prime}=[G^{0\prime},G^{0\prime}]$ of $G^{0\prime}$ has order 672 and it is a quotient of $[\mathbb{T}(3,3,4),\mathbb{T}(3,3,4)]\cong\mathbb{T}(4,4,4)$. The following MAGMA computation

```
> Test([4,4,4], 672,0);
{ 1046, 1255 }
>
```

shows that only the groups $G(672,v)$ with $v\in\{1046,1255\}$ have a spherical system of generators of type $[4,4,4]$.

Now the proof continues exactly as the proof of Lemma 1.7: we have that $G^{0\prime}=G^{0\prime\prime}\rtimes\mathbb{Z}_3$ and we construct all the groups of this form up to isomorphism.

The following MAGMA script shows that no group isomorphic to $G^{0\prime}=G^{0\prime\prime}\rtimes\mathbb{Z}_3$, with $G^{0\prime\prime}=G(672,1046)$, has a spherical system of generators of type $[3,3,4]$:

```
> H2:=SmallGroup(672,1046);
> A2:=AutomorphismGroup(H2);
> Aut2:=AutGr(H2);
> R2:=ConjugCl(Aut2,3);
> C3:=CyclicGroup(3);
> R2[1+#R2]:=Id(A2);
> f2:=[]; for i in [1..#R2] do
for> f2[i]:=hom<C3->A2|R2[i]>;end for;
> h1:=[]; for i in [1..#R2] do
for> h1[i]:=SemidirectProduct(H2,C3,f2[i]);
for> i, ExVectGens(h1[i],[3,3,4],0); end for;
1 false
2 false
>
```

The following MAGMA script shows that two extensions $G^{0\prime}=G^{0\prime\prime}\rtimes\mathbb{Z}_3$, with $G^{0\prime\prime}=G(672,1255)$, have a spherical system of generators of type $[3,3,4]$; moreover this two extensions are isomorphic.

```
> H2:=SmallGroup(672,1255);
> A2:=AutomorphismGroup(H2);
```

```
> Aut2:=AutGr(H2);
> R2:=ConjugCl(Aut2,3);
> C3:=CyclicGroup(3);
> R2[1+#R2]:=Id(A2);
> f2:=[]; for i in [1..#R2] do
for> f2[i]:=hom<C3->A2|R2[i]>;end for;
> h1:=[]; for i in [1..#R2] do
for> h1[i]:=SemidirectProduct(H2,C3,f2[i]);
for> i, ExVectGens(h1[i],[3,3,4],0); end for;
1 true
2 false
3 true
4 false
> IsIsomorphic(h1[1],h1[3]);
true  Homomorphism of ...
>H1:=h1[1];
```

It can be proved, in a similar way as for $G^{0\prime} \cong G^{0\prime\prime} \rtimes \mathbb{Z}_3$, that G^0 is isomorphic to a semidirect product $G^{0\prime} \rtimes \mathbb{Z}_2$.
The following MAGMA script (that continues the previous one) shows that $G^{0\prime}$=h1[1] has only one extension $G^{0\prime} \rtimes \mathbb{Z}_2$ with a spherical system of generators of type $(2,3,8)$:

```
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> R1:=ConjugCl(Aut1,2);
> R1[1+#R1]:=Id(A1);
> C2:=CyclicGroup(2);
> f1:=[]; for i in [1..#R1] do
for> f1[i]:=hom<C2->A1|R1[i]>;end for;
> h:=[]; for i in [1..#R1] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExVectGens(h[i],[2,3,8],0); end for;
1 false
2 false
3 false
4 true
5 false
6 false
7 false
8 false
9 false
>
> H:=h[4];
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,3,8]$ of G^0=h[4], the singularity test fails, and so also this case does not occur.

```
> SingularitiesY([{*1/8,1/4,5/8*},{**}],H,[2,3,8],0);
```

```
false
>
> SingularitiesY([{*1/4^^2*},{**}],H,[2,3,8],0);
false
>
> SingularitiesY([{*1/2*},{**}],H,[2,3,8],0);
false
```

□

Lemma 2.3. *No group of order 2560 has a pair of spherical system of generators of type* $[2,4,5]$ *which give the expected singularities on* Y*, i.e.* $\{2 \times C_{5,3}\}$.

Proof. Assume that G^0 is a group of order 2560 admitting a surjective homomorphism $\mathbb{T}(2,4,5) \to G^0$.

Since $\mathbb{T}(2,4,5)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2560, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1280 and it is a quotient of $[\mathbb{T}(2,4,5), \mathbb{T}(2,4,5)] \cong \mathbb{T}(2,5,5)$.

The following MAGMA computation

```
> TestBAD([2,5,5], 1280,0);
{ 1116310 }
>
```

shows that only the group $G(1280, 1116310)$ has a spherical system of generators of type $[2,5,5]$.

Now the proof continues exactly as the proof of Lemma 1.7: we have that $G^0 = G^{0\prime} \rtimes \mathbb{Z}_2$ and we construct all the groups of this form up to isomorphism. Among these groups only one has a spherical system of generators of type $[2,4,5]$ as the following MAGMA script shows:

```
> H1:=SmallGroup(1280,1116310);
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> C2:=CyclicGroup(2);
> R:=ConjugCl(Aut1,2);
> R[1+#R]:=Id(A1);
> f1:=[]; for i in [1..#R] do
for> f1[i]:=hom<C2->A1|R[i]>; end for;
> h:=[]; for i in [1..#R] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExVectGens(h[i],[2,4,5],0); end for;
1 true
2 false
3 false
4 false
5 false
6 false
7 false
8 false
9 false
```

```
10 false
11 false
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,4,5]$ of G^0=h[1], the singularities test fails, and so also this case does not occur.

```
> H:=h[1];
> SingularitiesY([{* 3/5 *},{* *}],H,[2,4,5],0);
false
>
```

□

Lemma 2.4. *No group of order 3072 has a pair of spherical system of generators of type* $[2,3,8]$ *which give the expected singularities on* Y*, i.e.* $\{2 \times C_{3,1}, 2 \times C_{3,2}\}$.

Proof. Assume that G^0 is a group of order 3072 admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 3072, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1536 and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8)] \cong \mathbb{T}(3,3,4)$.

The following MAGMA computation

```
> TestBAD([3,3,4], 1536,0);
{ 408526602 }
>
```

shows that only the group $G(1536, 408526602)$ has a spherical system of generators of type $[3,3,4]$.

Now the proof is the same of Lemma 2.3: we have that $G^0 = G^{0\prime} \rtimes \mathbb{Z}_2$ and we construct all the groups of this form up to isomorphism. Among these groups only one has a spherical system of generators of type $[2,3,8]$ as the following MAGMA script shows:

```
> H1:=SmallGroup(1536,408526602);
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> C2:=CyclicGroup(2);
> R:=ConjugCl(Aut1,2);
> R[1+#R]:=Id(A1);
> f1:=[]; for i in [1..#R] do
for> f1[i]:=hom<C2->A1|R[i]>; end for;
> h:=[]; for i in [1..#R] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExVectGens(h[i],[2,3,8],0); end for;
1 false
2 true
3 false
4 false
5 false
```

```
6 false
7 false
8 false
9 false
10 false
11 false
12 false
13 false
14 false
15 false
16 false
>
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,3,8]$ of G^0=h[2], the singularities test fails, and so also this case does not occur.

```
> H:=h[2];
> SingularitiesY([{* 1/3, 2/3 *}, {* *}], H, [2,3,8],0);
false
>
```

□

Lemma 2.5. *No group of order 4896 has a pair of spherical system of generators of type* $[2,3,8]$ *which give the expected singularities on* Y*, i.e.* $\{2 \times C_{8,1}, 2 \times C_{8,3}\}$.

Proof. Assume that G^0 is a group of order 4896 admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

It holds $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$. There is only one perfect group of order 4896, we denote it by H. $H = 2.L_2(17)$ has the following MAGMA representation:

```
> F<w>:=GF(9);
>
> x:=CambridgeMatrix(1,F,8,[
> "01000000",
> "20000000",
> "00010000",
> "00200000",
> "00000100",
> "00002000",
> "83300083",
> "37420004"]);
>
> y:=CambridgeMatrix(1,F,8,[
> "62000000",
> "00100000",
> "48300000",
> "00001000",
> "00000010",
```

```
> "00000001",
> "00010000",
> "46466262"]);
>
> H<x,y>:=MatrixGroup<8,F|x,y>;
> IsPerfect(H);
true
> #H;
4896
>
```

The following MAGMA computation

```
> ExVectGens(H,[2,4,5],0);
false
```

shows that H does not have a spherical system of generators of type $[2, 3, 8]$ If G^0 is a group of order 4896 with a spherical system of generators of type $[2, 3, 8]$, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2448 and it is a quotient of $[\mathbb{T}(2, 3, 8), \mathbb{T}(2, 3, 8)] \cong \mathbb{T}(3, 3, 4)$.

It holds $\mathbb{T}(3, 3, 4)^{\text{ab}} \cong \mathbb{Z}_3$ and there is only one perfect group of order 2448, we denote it by H', and we will analyze it later.

If G^0 is a group of order 2448 ($G^0 \neq H'$) with a spherical system of generators of type $[3, 3, 4]$, the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 816 and it is a quotient of $[\mathbb{T}(3, 3, 4), \mathbb{T}(3, 3, 4)] \cong \mathbb{T}(4, 4, 4)$. The following MAGMA computation

```
> Test([4,4,4], 816,0);
{}
>
```

shows that there are no groups of order 816 with a spherical system of generators of type $[4, 4, 4]$.

Now we go back to H'. $H' = 2.L_2(17)$ has the following MAGMA representation:

```
> F:=GF(17);
>
> x:=CambridgeMatrix(3,F,3,\[
> 1,0,0,
> 3,16,0,
> 3,0,16]);
>
> y:=CambridgeMatrix(3,F,3,\[
> 0,1,0,
> 0,0,1,
> 1,0,0]);
>
> H1<x,y>:=MatrixGroup<3,F|x,y>;
> IsPerfect(H1);
true
```

```
> #H1;
2448
>
```

The following MAGMA script

```
> ExVectGens(H1,[3,3,4],0);
true
>
```

shows that this group has a spherical system of generators of type $[3, 3, 4]$.

Now the proof continues exactly as the proof of Lemma 2.3: we have that $G^0 = G^{0\prime} \rtimes \mathbb{Z}_2$ and we construct all the groups of this form up to isomorphism. Among these groups only one has a spherical system of generators of type $[2, 3, 8]$ as the following MAGMA script shows:

```
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> C2:=CyclicGroup(2);
> R:=ConjugCl(Aut1,2);
> R[1+#R]:=Id(A1);
> f1:=[]; for i in [1..#R] do
for> f1[i]:=hom<C2->A1|R[i]>; end for;
> h:=[]; for i in [1..#R] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExVectGens(h[i],[2,3,8],0); end for;
1 false
2 true
3 true
> IsIsomorphic(h[2],h[3]);
true  Homomorphism of ...
> H:=h[2];
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2, 3, 8]$ of G^0=h[2], the singularity test fails, and so also this case does not occur.

```
> SingularitiesY([{* 3/5 *},{* *}],H,[2,4,5],0);
false
>
```

$\square$

We recall that a pair of spherical systems generators (T_1, T_2) is *disjoint* if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\} .$$

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are a disjoint pair of spherical system of generators of type $[2, 3, 9]$ for G^0, then $(a_i, b_i a_i b_i^{-1}, b_i^2 a_i b_i^{-2}, c_i^3)$, for $i = 1, 2$, are spherical system of generators of type $[2, 2, 2, 3]$ for $G^{0\prime} = [G^0, G^0]$; moreover these two systems are disjoint (see [BCG08, Lemma 4.3]).

We note that a pair of spherical system of generators are disjoint if and only if the basket of singularities that they induce is empty.

Lemma 2.6. *No group of order 2592 has a disjoint pair of spherical systems of generators of type* $[2, 3, 9]$.

Proof. Assume that G^0 is a group of order 2592 admitting a surjective homomorphism $\mathbb{T}(2, 3, 9) \to G^0$.

Since $\mathbb{T}(2, 3, 9)^{\text{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2592, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 864 and it is a quotient of $[\mathbb{T}(2, 3, 9), \mathbb{T}(2, 3, 9] \cong \mathbb{T}(2, 2, 2, 3)$. The following MAGMA computation

```
> Test([2,2,2,3], 864,0);
{2225, 4175}
>
```

shows that only the groups $G(864, v)$ with $v \in \{2225, 4175\}$ have a spherical system of generators of type $[2, 2, 2, 3]$.

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are a disjoint pair of spherical system of generators of type $[2, 3, 9]$ for G^0, then $(a_i,\ b_i a_i b_i^{-1},\ b_i^2 a_i b_i^{-2},\ c_i^3)$, for $i = 1, 2$, are spherical system of generators of type $[2, 2, 2, 3]$ for $G^{0\prime} = [G^0, G^0]$; moreover these two systems are disjoint.

The following MAGMA computations

```
> SingularitiesY([{**},{**}],SmallGroup(864,2225),[2,2,2,3],0);
false
> SingularitiesY([{**},{**}],SmallGroup(864,4175),[2,2,2,3],0);
false
```

show that the groups $G(864, 2225)$ and $G(864, 4175)$ do not have a disjoint pair of spherical system of generators of type $[2, 2, 2, 3]$, a contradiction. $\square$