Surfaces of general type with geometric genus zero: a survey

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Abstract In the last years there have been several new constructions of surfaces of general type with $p_g = 0$, and important progress on their classification. The present paper presents the status of the art on surfaces of general type with $p_g = 0$, and gives an updated list of the existing surfaces, in the case where $K^2 = 1, ..., 7$. It also focuses on certain important aspects of this classification.

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Introduction

It is nowadays well known that minimal surfaces of general type with $p_g(S) = 0$ have invariants $p_g(S) = q(S) = 0, 1 \leq K_S^2 \leq 9$, hence they yield a finite number of irreducible components of the moduli space of surfaces of general type.

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At first glance this class of surfaces seems rather narrow, but we want to report on recent results showing how varied and rich is the botany of such surfaces, for which a complete classification is still out of reach.

These surfaces represent for algebraic geometers an almost prohibitive test case about the possibility of extending the fine Enriques classification of special surfaces to surfaces of general type.

On the one hand, they are the surfaces of general type which achieve the minimal value 1 for the holomorphic Euler-Poincaré characteristic $\chi(S) := p_g(S) - q(S) + 1$, so a naive (and false) guess is that they should be "easier" to understand than other surfaces with higher invariants; on the other hand, there are pathologies (especially concerning the pluricanonical systems) or problems (cf. the Bloch conjecture ([Blo75]) asserting that for surfaces with $p_g(S) = q(S) = 0$ the group of zero cycles modulo rational equivalence should be isomorphic to $\mathbb{Z}$), which only occur for surfaces with $p_g = 0$.

Surfaces with $p_g(S) = q(S) = 0$ have a very old history, dating back to 1896 ([Enr96], see also [EnrMS], I, page 294, and [Cas96]) when Enriques constructed the so called Enriques surfaces in order to give a counterexample to the conjecture of Max Noether that any such surface should be rational, immediately followed by Castelnuovo who constructed a surface with $p_g(S) = q(S) = 0$ whose bicanonical pencil is elliptic.

The first surfaces of general type with $p_g = q = 0$ were constructed in the 1930' s by Luigi Campedelli and by Lucien Godeaux (cf. [Cam32], [God35]): in their honour minimal surfaces of general type with $K_S^2 = 1$ are called numerical Godeaux surfaces, and those with $K_S^2 = 2$ are called numerical Campedelli surfaces.

In the 1970's there was a big revival of interest in the construction of these surfaces and in a possible attempt to classification.

After rediscoveries of these and other old examples a few new ones were found through the efforts of several authors, in particular Rebecca Barlow ([Bar85a]) found a simply connected numerical Godeaux surface, which played a decisive role in the study of the differential topology of algebraic surfaces and 4-manifolds (and also in the discovery of Kähler Einstein metrics of opposite sign on the same manifold, see [CL97]).

A (relatively short) list of the existing examples appeared in the book [BPV84], (see [BPV84], VII, 11 and references therein, and see also [BHPV04] for an updated slightly longer list).

There has been recently important progress on the topic, and the goal of the present paper is to present the status of the art on surfaces of general type with $p_g = 0$, of course focusing only on certain aspects of the story.

Our article is organized as follows: in the first section we explain the "fine" classification problem for surfaces of general type with $p_g = q = 0$. Since the solution to this problem is far from sight we pose some easier problems which could have a greater chance to be solved in the near future.

Moreover, we try to give an update on the current knowledge concerning surfaces with $p_g = q = 0$.

In the second section, we shortly review several reasons why there has been a lot of attention devoted to surfaces with geometric genus p_g equal to zero: Bloch's conjecture, the exceptional behaviour of the pluricanonical maps and the interesting questions whether there are surfaces of general type homeomorphic to Del Pezzo surfaces. It is not possible that a surface of general type be diffeomorphic to a rational surface. This follows from Seiberg-Witten theory which brought a breakthrough establishing in particular that the Kodaira dimension is a differentiable invariant of the 4-manifold underlying an algebraic surface.

Since the first step towards a classification is always the construction of as many examples as possible, we describe in section three various construction methods for algebraic surfaces, showing how they lead to surfaces of general type with $p_g = 0$. Essentially, there are two different approaches, one is to take quotients, by a finite or infinite group, of known (possibly non-compact) surfaces, and the other is in a certain sense the dual one, namely constructing the surfaces as Galois coverings of known surfaces.

The first approach (i.e., taking quotients) seems at the moment to be far more successful concerning the number of examples that have been constructed by this method. On the other hand, the theory of abelian coverings seems much more useful to study the deformations of the constructed surfaces, i.e., to get hold of the irreducible, resp. connected components of the corresponding moduli spaces.

In the last section we review some recent results which have been obtained by the first two authors, concerning the connected components of the moduli spaces corresponding to Keum-Naie, respectively primary Burniat surfaces.

1 Notation

For typographical reasons, especially lack of space inside the tables, we shall use the following non standard notation for a finite cyclic group of order m:

$$\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m.$$

Furthermore Q_8 will denote the quaternion group of order 8,

$$Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}.$$

As usual, $\mathfrak{S}_n$ is the symmetric group in n letters, $\mathfrak{A}_n$ is the alternating subgroup. $D_{p,q,r}$ is the generalized dihedral group admitting the following presentation:

$$D_{p,q,r} = \langle x, y | x^p, y^q, xyx^{-1}y^{-r} \rangle,$$

while $D_n = D_{2,n,-1}$ is the usual dihedral group of order $2n$.

$G(n,m)$ denotes the m-th group of order n in the MAGMA database of small groups.

Finally, we have semidirect products $H \rtimes \mathbb{Z}_r$; to specify them, one should indicate the image $\varphi \in Aut(H)$ of the standard generator of $\mathbb{Z}_r$ in $Aut(H)$. There is no space in the tables to indicate φ, hence we explain here which automorphism φ will be in the case of the semidirect products occurring as fundamental groups.

For $H = \mathbb{Z}^2$ either r is even, and then φ is $-Id$, or $r = 3$ and φ is the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Else H is finite and $r = 2$; for $H = \mathbb{Z}_3^2$, φ is $-Id$; for $H = \mathbb{Z}_2^4$, φ is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Concerning the case where the group G is a semidirect product, we simply refer to [BCGP08] for more details.

Finally, Π_g is the fundamental group of a compact Riemann surface of genus g.

2 The classification problem and "simpler" sub-problems

The history of surfaces with geometric genus equal to zero starts about 120 years ago with a question posed by Max Noether.

Assume that $S \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth projective surface. Recall that the *geometric genus* of S:

$$p_g(S) := h^0(S, \Omega^2_S) := \dim H^0(S, \Omega^2_S),$$

and the *irregularity* of S:

$$q(S) := h^0(S, \Omega^1_S) := \dim H^0(S, \Omega^1_S),$$

are *birational invariants* of S.

Trying to generalize the one dimensional situation, Max Noether asked the following:

Question 2.1. Let S be a smooth projective surface with $p_g(S) = q(S) = 0$. Does this imply that S is rational?

The first negative answer to this question is, as we already wrote, due to Enriques ([Enr96], see also [EnrMS], I, page 294) and Castelnuovo, who constructed counterexamples which are surfaces of special type (this means, with Kodaira dimension ≤ 1. *Enriques surfaces* have Kodaira dimension equal to 0, Castelnuovo surfaces have instead Kodaira dimension 1).

After the already mentioned examples by Luigi Campedelli and by Lucien Godeaux and the new examples found by Pol Burniat ([Bur66]), and by many other authors, the discovery and understanding of surfaces of general type with $p_g = 0$ was considered as a challenging problem (cf. [Dol77]): a complete fine classification however soon seemed to be far out of reach.

Maybe this was the motivation for D. Mumford to ask the following provocative

Question 2.2 (Montreal 1980). Can a computer classify all surfaces of general type with $p_g = 0$?

Before we comment more on Mumford's question, we shall recall some basic facts concerning surfaces of general type.

Let S be a *minimal* surface of general type, i.e., S does not contain any rational curve of self intersection (-1), or equivalently, the canonical divisor K_S of S is nef and big ($K_S^2 > 0$). Then it is well known that

$$K_S^2 \geq 1, \ \chi(S) := 1 - q(S) + p_g(S) \geq 1.$$

In particular, $p_g(S) = 0 \implies q(S) = 0$. Moreover, we have a coarse moduli space parametrizing minimal surfaces of general type with fixed χ and K^2.

Theorem 2.3. *For each pair of natural numbers (x,y) we have the Gieseker moduli space $\mathfrak{M}^{can}_{(x,y)}$, whose points correspond to the isomorphism classes of minimal surfaces S of general type with $\chi(S) = x$ and $K_S^2 = y$.*

It is a quasi projective scheme which is a coarse moduli space for the canonical models of minimal surfaces S of general type with $\chi(S) = x$ and $K_S^2 = y$.

An upper bound for K_S^2 is given by the famous Bogomolov-Miyaoka-Yau inequality:

Theorem 2.4 ([Miy77b], [Yau77], [Yau78], [Miy82]). *Let S be a smooth surface of general type. Then*

$$K_S^2 \leq 9\chi(S),$$

and equality holds if and only if the universal covering of S is the complex ball $\mathbb{B}_2 := \{(z,w) \in \mathbb{C}^2 | |z|^2 + |w|^2 < 1\}$.

As a note for the non experts: Miyaoka proved in the first paper the general inequality, which Yau only proved under the assumption of ampleness of the canonical divisor K_S. But Yau showed that if equality holds, and K_S is ample, then the universal cover is the ball; in the second paper Miyaoka showed that if equality holds, then necessarily K_S is ample.

Remark 2.5. Classification of surfaces of general type with $p_g = 0$ means therefore to "understand" the nine moduli spaces $\mathfrak{M}^{can}_{(1,n)}$ for $1 \leq n \leq 9$, in particular, the connected components of each $\mathfrak{M}^{can}_{(1,n)}$ corresponding to surfaces with $p_g = 0$. Here, understanding means to describe the connected and irreducible components and their respective dimensions.

Even if this is the "test-case" with the lowest possible value for the invariant $\chi(S)$ for surfaces of general type, still nowadays we are quite far from realistically seeing how this goal can be achieved. It is in particular a quite non trivial question, given two explicit surfaces with the same invariants (χ, K^2), to decide whether they are in the same connected component of the moduli space.

An easy observation, which indeed is quite useful, is the following:

Remark 2.6. Assume that S, S' are two minimal surfaces of general type which are in the same connected component of the moduli space. Then S and S' are orientedly diffeomorphic through a diffeomorphism preserving the Chern class of the canonical divisor; whence S and S' are homeomorphic, in particular they have the same (topological) fundamental group.

Thus the fundamental group π_1 is the simplest invariant which distinguishes connected components of the moduli space $\mathfrak{M}^{can}_{(x,y)}$.

So, it seems natural to pose the following questions which sound ”easier” to solve than the complete classification of surfaces with geometric genus zero.

Question 2.7. What are the topological fundamental groups of surfaces of general type with $p_g = 0$ and $K_S^2 = y$?

Question 2.8. Is $\pi_1(S) =: \Gamma$ residually finite, i.e., is the natural homomorphism $\Gamma \to \hat{\Gamma} = \lim_{H \triangleleft_f \Gamma}(\Gamma/H)$ from Γ to its profinite completion $\hat{\Gamma}$ injective?

Remark 2.9. 1) Note that in general fundamental groups of algebraic surfaces are not residually finite, but all known examples have $p_g > 0$ (cf. [Tol93], [CK92]).

2) There are examples of surfaces S, S' with non isomorphic topological fundamental groups, but whose profinite completions are isomorphic (cf. [Serre64], [BCG07]).

Question 2.10. What are the best possible positive numbers a, b such that

- $K_S^2 \leq a \implies |\pi_1(S)| < \infty$,
- $K_S^2 \geq b \implies |\pi_1(S)| = \infty$?

In fact, by Yau's theorem $K_S^2 = 9 \implies |\pi_1(S)| = \infty$. Moreover by [BCGP08] there exists a surface S with $K_S^2 = 6$ and finite fundamental group, so $b \geq 7$. On the other hand, there are surfaces with $K^2 = 4$ and infinite fundamental group (cf. [Keu88], [Nai99]), whence $a \leq 3$.

Note that all known minimal surfaces of general type S with $p_g = 0$ and $K_S^2 = 8$ are uniformized by the bidisk $\mathbb{B}_1 \times \mathbb{B}_1$.

Question 2.11. Is the universal covering of S with $K_S^2 = 8$ always $\mathbb{B}_1 \times \mathbb{B}_1$?

An affirmative answer to the above question would give a negative answer to the following question of F. Hirzebruch:

Question 2.12 (F. Hirzebruch). Does there exist a surface of general type homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$?

Or homeomorphic to the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ in one point ?

In the other direction, for $K_S^2 \leq 2$ it is known that the profinite completion $\hat{\pi}_1$ is finite. There is the following result:

Theorem 2.13. *1) $K_S^2 = 1 \implies \hat{\pi}_1 \cong \mathbb{Z}_m$ for $1 \leq m \leq 5$ (cf. [Rei78]).*
2) $K_S^2 = 2 \implies |\hat{\pi}_1| \leq 9$ (cf. [Rei], [Xia85a]).

The bounds are sharp in both cases, indeed for the case $K_S^2 = 1$ there are examples with $\pi_1(S) \cong \mathbb{Z}_m$ for all $1 \leq m \leq 5$ and there is the following conjecture

Conjecture 2.14 (M. Reid). $\mathfrak{M}^{can}_{(1,1)}$ has exactly five irreducible components corresponding to each choice $\pi_1(S) \cong \mathbb{Z}_m$ for all $1 \leq m \leq 5$.

This conjecture is known to hold true for $m \geq 3$ (cf. [Rei78]).
One can ask similar questions:

Question 2.15. 2) Does $K_S^2 = 2$, $p_g(S) = 0$ imply that $|\pi_1(S)| \leq 9$?
3) Does $K_S^2 = 3$ (and $p_g(S) = 0$) imply that $|\pi_1(S)| \leq 16$?

2.1 Update on surfaces with $p_g = 0$

There has been recently important progress on surfaces of general type with $p_g = 0$ and the current situation is as follows:

$\underline{K_S^2 = 9}$: these surfaces have the unit ball in $\mathbb{C}^2$ as universal cover, and their fundamental group is an arithmetic subgroup Γ of $SU(2,1)$.

This case seems to be completely classified through exciting new work of Prasad and Yeung and of Cartright and Steger ([PY07], [PY09], [CS]) asserting that the moduli space consists exactly of 100 points, corresponding to 50 pairs of complex conjugate surfaces (cf. [KK02]).

$\underline{K_S^2 = 8}$: we posed the question whether in this case the universal cover must be the bidisk in $\mathbb{C}^2$.

Assuming this, a complete classification should be possible.

The classification has already been accomplished in [BCG08] for the reducible case where there is a finite étale cover which is isomorphic to a product of curves. In this case there are exactly 18 irreducible connected components of the moduli space: in fact, 17 such components are listed in [BCG08], and recently Davide Frapporti ([Frap10]), while rerunning the classification program, found one more family whose existence had been excluded by an incomplete analysis. There are many examples, due to Kuga and Shavel ([Kug75], [Sha78]) for the irreducible case, which yield (as in the case $K_S^2 = 9$) rigid surfaces (by results of Jost and Yau [JT85]); but a complete classification of this second case is still missing.

The constructions of minimal surfaces of general type with $p_g = 0$ and with $K_S^2 \leq 7$ available in the literature (to the best of the authors' knowledge, and excluding the recent results of the authors, which will be described later) are listed in table 1.

We proceed to a description, with the aim of putting the recent developments in proper perspective.

$\underline{K_S^2 = 1}$, i.e., *numerical Godeaux surfaces*: recall that by conjecture 2.14 the moduli space should have exactly five irreducible connected components, distinguished by the order of the fundamental group, which should be cyclic of order at most 5

Table 1 Minimal surfaces of general type with $p_g = 0$ and $K^2 \leq 7$ available in the literature

K^2	π_1	π_1^{alg}	H_1	References
1	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[God34][Rei78][Miy76]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[Rei78][OP81][Bar84][Nai94]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[Rei78]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Bar84][Ino94][KL10]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Wer94][Wer97]
	$\{1\}$	$\{1\}$	$\{0\}$	[Bar85a][LP07]
	?	$\{1\}$	$\{0\}$	[CG94][DW99]
2	$\mathbb{Z}_9$	$\mathbb{Z}_9$	$\mathbb{Z}_9$	[MP08]
	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	[Xia85a][MP08]
	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	[Cam32][Rei][Pet76][Ino94][Nai94]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[Rei][Nai94][Keu88]
	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	[Rei]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[Rei] [Bea96]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[Rei91]
	?	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[NP09]
	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[Cat81][Sup98]
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	[Ino94][Keu88]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[LP09]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KL10]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[LP09]
	$\{1\}$	$\{1\}$	$\{0\}$	[LP07]
3	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[Nai94] [Keu88] [MP04a]
	$Q_8 \times \mathbb{Z}_2$	$Q_8 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$	[Bur66][Pet77] [Ino94]
	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	[CS]
	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}$	[CS]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[CS]
	D_4	D_4	$\mathbb{Z}_2^2$	[CS]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[CS]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[CS]
	$\mathfrak{S}_3$	$\mathfrak{S}_3$	$\mathbb{Z}_2$	[CS]
	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[CS]
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	[CS]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[CS]
	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[CS]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KL10][CS]
	?	?	$\mathbb{Z}_2$	[PPS08a]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS09a][CS]
4	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	[Nai94][Keu88]
	$Q_8 \times \mathbb{Z}_2^2$	$Q_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	[Bur66][Pet77][Ino94]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Par10]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS09b]
5	$Q_8 \times \mathbb{Z}_2^3$	$Q_8 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	[Bur66][Pet77][Ino94]
	?	?	?	[Ino94]
6	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^6$	[Bur66][Pet77][Ino94]
	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_3^3 \subset H_1$	[Kul04]
	?	?	?	[Ino94][MP04b]
7	$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	?	[Ino94][MP01a] [BCC10]

([Rei78] settled the case where the order of the first homology group is at least 3; [Bar85a], [Bar84] and [Wer94] were the first to show the occurrence of the two other groups).

$K_S^2 = 2$, i.e., *numerical Campedelli surfaces*: here, it is known that the order of the algebraic fundamental group is at most 9, and the cases of order $8, 9$ have been classified by Mendes Lopes, Pardini and Reid ([MP08], [MPR09], [Rei]), who showed in particular that the fundamental group equals the algebraic fundamental group and cannot be the dihedral group D_4 of order 8. Naie ([Nai99]) showed that the group D_3 of order 6 cannot occur as the fundamental group of a numerical Campedelli surface. By the work of Lee and Park ([LP07]), one knows that there exist simply connected numerical Campedelli surfaces.

Recently, in [BCGP08], [BP10], the construction of eight families of numerical Campedelli surfaces with fundamental group $\mathbb{Z}_3$ was given. Neves and Papadakis ([NP09]) constructed a numerical Campedelli surface with algebraic fundamental group $\mathbb{Z}_6$, while Lee and Park ([LP09]) constructed one with algebraic fundamental group $\mathbb{Z}_2$, and one with algebraic fundamental group $\mathbb{Z}_3$ was added in the second version of the same paper. Finally Keum and Lee ([KL10]) constructed examples with topological fundamental group $\mathbb{Z}_2$.

Open conjectures are:

Conjecture 2.16. Is the fundamental group $\pi_1(S)$ of a numerical Campedelli surface finite?

Question 2.17. Does every group of order ≤ 9 except D_4 and D_3 occur as topological fundamental group (not only as algebraic fundamental group)?

The answer to question 2.17 is completely open for $\mathbb{Z}_4$; for $\mathbb{Z}_6$ one suspects that this fundamental group is realized by the Neves-Papadakis surfaces.

Note that the existence of the case where $\pi_1(S) = \mathbb{Z}_7$ is shown in the paper [Rei91] (where the result is not mentioned in the introduction).

$K_S^2 = 3$: here there were two examples of non trivial fundamental groups, the first one due to Burniat and Inoue, the second one to Keum and Naie ([Bur66], [Ino94], [Keu88], [Nai94]).

It is conjectured that for $p_g(S) = 0, K_S^2 = 3$ the algebraic fundamental group is finite, and one can ask as in 1) above whether also $\pi_1(S)$ is finite. Park, Park and Shin ([PPS09a]) showed the existence of simply connected surfaces, and of surfaces with torsion $\mathbb{Z}_2$ ([PPS08a]). More recently Keum and Lee ([KL10]) constructed an example with $\pi_1(S) = \mathbb{Z}_2$.

Other constructions were given in [Cat98], together with two more examples with $p_g(S) = 0, K^2 = 4, 5$: these turned out however to be the same as the Burniat surfaces.

In [BP10], the existence of four new fundamental groups is shown. Then new fundamental groups were shown to occur by Cartright and Steger, while considering quotients of a fake projective plane by an automorphism of order 3.

With this method Cartright and Steger produced also other examples with $p_g(S) = 0$, $K_S^2 = 3$, and trivial fundamental group, or with $\pi_1(S) = \mathbb{Z}_2$.

$K_S^2 = 4$: there were known up to now three examples of fundamental groups, the trivial one (Park, Park and Shin, [PPS09b]), a finite one, and an infinite one. In [BCGP08], [BP10] the existence of 10 new groups, 6 finite and 4 infinite, is shown: thus minimal surfaces with $K_S^2 = 4$, $p_g(S) = q(S) = 0$ realize at least 13 distinct topological types. Recently, H. Park constructed one more example in [Par10] raising the number of topological types to 14.

$K_S^2 = 5,6,7$: there was known up to now only one example of a fundamental group for $K_S^2 = 5,7$.

Instead for $K_S^2 = 6$, there are the Inoue-Burniat surfaces and an example due to V. Kulikov (cf. [Kul04]), which contains $\mathbb{Z}_3^3$ in its torsion group. Like in the case of primary Burniat surfaces one can see that the fundamental group of the Kulikov surface fits into an exact sequence

$$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1.$$

$K_S^2 = 5$: in [BP10] the existence of 7 new groups, four of which are finite, is shown: thus minimal surfaces with $K_S^2 = 5$, $p_g(S) = q(S) = 0$ realize at least 8 distinct topological types.

$K_S^2 = 6$: in [BCGP08] the existence of 6 new groups, three of which finite, is shown: thus minimal surfaces with $K_S^2 = 6$, $p_g(S) = q(S) = 0$ realize at least 7 distinct topological types.

$K_S^2 = 7$: we shall show elsewhere ([BCC10]) that these surfaces, constructed by Inoue in [Ino94], have a fundamental group fitting into an exact sequence

$$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1.$$

This motivates the following further question (cf. question 2.10).

Question 2.18. Is it true that fundamental groups of surfaces of general type with $q = p_g = 0$ are finite for $K_S^2 \leq 3$, and infinite for $K_S^2 \geq 7$?

3 Other reasons why surfaces with $p_g = 0$ have been of interest in the last 30 years

3.1 Bloch's conjecture

Another important problem concerning surfaces with $p_g = 0$ is related to the problem of rational equivalence of 0-cycles.

Recall that, for a nonsingular projective variety X, $A_0^i(X)$ is the group of rational equivalence classes of zero cycles of degree i.

Conjecture 3.1. Let S be a smooth surface with $p_g = 0$. Then the kernel $T(S)$ of the natural morphism (the so-called *Abel-Jacobi map*) $A_0^0(S) \to \mathrm{Alb}(S)$ is trivial.

By a beautiful result of D. Mumford ([Mum68]), the kernel of the Abel-Jacobi map is infinite dimensional for surfaces S with $p_g \neq 0$.

The conjecture has been proven for $\kappa(S) < 2$ by Bloch, Kas and Liebermann (cf. [BKL76]). If instead S is of general type, then $q(S) = 0$, whence Bloch's conjecture asserts for those surfaces that $A_0(S) \cong \mathbb{Z}$.

Inspite of the efforts of many authors, there are only few cases of surfaces of general type for which Bloch's conjecture has been verified (cf. e.g. [IM79], [Bar85b], [Keu88], [Voi92]).

Recently S. Kimura introduced the following notion of *finite dimensionality* of motives ([Kim05]).

Definition 3.2. Let M be a motive.

Then M is *evenly finite dimensional* if there is a natural number $n \geq 1$ such that $\wedge^n M = 0$.

M is *oddly finite dimensional* if there is a natural number $n \geq 1$ such that $\mathrm{Sym}^n M = 0$.

And, finally, M is *finite dimensional* if $M = M^+ \oplus M^-$, where M^+ is evenly finite dimensional and M^- is oddly finite dimensional.

Using this notation, he proves the following

Theorem 3.3. *1) The motive of a smooth projective curve is finite dimensional ([Kim05], cor. 4.4.).*

2) The product of finite dimensional motives is finite dimensional (loc. cit., cor. 5.11.).

3) Let $f: M \to N$ be a surjective morphism of motives, and assume that M is finite dimensional. Then N is finite dimensional (loc. cit., prop. 6.9.).

4) Let S be a surface with $p_g = 0$ and suppose that the Chow motive of X is finite dimensional. Then $T(S) = 0$ (loc.cit., cor. 7.7.).

Using the above results we obtain

Theorem 3.4. *Let S be the minimal model of a product-quotient surface (i.e., birational to $(C_1 \times C_2)/G$, where G is a finite group acting effectively on a product of two compact Riemann surfaces of respective genera $g_i \geq 2$) with $p_g = 0$.*

Then Bloch's conjecture holds for S, namely, $A_0(S) \cong \mathbb{Z}$.

Proof. Let S be the minimal model of $X = (C_1 \times C_2)/G$. Since X has rational singularities $T(X) = T(S)$.

By thm. 3.3, 2), 3) we have that the motive of X is finite dimensional, whence, by 4), $T(S) = T(X) = 0$.

Since S is of general type we have also $q(S) = 0$, hence $A_0^0(S) = T(S) = 0$.

Corollary 3.5. *All the surfaces in table 2, 3, and all the surfaces in [BC04], [BCG08] satisfy Bloch's conjecture.*

3.2 Pluricanonical maps

A further motivation for the study of surfaces with $p_g = 0$ comes from the behavior of the pluricanonical maps of surfaces of general type.

Definition 3.6. The n-th pluricanonical map

$$\varphi_n := \varphi_{|nK_S|} : S \dashrightarrow \mathbb{P}^{P_n - 1}$$

is the rational map associated to $H^0(\mathscr{O}_S(nK_S))$.

We recall that for a curve of general type φ_n is an embedding as soon as $n \geq 3$, and also for $n = 2$, if the curve is not of genus 2. The situation in dimension 2 is much more complicated. We recall:

Definition 3.7. The canonical model of a surface of general type is the normal surface

$$X := Proj(\bigoplus_{n=0}^{\infty} H^0(\mathscr{O}_S(nK_S))),$$

the projective spectrum of the (finitely generated) canonical ring.

X is obtained from its minimal model S by contracting all the curves C with $K_S \cdot C = 0$, i.e., all the smooth rational curves with self intersection equal to -2.

The n-th pluricanonical map $\varphi_{|nK_S|}$ of a surface of general type is the composition of the projection onto its canonical model X with $\psi_n := \varphi_{|nK_X|}$. So it suffices to study this last map.

This was done by Bombieri, whose results were later improved by the work of several authors. We summarize these efforts in the following theorem.

Theorem 3.8 ([Bom73], [Miy76], [BC78], [Cat77], [Reider88], [Fran88], [CC88], [CFHR99]).
Let X be the canonical model of a surface of general type. Then

i) $\varphi_{|nK_X|}$ is an embedding for all $n \geq 5$;
ii) $\varphi_{|4K_X|}$ is an embedding if $K_X^2 \geq 2$;
iii) $\varphi_{|3K_X|}$ is a morphism if $K_X^2 \geq 2$ and an embedding if $K_X^2 \geq 3$;
iv) $\varphi_{|nK_X|}$ is birational for all $n \geq 3$ unless
a) either $K^2 = 1$, $p_g = 2$, $n = 3$ or 4.
In this case X is a hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,1,2,5)$, a finite double cover of the quadric cone $Y := \mathbb{P}(1,1,2)$, $\varphi_{|3K_X|}(X)$ is birational to Y and isomorphic to an embedding of the surface $\mathbb{F}_2$ in $\mathbb{P}^5$, while $\varphi_{|4K_X|}(X)$ is an embedding of Y in $\mathbb{P}^8$.
b) Or $K^2 = 2$, $p_g = 3$, $n = 3$ (in this case X is a double cover of $\mathbb{P}^2$ branched on a curve of degree 8, and $\varphi_{|3K_X|}(X)$ is the image of the Veronese embedding $\nu_3 : \mathbb{P}^2 \to \mathbb{P}^9$).
v) $\varphi_{|2K_X|}$ is a morphism if $K_X^2 \geq 5$ or if $p_g \neq 0$.

vi) If $K_X^2 \geq 10$ then $\varphi_{|2K_X|}$ is birational if and only if X does not admit a morphism onto a curve with general fibre of genus 2.

The surfaces with $p_g = 0$ arose as the difficult case for the understanding of the tricanonical map, because, in the first version of his theorem, Bombieri could not determine whether the tricanonical and quadricanonical map of the numerical Godeaux and of the numerical Campedelli surfaces had to be birational. This was later proved in [Miy76], in [BC78], and in [Cat77].

It was already known to Kodaira that a morphism onto a smooth curve with general fibre of genus 2 forces the bicanonical map to factor through the hyperelliptic involution of the fibres: this is called the *standard case* for the nonbirationality of the bicanonical map. Part vi) of Theorem 3.8 shows that there are finitely many families of surfaces of general type with bicanonical map nonbirational which do not present the standard case. These interesting families have been classified under the hypothesis $p_g > 1$ or $p_g = 1$, $q \neq 1$: see [BCP06] for a more precise account on this results.

Again, the surfaces with $p_g = 0$ are the most difficult and hence the most interesting, since there are ”pathologies” which can happen only for surfaces with $p_g = 0$.

For example, the bicanonical system of a numerical Godeaux surface is a pencil, and therefore maps the surface onto $\mathbb{P}^1$, while [Xia85b] showed that the bicanonical map of every other surface of general type has a two dimensional image. Moreover, obviously for a numerical Godeaux surface $\varphi_{|2K_X|}$ is not a morphism, thus showing that the condition $p_g \neq 0$ in the point v) of the Theorem 3.8 is sharp.

Recently, Pardini and Mendes Lopes (cf. [MP08]) showed that there are more examples of surfaces whose bicanonical map is not a morphism, constructing two families of numerical Campedelli surfaces whose bicanonical system has two base points.

What it is known on the degree of the bicanonical map of surfaces with $p_g = 0$ can be summarized in the following

Theorem 3.9 ([MP07a],[MLP02], [MP08]). *Let S be a surface with $p_g = q = 0$. Then*

- *if* $K_S^2 = 9 \Rightarrow \deg \varphi_{|2K_S|} = 1$,
- *if* $K_S^2 = 7, 8 \Rightarrow \deg \varphi_{|2K_S|} = 1$ *or* 2,
- *if* $K_S^2 = 5, 6 \Rightarrow \deg \varphi_{|2K_S|} = 1$, 2 *or* 4,
- *if* $K_S^2 = 3, 4 \Rightarrow \deg \varphi_{|2K_S|} \leq 5$*; if moreover* $\varphi_{|2K_S|}$ *is a morphism, then* $\deg \varphi_{|2K_S|} = 1$, 2 *or* 4,
- *if* $K_S^2 = 2$ *(since the image of the bicanonical map is* $\mathbb{P}^2$*, the bicanonical map is non birational), then* $\deg \varphi_{|2K_S|} \leq 8$*. In the known examples it has degree* 6 *(and the bicanonical system has two base points) or* 8 *(and the bicanonical system has no base points).*

3.3 Differential topology

The surfaces with $p_g = 0$ are very interesting also from the point of view of differential topology, in particular in the simply connected case. We recall Freedman's theorem.

Theorem 3.10 ([Fre82]). *Let M be an oriented, closed, simply connected topological manifold: then M is determined (up to homeomorphism) by its intersection form*

$$q\colon H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$$

and by the Kirby-Siebenmann invariant $\alpha(M) \in \mathbb{Z}_2$, which vanishes if and only if $M \times S^1$ admits a differentiable structure.

If M is a complex surface, the Kirby-Siebenmann invariant automatically vanishes and therefore the oriented homeomorphism type of M is determined by the intersection form.

Combining it with a basic result of Serre on indefinite unimodular forms, and since by [Yau77] the only simply connected compact complex surface whose intersection form is definite is $\mathbb{P}^2$ one concludes

Corollary 3.11. *The oriented homeomorphism type of any simply connected complex surface is determined by the rank, the index and the parity of the intersection form.*

This gives a rather easy criterion to decide whether two complex surfaces are orientedly homeomorphic; anyway two orientedly homeomorphic complex surfaces are not necessarily diffeomorphic.

In fact, Dolgachev surfaces ([Dol77], see also [BHPV04, IX.5]) give examples of infinitely many surfaces which are all orientedly homeomorphic, but pairwise not diffeomorphic; these are elliptic surfaces with $p_g = q = 0$.

As mentioned, every compact complex surface homeomorphic to $\mathbb{P}^2$ is diffeomorphic (in fact, algebraically isomorphic) to $\mathbb{P}^2$ (cf. [Yau77]), so one can ask a similar question (cf. e.g. Hirzebruch's question 2.12): if a surface is homeomorphic to a rational surface, is it also diffeomorphic to it?

Simply connected surfaces of general type with $p_g = 0$ give a negative answer to this question. Indeed, by Freedman's theorem each simply connected minimal surface S of general type with $p_g = 0$ is orientedly homeomorphic to a Del Pezzo surface of degree K_S^2. Still these surfaces are not diffeomorphic to a Del Pezzo surface because of the following

Theorem 3.12 ([FQ94]). *Let S be a surface of general type. Then S is not diffeomorphic to a rational surface.*

The first simply connected surface of general type with $p_g = 0$ was constructed by R. Barlow in the 80's, and more examples have been constructed recently by Y. Lee, J. Park, H. Park and D. Shin. We summarize their results in the following

Theorem 3.13 ([Bar85a], [LP07], [PPS09a], [PPS09b]). $\forall 1 \leq y \leq 4$ *there are minimal simply connected surfaces of general type with $p_g = 0$ and $K^2 = y$.*

4 Construction techniques

As already mentioned, a first step towards a classification is the construction of examples. Here is a short list of different methods for constructing surfaces of general type with $p_g = 0$.

4.1 Quotients by a finite (resp. : infinite) group

4.1.1 Ball quotients

By the Bogomolov-Miyaoka-Yau theorem, a surface of general type with $p_g = 0$ is uniformized by the two dimensional complex ball $\mathbb{B}_2$ if and only if $K_S^2 = 9$. These surfaces are classically called *fake projective planes*, since they have the same Betti numbers as the projective plane $\mathbb{P}^2$.

The first example of a fake projective plane was constructed by Mumford (cf. [Mum79]), and later very few other examples were given (cf.[IK98], [Keu06]).

Ball quotients $S = \mathbb{B}_2/\Gamma$, where $\Gamma \leq PSU(2,1)$ is a discrete, cocompact, torsionfree subgroup are strongly rigid surfaces in view of Mostow's rigidity theorem ([Mos73]).

In particular the moduli space $\mathfrak{M}_{(1,9)}$ consists of a finite number of isolated points.

The possibility of obtaining a complete list of these fake planes seemed rather unrealistic until a breakthrough came in 2003: a surprising result by Klingler (cf. [Kli03]) showed that the cocompact, discrete, torsionfree subgroups $\Gamma \leq PSU(2,1)$ having minimal Betti numbers, i.e., yielding fake planes, are indeed arithmetic.

This allowed a complete classification of these surfaces carried out by Prasad and Yeung, Steger and Cartright ([PY07], [PY09]): the moduli space contains exactly 100 points, corresponding to 50 pairs of complex conjugate surfaces.

4.1.2 Product quotient surfaces

In a series of papers the following construction was explored systematically by the authors with the help of the computer algebra program MAGMA (cf. [BC04], [BCG08], [BCGP08], [BP10]).

Let C_1, C_2 be two compact curves of respective genera $g_1, g_2 \geq 2$. Assume further that G is a finite group acting effectively on $C_1 \times C_2$.

In the case where the action of G is free, the quotient surface is minimal of general type and is said to be *isogenous to a product* (see [Cat00]).

If the action is not free we consider the minimal resolution of singularities S' of the normal surface $X := (C_1 \times C_2)/G$ and its minimal model S. The aim is to give a complete classification of those S obtained as above which are of general type and have $p_g = 0$.

One observes that, if the tangent action of the stabilizers is contained in $SL(2,\mathbb{C})$, then X has Rational Double Points as singularities and is the canonical model of a surface of general type. In this case S' is minimal.

Recall the definition of an orbifold surface group (here the word ‘surface’ stands for ‘Riemann surface’):

Definition 4.1. An *orbifold surface group* of genus g' and multiplicities $m_1,\dots m_r \in \mathbb{N}_{\geq 2}$ is the group presented as follows:

$$\mathbb{T}(g';m_1,\dots,m_r) := \langle a_1,b_1,\dots,a_{g'},b_{g'},\, c_1,\dots,c_r| \\ c_1^{m_1},\dots,c_r^{m_r},\prod_{i=1}^{g'}[a_i,b_i]\cdot c_1\cdot\dots\cdot c_r\rangle.$$

The sequence $(g';m_1,\dots m_r)$ is called the *signature* of the orbifold surface group.

Moreover, recall the following special case of *Riemann’s existence theorem*:

Theorem 4.2. *A finite group G acts as a group of automorphisms on a compact Riemann surface C of genus g if and only if there are natural numbers* $g',m_1,\dots,m_r$, *and an ‘appropriate’ orbifold homomorphism*

$$\varphi\colon\ \mathbb{T}(g';m_1,\dots,m_r)\to G$$

such that the Riemann - Hurwitz relation holds:

$$2g-2=|G|\left(2g'-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right).$$

”Appropriate” means that φ *is surjective and moreover that the image* $\gamma_i\in G$ *of a generator* c_i *has order exactly equal to* m_i *(the order of* c_i *in* $\mathbb{T}(g';m_1,\dots,m_r)$*).*

In the above situation g' is the genus of $C' := C/G$. The G-cover $C\to C'$ is branched in r points $p_1,\dots,p_r$ with branching indices $m_1,\dots,m_r$, respectively.

Denote as before $\varphi(c_i)$ by $\gamma_i\in G$ the image of c_i under φ: then the set of stabilizers for the action of G on C is the set

$$\Sigma(\gamma_1,\dots,\gamma_r) := \cup_{a\in G}\cup_{i=0}^{max\{m_i\}}\{a\gamma_1^i a^{-1},\dots a\gamma_r^i a^{-1}\}.$$

Assume now that there are two epimorphisms

$$\varphi_1\colon\ \mathbb{T}(g_1';m_1,\dots,m_r)\to G,$$

$$\varphi_2\colon\ \mathbb{T}(g_2';n_1,\dots,n_s)\to G,$$

determined by two Galois covers $\lambda_i\colon C_i\to C_i'$, $i=1,2$.

We will assume in the following that $g(C_1),\ g(C_2) \geq 2$, and we shall consider the diagonal action of G on $C_1 \times C_2$.

We shall say in this situation that the action of G on $C_1 \times C_2$ is of *unmixed* type (indeed, see [Cat00], there is always a subgroup of G of index at most 2 with an action of unmixed type).

Theorem 4.3 ([BC04], [BCG05] [BCGP08],[BP10]).
1) Surfaces S isogenous to a product with $p_g(S) = q(S) = 0$ form 17 irreducible connected components of the moduli space $\mathfrak{M}^{can}_{(1,8)}$.

2) Surfaces with $p_g = 0$, whose canonical model is a singular quotient $X := (C_1 \times C_2)/G$ by an unmixed action of G form 27 further irreducible families.

3) Minimal surfaces with $p_g = 0$ which are the minimal resolution of the singularities of $X := C_1 \times C_2/G$ such that the action is of unmixed type and X does not have canonical singularities form exactly further 32 irreducible families.

Moreover, $K_S^2 = 8$ if and only if S is isogenous to a product.

We summarize the above results in tables 2 and 3.

Remark 4.4. 1) Recall that, if a diagonal action of G on $C_1 \times C_2$ is not free, then G has a finite set of fixed points. The quotient surface $X := (C_1 \times C_2)/G$ has a finite number of singular points. These can be easily found by looking at the given description of the stabilizers for the action of G on each individual curve.

Assume that $x \in X$ is a singular point. Then it is a cyclic quotient singularity of type $\frac{1}{n}(1,a)$ with $g.c.d(a,n) = 1$, i.e., X is, locally around x, biholomorphic to the quotient of $\mathbb{C}^2$ by the action of a diagonal linear automorphism with eigenvalues $\exp(\frac{2\pi i}{n})$, $\exp(\frac{2\pi i a}{n})$. That $g.c.d(a,n) = 1$ follows since the tangent representation is faithful on both factors.

2) We denote by K_X the canonical (Weil) divisor on the normal surface corresponding to $i_*(\Omega^2_{X^0})$, $i\colon X^0 \to X$ being the inclusion of the smooth locus of X. According to Mumford we have an intersection product with values in $\mathbb{Q}$ for Weil divisors on a normal surface, and in particular we consider the selfintersection of the canonical divisor,

$$K_X^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} \in \mathbb{Q}, \tag{1}$$

which is not necessarily an integer.

K_X^2 is however an integer (equal indeed to K_S^2) if X has only RDP's as singularities.

3) The resolution of a cyclic quotient singularity of type $\frac{1}{n}(1,a)$ with $g.c.d(a,n) = 1$ is well known. These singularities are resolved by the so-called *Hirzebruch-Jung strings*. More precisely, let $\pi\colon S \to X$ be a minimal resolution of the singularities and let $E = \bigcup_{i=1}^m E_i = \pi^{-1}(x)$. Then E_i is a smooth rational curve with $E_i^2 = -b_i$ and $E_i \cdot E_j = 0$ if $|i-j| \geq 2$, while$E_i \cdot E_{i+1} = 1$ for $i \in \{1,\dots,m-1\}$.

The b_i's are given by the continued fraction

Table 2 Surfaces isogenous to a product and minimal standard isotrivial fibrations with $p_g = 0$, $K^2 \geq 4$

K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
8	$\emptyset$	$2,5^2$	3^4	$\mathfrak{A}_5$	1	$\mathbb{Z}_3^2 \times \mathbb{Z}_{15}$	$1 \to \Pi_{21} \times \Pi_4 \to \pi_1 \to G \to 1$
8	$\emptyset$	5^3	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_{10}^2$	$1 \to \Pi_6 \times \Pi_{13} \to \pi_1 \to G \to 1$
8	$\emptyset$	$3^2,5$	2^5	$\mathfrak{A}_5$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_6$	$1 \to \Pi_{16} \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2,4,6$	2^6	$\mathfrak{S}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	$1 \to \Pi_{25} \times \Pi_3 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2^2,4^2$	$2^3,4$	$\mathrm{G}(32,27)$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$	$1 \to \Pi_5 \times \Pi_9 \to \pi_1 \to G \to 1$
8	$\emptyset$	5^3	5^3	$\mathbb{Z}_5^2$	2	$\mathbb{Z}_5^2$	$1 \to \Pi_6 \times \Pi_6 \to \pi_1 \to G \to 1$
8	$\emptyset$	$3,4^2$	2^6	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^4 \times \mathbb{Z}_8$	$1 \to \Pi_{13} \times \Pi_3 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2^2,4^2$	$2^2,4^2$	$\mathrm{G}(16,3)$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$	$1 \to \Pi_5 \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2^3,4$	2^6	$\mathrm{D}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$	$1 \to \Pi_9 \times \Pi_3 \to \pi_1 \to G \to 1$
8	$\emptyset$	2^5	2^5	$\mathbb{Z}_2^4$	1	$\mathbb{Z}_2^4$	$1 \to \Pi_5 \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	3^4	3^4	$\mathbb{Z}_3^2$	1	$\mathbb{Z}_3^4$	$1 \to \Pi_4 \times \Pi_4 \to \pi_1 \to G \to 1$
8	$\emptyset$	2^5	2^6	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^6$	$1 \to \Pi_3 \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	mixed		$\mathrm{G}(256,3678)$	3		
8	$\emptyset$	mixed		$\mathrm{G}(256,3679)$	1		
8	$\emptyset$	mixed		$\mathrm{G}(64,92)$	1		
6	$1/2^2$	$2^3,4$	$2^4,4$	$\mathbb{Z}_2 \times D_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$1 \to \mathbb{Z}^2 \times \Pi_2 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
6	$1/2^2$	$2^4,4$	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \Pi_2 \to \pi_1 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1$
6	$1/2^2$	$2,5^2$	$2,3^3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_3 \times \mathbb{Z}_{15}$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_{15}$
6	$1/2^2$	$2,4,10$	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathfrak{S}_3 \times D_{4,5,-1}$
6	$1/2^2$	$2,7^2$	$3^2,4$	PSL(2,7)	2	$\mathbb{Z}_{21}$	$\mathbb{Z}_7 \times \mathfrak{A}_4$
6	$1/2^2$	$2,5^2$	$3^2,4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_{15}$	$\mathbb{Z}_5 \times \mathfrak{A}_4$
5	$1/3,2/3$	$2,4,6$	$2^4,3$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^2 \to \pi_1 \to D_{2,8,3} \to 1$
5	$1/3,2/3$	$2^4,3$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$
5	$1/3,2/3$	$4^2,6$	$2^3,3$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$
5	$1/3,2/3$	$2,5,6$	$3,4^2$	$\mathfrak{S}_5$	1	$\mathbb{Z}_8$	$D_{8,5,-1}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_5 \times Q_8$
5	$1/3,2/3$	$2^3,3$	$3,4^2$	$\mathbb{Z}_2^4 \rtimes \mathfrak{S}_3$	1	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$D_{8,4,3}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$
4	$1/2^4$	2^5	2^5	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
4	$1/2^4$	$2^2,4^2$	$2^2,4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
4	$1/2^4$	2^5	$2^3,4$	$\mathbb{Z}_2 \times D_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1$
4	$1/2^4$	$3,6^2$	$2^2,3^2$	$\mathbb{Z}_3 \times \mathfrak{S}_3$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$
4	$1/2^4$	$3,6^2$	$2,4,5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$
4	$1/2^4$	2^5	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$
4	$1/2^4$	$2^2,4^2$	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$
4	$1/2^4$	2^5	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$
4	$1/2^4$	$2^3,4$	$2^3,4$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	1	$\mathbb{Z}_4^2$	$G(32,2)$
4	$1/2^4$	$2,5^2$	$2^2,3^2$	$\mathfrak{A}_5$	1	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$
4	$1/2^4$	$2^2,3^2$	$2^2,3^2$	$\mathbb{Z}_3^2 \rtimes Z_2$	1	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$
4	$2/5^2$	$2^3,5$	$3^2,5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
4	$2/5^2$	$2,4,5$	$4^2,5$	$\mathbb{Z}_2^4 \rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
4	$2/5^2$	$2,4,5$	$3^2,5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$

Table 3 Minimal standard isotrivial fibrations with $p_g = 0$, $K^2 \leq 3$

K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
3	$1/5, 4/5$	$2^3, 5$	$3^2, 5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
3	$1/5, 4/5$	$2, 4, 5$	$4^2, 5$	$\mathbb{Z}_2^4 \rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
3	$1/3, 1/2^2, 2/3$	$2^2, 3, 4$	$2, 4, 6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
3	$1/5, 4/5$	$2, 4, 5$	$3^2, 5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$
2	$1/3^2, 2/3^2$	$2, 6^2$	$2^2, 3^2$	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3$	1	$\mathbb{Z}_2^2$	Q_8
2	$1/2^6$	4^3	4^3	$\mathbb{Z}_4^2$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
2	$1/2^6$	$2^3, 4$	$2^3, 4$	$\mathbb{Z}_2 \times D_4$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
2	$1/3^2, 2/3^2$	$2^2, 3^2$	$3, 4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_8$	$\mathbb{Z}_8$
2	$1/3^2, 2/3^2$	$3^2, 5$	$3^2, 5$	$\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$	2	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2, 5^2$	$2^3, 3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2^3, 4$	$2, 4, 6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/3^2, 2/3^2$	$3^2, 5$	$2^3, 3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2, 3, 7$	4^3	PSL(2,7)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2, 6^2$	$2^3, 3$	$\mathfrak{S}_3 \times \mathfrak{S}_3$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/2^6$	$2, 6^2$	$2, 4, 5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4, 1/2^2, 3/4$	$2, 4, 7$	$3^2, 4$	PSL(2,7)	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4, 1/2^2, 3/4$	$2, 4, 5$	$3^2, 4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4, 1/2^2, 3/4$	$2, 4, 6$	$2, 4, 5$	$\mathfrak{S}_5$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
1	$1/3, 1/2^4, 2/3$	$2^3, 3$	$3, 4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$
1	$1/3, 1/2^4, 2/3$	$2, 3, 7$	$3, 4^2$	PSL(2,7)	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^3, 3$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$

$$\frac{n}{a} = b_1 - \cfrac{1}{b_2 - \frac{1}{b_3 - \dots}}.$$

Since the minimal resolution $S' \to X$ of the singularities of X replaces each singular point by a tree of smooth rational curves, we have, by van Kampen's theorem, that $\pi_1(X) = \pi_1(S') = \pi_1(S)$.

Moreover, we can read off all invariants of S' from the group theoretical data. For details and explicit formulae we refer to [BP10].

Among others, we also prove the following lemma:

Lemma 4.5. *There exist positive numbers D, M, R, B, which depend explicitly (and only) on the singularities of X such that:*

1. $\chi(S') = 1 \implies K_{S'}^2 = 8 - B$;
2. *for the corresponding signatures* $(0; m_1, \dots, m_r)$ *and* $(0; n_1, \dots, n_s)$ *of the orbifold surface groups we have* $r, s \leq R$, $\forall\ i\ m_i, n_i \leq M$;
3. $|G| = \frac{K_{S'} + D}{2(-2+\sum_1^r (1-\frac{1}{m_i}))(-2+\sum_1^s (1-\frac{1}{n_i}))}$.

Remark 4.6. The above lemma 4.5 implies that there is an algorithm which computes all such surfaces S' with $p_g = q = 0$ and fixed $K_{S'}^2$:

a) find all possible configurations (= ”baskets”) $\mathscr{B}$ of singularities with $B = 8 - K_{S'}^2$;
b) for a fixed basket $\mathscr{B}$ find all signatures $(0; m_1, \dots, m_r)$ satisfying 2);
c) for each pair of signatures check all groups G of order given by 3), whether there are surjective homomorphisms $\mathbb{T}(0; m_i) \to G$, $\mathbb{T}(0; n_i) \to G$;
d) check whether the surfaces $X = (C_1 \times C_2)/G$ thus obtained have the right singularities.

Still this is not yet the solution of the problem and there are still several difficult problems to be overcome:

- We have to check whether the groups of a given order admit certain systems of generators of prescribed orders, and satisfying moreover certain further conditions (forced by the basket of singularities); we encounter in this way groups of orders 512, 1024, 1536: there are so many groups of these orders that the above investigation is not feasible for naive computer calculations. Moreover, we have to deal with groups of orders > 2000: they are not listed in any database
- If X is singular, we only get subfamilies, not a whole irreducible component of the moduli space. There remains the problem of studying the deformations of the minimal models S obtained with the above construction.
- The algorithm is heavy for K^2 small. In [BP10] we proved and implemented much stronger results on the singularities of X and on the possible signatures, which allowed us to obtain a complete list of surfaces with $K_S^2 \geq 1$.
- We have not yet answered completely the original question. Since, if X does not have canonical singularities, it may happen that $K_{S'}^2 \leq 0$ (recall that S' is the minimal resolution of singularities of X, which is not necessarily minimal!).

Concerning product quotient surfaces, we have proven (in a much more general setting, cf. [BCGP08]) a structure theorem for the fundamental group, which helps us to explicitly identify the fundamental groups of the surfaces we constructed. In fact, it is not difficult to obtain a presentation for these fundamental groups, but as usual having a presentation is not sufficient to determine the group explicitly.

We first need the following

Definition 4.7. We shall call the fundamental group $\Pi_g := \pi_1(C)$ of a smooth compact complex curve of genus g a *(genus g) surface group*.

Note that we admit also the “degenerate cases” $g = 0, 1$.

Theorem 4.8. *Let $C_1, \dots, C_n$ be compact complex curves of respective genera $g_i \geq 2$ and let G be a finite group acting faithfully on each C_i as a group of biholomorphic transformations.*

Let $X = (C_1 \times \dots \times C_n)/G$, and denote by S a minimal desingularisation of X. Then the fundamental group $\pi_1(X) \cong \pi_1(S)$ has a normal subgroup $\mathscr{N}$ of finite index which is isomorphic to the product of surface groups, i.e., there are natural numbers $h_1, \dots, h_n \geq 0$ such that $\mathscr{N} \cong \Pi_{h_1} \times \dots \times \Pi_{h_n}$.

Remark 4.9. In the case of dimension $n = 2$ there is no loss of generality in assuming that G acts faithfully on each C_i (see [Cat00]). In the general case there will be a group G_i, quotient of G, acting faithfully on C_i, hence the strategy has to be slightly changed in the general case. The generalization of the above theorem, where the assumption that G acts faithfully on each factor is removed, has been proven in [DP10].

We shall now give a short outline of the proof of theorem 4.8 in the case $n = 2$ (the case of arbitrary n is exactly the same).

We have two appropriate orbifold homomorphisms

$$\varphi_1 \colon \mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \dots, m_r) \to G,$$

$$\varphi_2 \colon \mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \dots, n_s) \to G.$$

We define the fibre product $\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2)$ as

$$\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2) := \{ (x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid \varphi_1(x) = \varphi_2(y) \}. \tag{2}$$

Then the exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{T}_1 \times \mathbb{T}_2 \to G \times G \to 1, \tag{3}$$

where $\Pi_{g_i} := \pi_1(C_i)$, induces an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{H}(G; \varphi_1, \varphi_2) \to G \cong \Delta_G \to 1. \tag{4}$$

Here $\Delta_G \subset G \times G$ denotes the diagonal subgroup.

Definition 4.10. Let H be a group. Then its *torsion subgroup* $\mathrm{Tors}(H)$ is the normal subgroup generated by all elements of finite order in H.

The first observation is that one can calculate our fundamental groups via a simple algebraic recipe:

$$\pi_1((C_1 \times C_2)/G) \cong \mathbb{H}(G; \varphi_1, \varphi_2)/\mathrm{Tors}(\mathbb{H}).$$

The strategy is then the following: using the structure of orbifold surface groups we construct an exact sequence

$$1 \to E \to \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to \Psi(\hat{\mathbb{H}}) \to 1,$$

where

i) E is finite,
ii) $\Psi(\hat{\mathbb{H}})$ is a subgroup of finite index in a product of orbifold surface groups.

Condition $ii)$ implies that $\Psi(\hat{\mathbb{H}})$ is residually finite and “good” according to the following

Definition 4.11 (J.-P. Serre). Let $\mathbb{G}$ be a group, and let $\tilde{\mathbb{G}}$ be its profinite completion. Then $\mathbb{G}$ is said to be *good* iff the homomorphism of cohomology groups

$$H^k(\tilde{\mathbb{G}}, M) \to H^k(\mathbb{G}, M)$$

is an isomorphism for all $k \in \mathbb{N}$ and for all finite $\mathbb{G}$ - modules M.

Then we use the following result due to F. Grunewald, A. Jaikin-Zapirain, P. Zalesski.

Theorem 4.12. ([GJZ08]) *Let G be residually finite and good, and let $\varphi\colon H \to G$ be surjective with finite kernel. Then H is residually finite.*

The above theorem implies that $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$ is residually finite, whence there is a subgroup $\Gamma \leq \mathbb{H}/\mathrm{Tors}(\mathbb{H})$ of finite index such that

$$\Gamma \cap E = \{1\}.$$

Now, $\Psi(\Gamma)$ is a subgroup of $\Psi(\hat{\mathbb{H}})$ of finite index, whence of finite index in a product of orbifold surface groups, and $\Psi|\Gamma$ is injective. This easily implies our result.

Remark 4.13. Note that theorem 4.8 in fact yields a geometric statement in the case where the genera of the surface groups are at least 2. Again, for simplicity, we assume that $n = 2$, and suppose that $\pi_1(S)$ has a normal subgroup $\mathscr{N}$ of finite index isomorphic to $\Pi_g \times \Pi_{g'}$, with $g, g' \geq 2$. Then there is an unramified Galois covering $\hat{S}$ of S such that $\pi_1(\hat{S}) \cong \Pi_g \times \Pi_{g'}$. This implies (see [Cat00]) that there is a finite morphism $\hat{S} \to C \times C'$, where $g(C) = g$, $g(C') = g'$.

Understanding this morphism can lead to the understanding of the irreducible or even of the connected component of the moduli space containing the isomorphism class $[S]$ of S. The method can also work in the case where we only have $g, g' \geq 1$. We shall explain how this method works in section 5.

We summarize the consequences of theorem 4.3 in terms of ”new” fundamental groups of surfaces with $p_g = 0$, respectively ”new” connected components of their moduli space.

Theorem 4.14. *There exist eight families of product-quotient surfaces of unmixed type yielding numerical Campedelli surfaces (i.e., minimal surfaces with $K_S^2 = 2, p_g(S) = 0$) having fundamental group $\mathbb{Z}/3$.*

Our classification also shows the existence of families of product-quotient surfaces yielding numerical Campedelli surfaces with fundamental groups $\mathbb{Z}/5$ (but numerical Campedelli surfaces with fundamental group $\mathbb{Z}/5$ had already been constructed in [Cat81]), respectively with fundamental group $(\mathbb{Z}/2)^2$ (but such fundamental group already appeared in [Ino94]), respectively with fundamental groups $(\mathbb{Z}/2)^3$, Q_8, $\mathbb{Z}/8$ and $\mathbb{Z}/2 \times \mathbb{Z}/4$.

Theorem 4.15. *There exist six families of product-quotient surfaces yielding minimal surfaces with* $K_S^2 = 3, p_g(S) = 0$ *realizing four new finite fundamental groups,* $\mathbb{Z}/2 \times \mathbb{Z}/6$, $\mathbb{Z}/8$, $\mathbb{Z}/6$ *and* $\mathbb{Z}/2 \times \mathbb{Z}/4$.

Theorem 4.16. *There exist sixteen families of product-quotient surfaces yielding minimal surfaces with* $K_S^2 = 4$, $p_g(S) = 0$. *Eight of these families realize 6 new finite fundamental groups,* $\mathbb{Z}/15$, $G(32,2)$, $(\mathbb{Z}/3)^3$, $\mathbb{Z}/2 \times \mathbb{Z}/6$, $\mathbb{Z}/8$, $\mathbb{Z}/6$. *Eight of these families realize 4 new infinite fundamental groups.*

Theorem 4.17. *There exist seven families of product-quotient surfaces yielding minimal surfaces with* $K_S^2 = 5$, $p_g(S) = 0$. *Four of these families realize four new finite fundamental groups,* $D_{8,5,-1}$, $\mathbb{Z}/5 \times Q_8$, $D_{8,4,3}$, $\mathbb{Z}/2 \times \mathbb{Z}/10$. *Three of these families realize three new infinite fundamental groups.*

Theorem 4.18. *There exist eight families of product-quotient surfaces yielding minimal surfaces with* $K_S^2 = 6$, $p_g(S) = 0$ *and realizing 6 new fundamental groups, three of them finite and three of them infinite. In particular, there exist minimal surfaces of general type with* $p_g = 0$, $K^2 = 6$ *and with finite fundamental group.*

4.2 Galois coverings and their deformations

Another standard method for constructing new algebraic surfaces is to consider abelian Galois-coverings of known surfaces.

We shall in the sequel recall the structure theorem on normal finite $\mathbb{Z}_2^r$-coverings, $r \geq 1$, of smooth algebraic surfaces Y. In fact (cf. [Par91], or [BC08] for a more topological approach) this theory holds more generally for any G-covering, with G a finite abelian group.

Since however we do not want here to dwell too much into the general theory and, in most of the applications we consider here only the case $\mathbb{Z}_2^2$ is used, we restrict ourselves to this more special situation.

We shall denote by $G := \mathbb{Z}_2^r$ the Galois group and by $G^* := \mathrm{Hom}(G, \mathbb{C}^*)$ its dual group of characters which we identify to $G^* := \mathrm{Hom}(G, \mathbb{Z}/2)$.

Since Y is smooth any finite abelian covering $f \colon X \to Y$ is flat hence in the eigensheaves splitting

$$f_* \mathscr{O}_X = \bigoplus_{\chi \in G^*} \mathscr{L}_\chi^* = \mathscr{O}_Y \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} \mathscr{O}_Y(-L_\chi).$$

each rank 1 sheaf $\mathscr{L}_\chi^*$ is invertible and corresponds to a Cartier divisor $-L_\chi$.

For each $\sigma \in G$ let $R_\sigma \subset X$ be the divisorial part of the fixed point set of σ. Then one associates to σ a divisor D_σ given by $f(R_\sigma) = D_\sigma$; let x_σ be a section such that $\mathrm{div}(x_\sigma) = D_\sigma$.

Then the algebra structure on $f_* \mathscr{O}_X$ is given by the following (symmetric, bilinear) multiplication maps:

$$\mathscr{O}_Y(-L_\chi) \otimes \mathscr{O}_Y(-L_\eta) \to \mathscr{O}_Y(-L_{\chi+\eta}),$$

given by the section $x_{\chi,\eta} \in H^0(Y, \mathscr{O}_Y(L_\chi + L_\eta - L_{\chi+\eta}))$, defined by

$$x_{\chi,\eta} := \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

It is now not difficult in this case to show directly the associativity of the multiplication defined above (cf. [Par05] for the general case of an abelian cover).

In particular, the G-covering $f\colon X \to Y$ is embedded in the vector bundle $\mathbb{V} := \bigoplus_{\chi \in G^*} \mathbb{L}_\chi$, where $\mathbb{L}_\chi$ is the geometric line bundle whose sheaf of sections is $\mathscr{O}_Y(L_\chi)$, and is there defined by the equations:

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

Note the special case where $\chi = \eta$, when $\chi + \eta$ is the trivial character 1, and $z_1 = 1$. In particular, let $\chi_1, \dots, \chi_r$ be a basis of $G^* = \mathbb{Z}_2^r$, and set $z_i := z_{\chi_i}$. Then we get the following r equations

$$z_i^2 = \prod_{\chi_i(\sigma)=1} x_\sigma. \tag{5}$$

These equations determine the extension of the function fields, hence one gets X as the normalization of the Galois covering given by (5). The main point however is that the previous formulae yield indeed the normalization explicitly under the conditions summarized in the following

Proposition 4.19. *A normal finite $G \cong \mathbb{Z}_2^r$-covering of a smooth variety Y is completely determined by the datum of*

1. *reduced effective divisors D_σ, $\forall \sigma \in G$, which have no common components,*
2. *divisor classes $L_1, \dots L_r$, for $\chi_1, \dots \chi_r$ a basis of G^*, such that we have the following linear equivalence*

(#) $2L_i \equiv \sum_{\chi_i(\sigma)=1} D_\sigma$.

Conversely, given the datum of 1) and 2) such that #) holds, we obtain a normal scheme X with a finite $G \cong \mathbb{Z}_2^r$-covering $f\colon X \to Y$.

Proof (Idea of the proof.). It suffices to determine the divisor classes L_χ for the remaining elements of G^*. But since any χ is a sum of basis elements, it suffices to exploit the fact that the linear equivalences

$$L_{\chi+\eta} \equiv L_\eta + L_\chi - \sum_{\chi(\sigma)=\eta(\sigma)=1} D_\sigma$$

must hold, and apply induction. Since the covering is well defined as the normalization of the Galois cover given by (5), each L_χ is well defined. Then the above formulae determine explicitly the ring structure of $f_*\mathscr{O}_X$, hence X. Finally, condition 1 implies the normality of the cover.

A natural question is of course: when is the scheme X a variety? I.e., X being normal, when is X connected, or, equivalently, irreducible? The obvious answer is that X is irreducible if and only if the monodromy homomorphism

$$\mu\colon H_1(Y\setminus(\cup_\sigma D_\sigma),\mathbb{Z})\to G$$

is surjective.

Remark 4.20. From the extension of Riemann's existence theorem due to Grauert and Remmert ([GR58]) we know that μ determines the covering. It is therefore worthwhile to see how μ is related to the datum of 1) and 2).

Write for this purpose the branch locus $D := \sum_\sigma D_\sigma$ as a sum of irreducible components D_i. To each D_i corresponds a simple geometric loop γ_i around D_i, and we set $\sigma_i := \mu(\gamma_i)$. Then we have that $D_\sigma := \sum_{\sigma_i=\sigma} D_i$. For each character χ, yielding a double covering associated to the composition $\chi\circ\mu$, we must find a divisor class L_χ such that $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$.

Consider the exact sequence

$$H^{2n-2}(Y,\mathbb{Z})\to H^{2n-2}(D,\mathbb{Z})=\oplus_i\mathbb{Z}[D_i]\to H_1(Y\setminus D,\mathbb{Z})\to H_1(Y,\mathbb{Z})\to 0$$

and the similar one with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$. Denote by Δ the subgroup image of $\oplus_i\mathbb{Z}_2[D_i]$. The restriction of μ to Δ is completely determined by the knowledge of the σ_i's, and we have

$$0\to\Delta\to H_1(Y\setminus D,\mathbb{Z}_2)\to H_1(Y,\mathbb{Z}_2)\to 0.$$

Dualizing, we get

$$0\to H^1(Y,\mathbb{Z}_2)\to H^1(Y\setminus D,\mathbb{Z}_2)\to \mathrm{Hom}(\Delta,\mathbb{Z}_2)\to 0.$$

The datum of $\chi\circ\mu$, extending $\chi\circ\mu|_\Delta$ is then seen to correspond to an affine space over the vector space $H^1(Y,\mathbb{Z}_2)$: and since $H^1(Y,\mathbb{Z}_2)$ classifies divisor classes of 2-torsion on Y, we infer that the different choices of L_χ such that $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$ correspond bijectively to all the possible choices for $\chi\circ\mu$.

Applying this to all characters, we find how μ determines the building data.

Observe on the other hand that if μ is not surjective, then there is a character χ vanishing on the image of μ, hence the corresponding double cover is disconnected.

But the above discussion shows that $\chi\circ\mu$ is trivial iff this covering is disconnected, if and only if the corresponding element in $H^1(Y\setminus D,\mathbb{Z}_2)$ is trivial, or, equivalently, iff the divisor class L_χ is trivial.

We infer then

Corollary 4.21. *Use the same notation as in prop. 4.19. Then the scheme X is irreducible if $\{\sigma|D_\sigma>0\}$ generates G.*

Or, more generally, if for each character χ the class in $H^1(Y\setminus D,\mathbb{Z}_2)$ corresponding to $\chi\circ\mu$ is nontrivial, or, equivalently, the divisor class L_χ is nontrivial.

Proof. We have seen that if $D_\sigma \geq D_i \neq 0$, then $\mu(\gamma_i) = \sigma$, whence we infer that μ is surjective.

An important role plays here once more the concept of *natural deformations*. This concept was introduced for bidouble covers in [Cat84], definition 2.8, and extended to the case of abelian covers in [Par91], definition 5.1. The two definitions do not exactly coincide, because Pardini takes a much larger parameter space: however, the deformations appearing with both definitions are the same. To avoid confusion we call Pardini's case the case of *extended natural deformations*.

Definition 4.22. Let $f\colon X \to Y$ be a finite $G \cong \mathbb{Z}_2^r$ covering with Y smooth and X normal, so that X is embedded in the vector bundle $\mathbb{V}$ defined above and is defined by equations

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma .$$

Let $\psi_{\sigma,\chi}$ be a section $\psi_{\sigma,\chi} \in H^0(Y, \mathscr{O}_Y(D_\sigma - L_\chi))$, given $\forall \sigma \in G, \chi \in G^*$. To such a collection we associate an *extended natural deformation*, namely, the subscheme of $\mathbb{V}$ defined by equations

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} \left(\sum_\theta \psi_{\sigma,\theta} \cdot z_\theta \right).$$

We have instead a (restricted) *natural deformation* if we restrict ourselves to the θ's such that $\theta(\sigma) = 0$,and we consider only an equation of the form

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} \left(\sum_{\theta(\sigma)=0} \psi_{\sigma,\theta} \cdot z_\theta \right).$$

One can generalize some results, even removing the assumption of smoothness of Y, if one assumes the $G \cong \mathbb{Z}_2^r$-covering to be *locally simple*, i.e., to enjoy the property that for each point $y \in Y$ the σ's such that $y \in D_\sigma$ are a linearly independent set. This is a good notion since (compare [Cat84], proposition 1.1) if also X is smooth the covering is indeed locally simple.

One has for instance the following result (see [Man01], section 3):

Proposition 4.23. *Let $f : X \to Y$ be a locally simple $G \cong \mathbb{Z}_2^r$ covering with Y smooth and X normal. Then we have the exact sequence*

$$\oplus_{\chi(\sigma)=0}(H^0(\mathscr{O}_{D_\sigma}(D_\sigma - L_\chi))) \to \mathrm{Ext}^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X) \to \mathrm{Ext}^1_{\mathscr{O}_X}(f^*\Omega^1_Y, \mathscr{O}_X).$$

In particular, every small deformation of X is a natural deformation if

1. $H^1(\mathscr{O}_Y(-L_\chi)) = 0$,
2. $\mathrm{Ext}^1_{\mathscr{O}_X}(f^*\Omega^1_Y, \mathscr{O}_X) = 0$.

If moreover

3. $H^0(\mathscr{O}_Y(D_\sigma - L_\chi)) = 0 \; \forall \sigma \in G, \chi \in G^*$,

every small deformation of X is again a $G \cong \mathbb{Z}_2^r$-covering.

Proof (Comments on the proof.).

In the above proposition condition 1) ensures that

$$H^0(\mathscr{O}_Y(D_\sigma - L_\chi)) \to H^0(\mathscr{O}_{D_\sigma}(D_\sigma - L_\chi))$$

is surjective.

Condition 2 and the above exact sequence imply then that the natural deformations are parametrized by a smooth manifold and have surjective Kodaira-Spencer map, whence they induce all the infinitesimal deformations.

Remark 4.24. In the following section we shall see examples where surfaces with $p_g = 0$ arise as double covers and as bidouble covers. In fact there are many more surfaces arising this way, see e.g. [Cat98].

5 Keum-Naie surfaces and primary Burniat surfaces

In the nineties J.H. Keum and D. Naie (cf. [Nai94], [Keu88]) constructed a family of surfaces with $K_S^2 = 4$ and $p_g = 0$ as double covers of an Enriques surface with eight nodes and calculated their fundamental group.

We want here to describe explicitly the moduli space of these surfaces.
The motivation for this investigation arose as follows: consider the following two cases of table 2 whose fundamental group has the form

$$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}_2^2 \to 0.$$

These cases yield 2 families of respective dimensions 2 and 4, which can also be seen as $\mathbb{Z}_4 \times \mathbb{Z}_2$, resp. $\mathbb{Z}_2^3$, coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ branched in a divisor of type $(4,4)$, resp. $(5,5)$, consisting entirely of horizontal and vertical lines. It turns out that their fundamental groups are isomorphic to the fundamental groups of the surfaces constructed by Keum-Naie.

A straightforward computation shows that our family of dimension 4 is equal to the family constructed by Keum, and that both families are subfamilies of the one constructed by Naie.

As a matter of fact each surface of our family of $\mathbb{Z}_2^3$ - coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ has 4 nodes. These nodes can be smoothened simultaneously in a 5 - dimensional family of $\mathbb{Z}_2^3$ - Galois coverings of $\mathbb{P}^1 \times \mathbb{P}^1$.

It suffices to take a smoothing of each D_i, which before the smoothing consisted of a vertical plus a horizontal line.The full six dimensional component is obtained then as the family of natural deformations of these Galois coverings.

It is a standard computation in local deformation theory to show that the six dimensional family of natural deformations of smooth $\mathbb{Z}_2^3$ - Galois coverings of $\mathbb{P}^1 \times$

$\mathbb{P}^1$ is an irreducible component of the moduli space. We will not give the details of this calculation, since we get a stronger result by another method.

In fact, the main result of [BC09a] is the following:

Theorem 5.1. *Let S be a smooth complex projective surface which is homotopically equivalent to a Keum-Naie surface. Then S is a Keum-Naie surface.*

The moduli space of Keum-Naie surfaces is irreducible, unirational of dimension equal to six. Moreover, the local moduli space of a Keum-Naie surface is smooth.

The proof resorts to a slightly different construction of Keum-Naie surfaces. We study a $\mathbb{Z}_2^2$-action on the product of two elliptic curves $E_1' \times E_2'$. This action has 16 fixed points and the quotient is an 8-nodal Enriques surface. Constructing S as a double cover of the Enriques surface is equivalent to constructing an étale $\mathbb{Z}_2^2$-covering $\hat{S}$ of S, whose existence can be inferred from the structure of the fundamental group, and which is obtained as a double cover of $E_1' \times E_2'$ branched in a $\mathbb{Z}_2^2$-invariant divisor of type $(4,4)$. Because $S = \hat{S}/\mathbb{Z}_2^2$.

The structure of this étale $\mathbb{Z}_2^2$-covering $\hat{S}$ of S is essentially encoded in the fundamental group $\pi_1(S)$, which can be described as an affine group $\Gamma \in \mathbb{A}(2,\mathbb{C})$. The key point is that the double cover $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$ is the Albanese map of $\hat{S}$.

Assume now that S' is homotopically equivalent to a Keum-Naie surface S. Then the corresponding étale cover $\hat{S'}$ is homotopically equivalent to $\hat{S}$. Since we know that the degree of the Albanese map of $\hat{S}$ is equal to two (by construction), we can conlude the same for the Albanese map of $\hat{S'}$ and this allows to deduce that also $\hat{S'}$ is a double cover of a product of elliptic curves.

A calculation of the invariants of a double cover shows that the branch locus is a $\mathbb{Z}_2^2$-invariant divisor of type $(4,4)$.

We are going to sketch the construction of Keum-Naie surfaces and the proof of theorem 5.1 in the sequel. For details we refer to the original article [BC09a].

Let (E,o) be any elliptic curve, with a $G = \mathbb{Z}_2^2 = \{0, g_1, g_2, g_1+g_2\}$ action given by

$$g_1(z) := z + \eta, \quad g_2(z) = -z.$$

Remark 5.2. Let $\eta \in E$ be a 2 - torsion point of E. Then the divisor $[o]+[\eta] \in Div^2(E)$ is invariant under G, hence the invertible sheaf $\mathscr{O}_E([o]+[\eta])$ carries a natural G-linearization.

In particular, G acts on $H^0(E, \mathscr{O}_E([o]+[\eta]))$, and for the character eigenspaces, we have the following:

Lemma 5.3. *Let E be as above, then:*

$$H^0(E,\mathscr{O}_E([o]+[\eta])) = H^0(E,\mathscr{O}_E([o]+[\eta]))^{++} \oplus H^0(E,\mathscr{O}_E([o]+[\eta]))^{--}.$$

I.e., $H^0(E,\mathscr{O}_E([o]+[\eta]))^{+-} = H^0(E,\mathscr{O}_E([o]+[\eta]))^{-+} = 0.$

Remark 5.4. Our notation is self explanatory, e.g.

$$H^0(E,\mathscr{O}_E([o]+[\eta]))^{+-} = H^0(E,\mathscr{O}_E([o]+[\eta]))^{\chi},$$

where χ is the character of G with $\chi(g_1) = 1$, $\chi(g_2) = -1$.

Let now $E_i' := \mathbb{C}/\Lambda_i$, $i = 1,2$, where $\Lambda_i := \mathbb{Z}e_i \oplus \mathbb{Z}e_i'$, be two complex elliptic curves. We consider the affine transformations $\gamma_1,\ \gamma_2 \in \mathbb{A}(2,\mathbb{C})$, defined as follows:

$$\gamma_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + \frac{e_1}{2} \\ -z_2 \end{pmatrix},\ \ \gamma_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_1 \\ z_2 + \frac{e_2}{2} \end{pmatrix},$$

and let $\Gamma \leq \mathbb{A}(2,\mathbb{C})$ be the affine group generated by γ_1, γ_2 and by the translations e_1, e_1', e_2, e_2'.

Remark 5.5. i) Γ induces a $G := \mathbb{Z}_2^2$-action on $E_1' \times E_2'$.
ii) While $\gamma_1,\ \gamma_2$ have no fixed points on $E_1' \times E_2'$, the involution $\gamma_1\gamma_2$ has 16 fixed points on $E_1' \times E_2'$. It is easy to see that the quotient $Y := (E_1' \times E_2')/G$ is an Enriques surface having 8 nodes, with canonical double cover the Kummer surface $(E_1' \times E_2')/ < \gamma_1\gamma_2 >$.

We lift the G-action on $E_1' \times E_2'$ to an appropriate ramified double cover $\hat{S}$ such that G acts freely on $\hat{S}$.

To do this, consider the following geometric line bundle $\mathbb{L}$ on $E_1' \times E_2'$, whose invertible sheaf of sections is given by:

$$\mathcal{O}_{E_1' \times E_2'}(\mathbb{L}) := p_1^*\mathcal{O}_{E_1'}([o_1] + [\frac{e_1}{2}]) \otimes p_2^*\mathcal{O}_{E_2'}([o_2] + [\frac{e_2}{2}]),$$

where $p_i : E_1' \times E_2' \to E_i'$ is the projection to the i-th factor.

By remark 5.2, the divisor $[o_i] + [\frac{e_i}{2}] \in \mathrm{Div}^2(E_i')$ is invariant under G. Therefore, we get a natural G-linearization on the two line bundles $\mathcal{O}_{E_i'}([o_i] + [\frac{e_i}{2}])$, whence also on $\mathbb{L}$.

Any two G-linearizations of $\mathbb{L}$ differ by a character $\chi : G \to \mathbb{C}^*$. We twist the above obtained linearization of $\mathbb{L}$ with the character χ such that $\chi(\gamma_1) = 1$, $\chi(\gamma_2) = -1$.

Definition 5.6. Let

$$f \in H^0(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]) \otimes p_2^*\mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^G$$

be a G - invariant section of $\mathbb{L}^{\otimes 2}$ and denote by w a fibre coordinate of $\mathbb{L}$. Let $\hat{S}$ be the double cover of $E_1' \times E_2'$ branched in f, i.e.,

$$\hat{S} = \{w^2 = f(z_1, z_2)\} \subset \mathbb{L}.$$

Then $\hat{S}$ is a G - invariant hypersurface in $\mathbb{L}$, and we have a G - action on $\hat{S}$.
We call $S := \hat{S}/G$ a *Keum - Naie surface*, if

- G acts freely on $\hat{S}$, and
- $\{f = 0\}$ has only *non-essential singularities*, i.e., $\hat{S}$ has at most rational double points.

Remark 5.7. If

$$f \in H^0(E_1' \times E_2', p_1^* \mathscr{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]) \otimes p_2^* \mathscr{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^G$$

is such that $\{(z_1, z_2) \in E_1' \times E_2' \mid f(z_1, z_2) = 0\} \cap \mathrm{Fix}(\gamma_1 + \gamma_2) = \emptyset$, then G acts freely on $\hat{S}$.

Proposition 5.8. *Let S be a Keum - Naie surface. Then S is a minimal surface of general type with*

i) $K_S^2 = 4$,
ii) $p_g(S) = q(S) = 0$,
iii) $\pi_1(S) = \Gamma$.

i) is obvious, since $K_{\hat{S}}^2 = 16$,
ii) is verified via standard arguments of representation theory.
iii) follows since $\pi_1(\hat{S}) = \pi_1(E_1' \times E_2')$.

Let now S be a smooth complex projective surface with $\pi_1(S) = \Gamma$. Recall that $\gamma_i^2 = e_i$ for $i = 1, 2$. Therefore $\Gamma = \langle \gamma_1, e_1', \gamma_2, e_2' \rangle$ and we have the exact sequence

$$1 \to \mathbb{Z}^4 \cong \langle e_1, e_1', e_2, e_2' \rangle \to \Gamma \to \mathbb{Z}_2^2 \to 1,$$

where $e_i \mapsto \gamma_i^2$.

We set $\Lambda_i' := \mathbb{Z} e_i \oplus \mathbb{Z} e_i'$, hence $\pi_1(E_1' \times E_2') = \Lambda_1' \oplus \Lambda_2'$. We also have the two lattices $\Lambda_i := \mathbb{Z} \frac{e_i}{2} \oplus \mathbb{Z} e_i'$.

Remark 5.9. 1) Γ is a group of affine transformations on $\Lambda_1 \oplus \Lambda_2$.
2) We have an étale double cover $E_i' = \mathbb{C}/\Lambda_i' \to E_i := \mathbb{C}/\Lambda_i$, which is the quotient by a semiperiod of E_i'.

Γ has two subgroups of index two:

$$\Gamma_1 := \langle \gamma_1, e_1', e_2, e_2' \rangle, \quad \Gamma_2 := \langle e_1, e_1', \gamma_2, e_2' \rangle,$$

corresponding to two étale covers of S: $S_i \to S$, for $i = 1, 2$.

Then one can show:

Lemma 5.10. *The Albanese variety of S_i is E_i. In particular, $q(S_1) = q(S_2) = 1$.*

Let $\hat{S} \to S$ be the étale $\mathbb{Z}_2^2$-covering associated to $\mathbb{Z}^4 \cong \langle e_1, e_1', e_2, e_2' \rangle \triangleleft \Gamma$. Since $\hat{S} \to S_i \to S$, and S_i maps to E_i (via the Albanese map), we get a morphism

$$f : \hat{S} \to E_1 \times E_2 = \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2.$$

Then the covering of $E_1 \times E_2$ associated to $\Lambda_1' \oplus \Lambda_2' \leq \Lambda_1 \oplus \Lambda_2$ is $E_1' \times E_2'$, and since $\pi_1(\hat{S}) = \Lambda_1' \oplus \Lambda_2'$ we see that f factors through $E_1' \times E_2'$ and that the Albanese map of $\hat{S}$ is $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$.

The proof of the main result follows then from

Proposition 5.11. *Let S be a smooth complex projective surface, which is homotopically equivalent to a Keum - Naie surface. Let $\hat{S} \to S$ be the étale $\mathbb{Z}_2^2$-cover associated to $\langle e_1, e_1', e_2, e_2' \rangle \lhd \Gamma$ and let*

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\hat{\alpha}} & E_1' \times E_2' \\ & \searrow & \uparrow \varphi \\ & & Y \end{array}$$

be the Stein factorization of the Albanese map of $\hat{S}$.
Then φ has degree 2 and Y is a canonical model of $\hat{S}$.

More precisely, φ is a double cover of $E_1' \times E_2'$ branched on a divisor of type $(4,4)$.

The fact that S is homotopically equivalent to a Keum-Naie surface immediately implies that the degree of $\hat{\alpha}$ is equal to two.

The second assertion, i.e., that Y has only canonical singularities, follows instead from standard formulae on double covers (cf. [Hor75]).

The last assertion follows from $K_{\hat{S}}^2 = 16$ and $(\mathbb{Z}/2\mathbb{Z})^2$- invariance.

In fact, we conjecture a stronger statement to hold true:

Conjecture 5.12. Let S be a minimal smooth projective surface such that

i) $K_S^2 = 4$,
ii) $\pi_1(S) \cong \Gamma$.

Then S is a Keum-Naie surface.

We can prove

Theorem 5.13. *Let S be a minimal smooth projective surface such that*

i) $K_S^2 = 4$,
ii) $\pi_1(S) \cong \Gamma$,
iii)there is a deformation of S with ample canonical bundle.

Then S is a Keum-Naie surface.

We recall the following results:

Theorem 5.14 (Severi's conjecture, [Par05]). *Let S be a minimal smooth projective surface of maximal Albanese dimension (i.e., the image of the Albanese map is a surface), then $K_S^2 \geq 4\chi(S)$.*

M. Manetti proved Severi's inequality under the assumption that K_S is ample, but he also gave a description of the limit case $K_S^2 = 4\chi(S)$, which will be crucial for the above theorem 5.13.

Theorem 5.15 (M. Manetti, [Man03]). *Let S be a minimal smooth projective surface of maximal Albanese dimension with K_S ample then $K_S^2 \geq 4\chi(S)$, and equality holds if and only if $q(S) = 2$, and the Albanese map $\alpha : S \to \mathrm{Alb}(S)$ is a finite double cover.*

Proof (Proof of theorem 5.13). We know that there is an étale $\mathbb{Z}_2^2$-cover $\hat{S}$ of S with Albanese map $\hat{\alpha}: \hat{S} \to E_1' \times E_2'$. Note that $K_{\hat{S}}^2 = 4K_S^2 = 16$. By Severi's inequality, it follows that $\chi(S) \leq 4$, but since $1 \leq \chi(S) = \frac{1}{4}\chi(\hat{S})$, we have $\chi(S) = 4$. Since S deforms to a surface with K_S ample, we can apply Manetti's result and obtain that $\hat{\alpha}: \hat{S} \to E_1' \times E_2'$ has degree 2, and we conclude as before.

It seems reasonable to conjecture (cf. [Man03]) the following, which would immediately imply our conjecture 5.12.

Conjecture 5.16. Let S be a minimal smooth projective surface of maximal Albanese dimension. Then $K_S^2 = 4\chi(S)$ if and only if $q(S) = 2$, and the Albanese map has degree 2.

During the preparation of the article [BC09a] the authors realized that a completely similar argument applies to *primary Burniat surfaces*.

We briefly recall the construction of Burniat surfaces: for more details, and for the proof that Burniat surfaces are exactly certain Inoue surfaces we refer to [BC09b].

Burniat surfaces are minimal surfaces of general type with $K^2 = 6,5,4,3,2$ and $p_g = 0$, which were constructed in [Bur66] as singular bidouble covers (Galois covers with group $\mathbb{Z}_2^2$) of the projective plane branched on 9 lines.

Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points (which we assume to be the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$) and let's denote by $Y := \hat{\mathbb{P}}^2(P_1,P_2,P_3)$ the Del Pezzo surface of degree 6, blow up of $\mathbb{P}^2$ in P_1,P_2,P_3.

Y is 'the' smooth Del Pezzo surface of degree 6, and it is the closure of the graph of the rational map

$$\varepsilon : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that

$$\varepsilon(y_1 : y_2 : y_3) = ((y_2 : y_3), (y_3 : y_1), (y_1 : y_2)).$$

One sees immediately that $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the hypersurface of type $(1,1,1)$:

$$Y = \{((x_1' : x_1), (x_2' : x_2), (x_3' : x_3)) \mid x_1x_2x_3 = x_1'x_2'x_3'\}.$$

We denote by L the total transform of a general line in $\mathbb{P}^2$, by E_i the exceptional curve lying over P_i, and by $D_{i,1}$ the unique effective divisor in $|L - E_i - E_{i+1}|$, i.e., the proper transform of the line $y_{i-1} = 0$, side of the triangle joining the points P_i, P_{i+1}.

Consider on Y, for each $i \in \mathbb{Z}_3 \cong \{1,2,3\}$, the following divisors

$$D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2} \in |3L - 3E_i - E_{i+1} + E_{i+2}|,$$

where $D_{i,j} \in |L - E_i|$, for $\mathrm{j} = 2,3$, $\mathrm{D_{i,j}} \neq \mathrm{D_{i,1}}$, is the proper transform of another line through P_i and $D_{i,1} \in |L - E_i - E_{i+1}|$ is as above. Assume also that all the corresponding lines in $\mathbb{P}^2$ are distinct, so that $D := \sum_i D_i$ is a reduced divisor.

Note that, if we define the divisor $\mathscr{L}_i := 3L - 2E_{i-1} - E_{i+1}$, then

$$D_{i-1} + D_{i+1} = 6L - 4E_{i-1} - 2E_{i+1} \equiv 2\mathscr{L}_i,$$

and we can consider (cf. section 4, [Cat84] and [Cat98]) the associated bidouble cover $X' \to Y$ branched on $D := \sum_i D_i$ (but we take a different ordering of the indices of the fibre coordinates u_i, using the same choice as the one made in [BC09b], except that X' was denoted by X).

We recall that this precisely means the following: let $D_i = \mathrm{div}(\delta_i)$, and let u_i be a fibre coordinate of the geometric line bundle $\mathbb{L}_{i+1}$, whose sheaf of holomorphic sections is $\mathscr{O}_Y(\mathscr{L}_{i+1})$.

Then $X \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is given by the equations:

$$u_1u_2 = \delta_1 u_3, \;\; u_1^2 = \delta_3\delta_1;$$

$$u_2u_3 = \delta_2 u_1, \;\; u_2^2 = \delta_1\delta_2;$$

$$u_3u_1 = \delta_3 u_2, \;\; u_3^2 = \delta_2\delta_3.$$

From the birational point of view, as done by Burniat, we are simply adjoining to the function field of $\mathbb{P}^2$ two square roots, namely $\sqrt{\frac{\Delta_1}{\Delta_3}}$ and $\sqrt{\frac{\Delta_2}{\Delta_3}}$, where Δ_i is the cubic polynomial in $\mathbb{C}[x_0,x_1,x_2]$ whose zero set has $D_i - E_{i+2}$ as strict transform.

This shows clearly that we have a Galois cover $X' \to Y$ with group $\mathbb{Z}_2^2$.

The equations above give a biregular model X' which is nonsingular exactly if the divisor D does not have points of multiplicity 3 (there cannot be points of higher multiplicities!). These points give then quotient singularities of type $\frac{1}{4}(1,1)$, i.e., isomorphic to the quotient of $\mathbb{C}^2$ by the action of $\mathbb{Z}_4$ sending $(u,v) \mapsto (iu,iv)$ (or, equivalently, the affine cone over the 4-th Veronese embedding of $\mathbb{P}^1$).

Definition 5.17. A *primary Burniat surface* is a surface constructed as above, and which is moreover smooth. It is then a minimal surface S with K_S ample, and with $K_S^2 = 6$, $p_g(S) = q(S) = 0$.

A *secondary Burniat surface* is the minimal resolution of a surface X' constructed as above, and which moreover has $1 \le m \le 2$ singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, and with $K_S^2 = 6-m$, $p_g(S) = q(S) = 0$.

A *tertiary (respectively, quaternary) Burniat surface* is the minimal resolution of a surface X' constructed as above, and which moreover has $m = 3$ (respectively $m = 4$) singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, but not ample, and with $K_S^2 = 6-m$, $p_g(S) = q(S) = 0$.

Remark 5.18. 1) We remark that for $K_S^2 = 4$ there are two possible types of configurations. The one where there are three collinear points of multiplicity at least 3 for the plane curve formed by the 9 lines leads to a Burniat surface S which we call of *nodal type*, and with K_S not ample, since the inverse image of the line joining the 3 collinear points is a (-2)-curve (a smooth rational curve of self intersection -2).

In the other cases with $K_S^2 = 4,5,6$, K_S is instead ample.

2) In the nodal case, if we blow up the two $(1,1,1)$ points of D, we obtain a weak Del Pezzo surface $\tilde{Y}$, since it contains a (-2)-curve. Its anticanonical model Y' has a

node (an A_1-singularity, corresponding to the contraction of the (-2)-curve). In the non nodal case, we obtain a smooth Del Pezzo surface $\tilde{Y} = Y'$ of degree 4.

With similar methods as in [BC09a] (cf. [BC09b]) the first two authors proved

Theorem 5.19. *The subset of the Gieseker moduli space corresponding to primary Burniat surfaces is an irreducible connected component, normal, rational and of dimension four. More generally, any surface homotopically equivalent to a primary Burniat surface is indeed a primary Burniat surface.*

Remark 5.20. The assertion that the moduli space corresponding to primary Burniat surfaces is rational needs indeed a further argument, which is carried out in [BC09b].

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