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Introduction

In this thesis we face the problem of classification of numerical Godeaux surfaces. Recall that, by definition, a numerical Godeaux surface is a complex algebraic surface of general type with $p_g = 0$ and such that on its minimal model $K^2 = 1$. Recall also that (see, e.g., [**BPV**]) a numerical Godeaux surface is a regular surface ($q = 0$).

The first interest for surfaces with $p_g(S) = q(S) = 0$ comes from Castelnuovo's criterion on rationality (see, e.g., [**BPV**]): an algebraic surface with $P_2(S) = q(S) = 0$ is rational. Castelnuovo asked whether it is possible to improve the criterion: is a surface with $p_g(S) = q(S) = 0$ rational?

The first counterexample was given by Enriques, who constructed what are by now called the Enriques surfaces (cf. [**E1**], [**E2**], and [**BPV**] for further details and references); these have fundamental group $\mathbb{Z}/2\mathbb{Z}$, and Kodaira dimension 0 (so these are not of general type).

The first examples of surfaces of general type with $p_g(S) = q(S) = 0$ were constructed by Campedelli ([**Cam**], $K^2 = 2$), and by Godeaux ([**G1**], [**G2**], $K^2 = 1$) in the '30s.

Then Severi asked whether a simply connected surface with $p_g(S) = 0$ would be rational: the first counterexamples were constructed by Dolgachev in [**Dol2**]; these are elliptic surfaces, that are now called Dolgachev surfaces.

Later, the interest for surfaces of general type with $p_g(S) = q(S) = 0$ (already considered in Chapter VIII of Enriques' book [**E2**] about the properties of their pluricanonical systems), was revived by the fundamental Bombieri's article [**Bo**], where the author left open problems about their pluricanonical systems. Theorem 14 in that paper shows also that the torsion group of a numerical Godeaux surface has order smaller than 6, and there Bombieri also announced that he was able to show that this order is smaller than 5; an easy proof is in [**Cat1**], page 263.

After that, surfaces of general type with $p_g(S) = q(S) = 0$ and $K^2 = 1$ (for the minimal model) were called numerical Godeaux surfaces, while those with $p_g(S) = q(S) = 0, K^2 = 2$ were called numerical Campedelli surfaces.

In the following years there were several papers devoted to these two classes of surfaces, and to their tri- and quadri-canonical maps.

In particular, Reid ([**R2**]) showed that the torsion group is cyclic and completely described the geometry of the numerical Godeaux surfaces with torsion group (the torsion subgroup of the Picard group) of order ≥ 3, inverting the method that Godeaux used in order to costruct the first example. More precisely, the surface constructed by Godeaux was a quotient of a quintic in $\mathbb{P}^3$ by a free action of $\mathbb{Z}/5\mathbb{Z}$. Reid described the canonical ring of the covering of a numerical Godeaux surface with torsion $\mathbb{Z}/n\mathbb{Z}$, $n \geq 3$, induced by the torsion group and the action of this group on it (the invariant subring is the canonical ring of the original Godeaux surface).

Several examples and some families of numerical Godeaux surfaces with torsion $\mathbb{Z}/2\mathbb{Z}$ are constructed, as in [**CD**], [**Ba1**], [**Wer1**] and in [**Wer2**].

The first costruction of a numerical Godeaux surface with torsion $\{0\}$ was done by Rebecca Barlow in the early '80s ([**R3**], [**Ba2**]). She constructed a simply connected numerical Godeaux surface, with a variant of Godeaux's method, using a non free action of a dihedral group. In fact, Barlow method constructs a dimension 4 family of simply connected numerical Godeaux surface, with 4 distinct fundamental cycles (so that the canonical class is not ample).

The only other explicit example of a numerical Godeaux surface was produced by Craighero and Gattazzo in 1994 ([**CG**]), as minimal resolution of a normal singular surface. In [**DW**] was proved that the resulting numerical Godeaux surface is simply connected, and has K_S ample. It is an open problem whether the Barlow surface and the Craighero Gattazzo surface are diffeomorphic.

It came out that the Barlow surface is an useful example both for applications to the differential topology of 4-manifolds ([**Kot1**], [**OVdV**]), and for problems on Einstein metrics ([**CL**]). In fact, a classification of simply connected numerical Godeaux surfaces could produce new simply connected differentiable 4-manifolds with $b^+ = 1$. From this point of view we have to say that the surface constructed by Craighero and Gattazzo is a interesting example because of the ampleness of its canonical system.

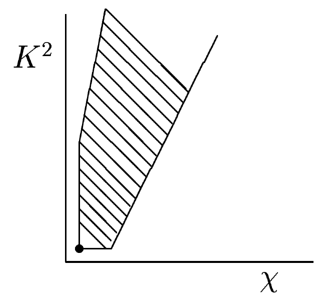

FIGURE 1. The allowed region

Another source of interest is the conjecture of Bloch (cf. [**Mu3**], [**BKL**], [**Blo**]) that for a surface with $q = p_g = 0$ the Chow group of degree zero 0-cycles is trivial (this has been settled only in few very special cases, cf. [**IM**], [**Ba3**], [**V**]).

Finally, another source of interest comes from the so called "geography" of surfaces of general type. Roughly speaking, the main problem of classification of surfaces can be summarized as follows: find all surfaces with given values of the invariants K^2 and χ (as usual, K^2 is the self intersection of the canonical class in the minimal model).

It is well known (see for details and references, [**BPV**], [**Cat4**]) that $K^2 \geq 1$, $\chi \geq 1$, $K^2 \leq 9\chi$ (B-M-Y inequality), $K^2 \geq 2\chi - 6$ (Noether inequality). These inequalities delimitate a region in the plane (K^2, χ), the "allowed region". We drew a picture of that in figure 1.

For every surface of general type the values of these two invariants give an integral point in this region. Conversely, results of Persson, Chen and others show that almost every point in this region corresponds to some of these surfaces.

In this picture, the numerical Godeaux surfaces live on the marked point, let's say the "vertex" of the allowed region. Most of the usual methods used in classification of surfaces work for high numbers of the invariants or near the special lines (B-M-Y, Noether and the Castelnuovo line, let us write $K^2 = 3\chi - 10$: in fact Castelnuovo work was for surfaces with $K^2 = 3p_g - 7$: a line in a picture similar to figure 1 one can draw with p_g instead of χ). The special position of this point indicates that we are, in some sense, in one of the harder cases of the classification of surfaces.

Now, let us write something about the method and the results.

For a numerical Godeaux surface the bicanonical system yields, on a suitable blow up $\tilde{S}$ of the minimal model S, a fibration $f : \tilde{S} \to \mathbb{P}^1$ whose fibres are curves of genus g, where g can only be $2, 3$, or 4.

In fact, if we write the bicanonical pencil as $|2K_S|$ as $|M| + F$, where F is the fixed part, we see that $KF = 0$, $KM = 2$. This gives four possibilities:

ia)	$M^2 = 4$	$F = 0$	$g = 4$	$\lvert M\rvert$ has 4 base points
ib)	$M^2 = 4$	$F = 0$	$g = 3$	$\lvert M\rvert$ has 1 double base point
ii)	$M^2 = 2$	$F^2 = -2$	$g = 3$	
iii)	$M^2 = 0$	$F^2 = -4$	$g = 2$.	

In every case we can consider a product rational mapping $\varphi = \varphi_1 \times \varphi_2 : S \dashrightarrow \mathbb{P}^{g-1} \times \mathbb{P}^1$, where φ_2 is the bicanonical map φ_{2K} and φ_1 maps every divisor of M in the canonical image of the corresponding fibre of f. We shall be more precise in chapter 1. Let us just remark that here the assumption $p_g = q = 0$ is crucial. A similar construction is possible also for more general surfaces but in those cases computations become harder because the target of φ is a suitable scroll (depending from the values of the invariants) instead of $\mathbb{P}^{g-1} \times \mathbb{P}^1$, and the fibres can be mapped with a subsystem of its canonical class.

We show that, in all the cases except (of course) the last one, the general fibre of f is not hyperelliptic. In fact, we show that for a numerical Godeaux surface, case iii) does not occur. This was already proved (unpublished) by Bombieri using the classification of reducible genus 2 fibres. Bombieri's proof takes more or less 40 pages, so we think is interesting to have a shorter one.

Therefore, in these cases φ yields a birational map, and indeed, on $\tilde{S}$, we get a product morphism $\psi_1 \times f$, which fails to be an embedding when we have a hyperelliptic fibre. So, for every birational class of numerical Godeaux surfaces, the map φ gives a (singular) representative contained in $\mathbb{P}^{g-1} \times \mathbb{P}^1$.

Every fibre of the projection $Y \to \mathbb{P}^1$ is a canonical curve. In the classical theory the ideal of a canonical curve C is given by the kernels of the maps $S^n(H^0(\omega_C)) \to H^0(\omega_C^n)$; it turns out that defining equations for Y are given by the kernels of the morphisms $\sigma_n : S^n(H^0(f_*\omega)) \to f_*\omega^n$, where ω is a line bundle on $\tilde{S}$ that induces by restriction the dualizing sheaf on every fibre of f.

Studying these maps, assuming that the torsion group is $\{0\}$, we prove that the conductor divisor of the normalization of Y is a sum of the hyperelliptic fibres of f (with certain multiplicities), and this allows us to completely describe the situation.

E.g., for case ia), summing up theorem 3.2.1 and proposition 4.3.1, we get the following (already in [**CP**])

THEOREM 0.0.1. *Assume that S is a numerical Godeaux surface with torsion $\{0\}$ and of type ia), i.e., s.t. the bicanonical pencil yields a genus 4 fibration f.*

Let $h = \sum_{C \text{ hyperelliptic}} mult(C)$: then a priori $0 \leq h \leq 3$; a posteriori the case with three distinct hyperelliptic fibres cannot occur.

Moreover $\exists \mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$, s.t. $Y := \varphi(S)$ is a divisor in the linear system $|\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ whose singular curves are exactly the twisted cubic curves image of the (honestly) hyperelliptic bicanonical divisors. Moreover, if $\mathcal{C}$ is the conductor ideal, $h^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$.

Viceversa, assume that $0 \leq h \leq 3$ and that $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$ is an irreducible divisor, and that in turn $Y \in |\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ is an irreducible divisor whose normalization is a surface $\tilde{X}$ with rational double points as the only singularities. Suppose moreover that the conductor ideal $\mathcal{C}$ defines a divisor on $\tilde{X}$ equal to h fibres (counted with multiplicity). Assume moreover that the singular curves of Y are (irreducible) twisted cubics, and that $h^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$. Then Y is the tri-bicanonical model of a numerical Godeaux surface with torsion $\{0\}$ and of type ia).

Similar theorems are given also in the other two cases (theorems 3.2.2 and 3.2.3).

Let us now describe the structure of this thesis.

In chapter 1 we describe the details of the strategy and state the first basic results. Then we show that for a fibration whose generic fibre is non hyperelliptic the sheaves cokernel of the morphism σ_n ($\mathcal{T}_n$) are skyscreaper sheaves supported on the image of the hyperelliptic fibres, and compute the length of the stalk of these sheaves at the points corresponding to some class of fibres.

In chapter 2 after some result on the canonical ring of a numerical Godeaux surface, we construct the fibration f, and we study which fibres are allowed (in both cases of torsion $\{0\}$ and $\mathbb{Z}/2\mathbb{Z}$). Then, computing the Euler characteristic of the sheaves $f_*\omega^n$ and by the results of chapter 1, we compute in case of torsion $\{0\}$, the sheaves $\mathcal{L}_n$ (kernel of σ_n). The degrees of these sheaves give us the linear system in which Y is a divisor (more precisely, in case ia) Y is a divisor in an hypersurface, so in another divisor; we compute the linear systems of both). Finally, a precise computation of the sheaves $f_*\omega^n$ allows us to exclude case iii) (without hypotheses on the torsion group).

In chapter 3 we compute the adjunction conditions, (e.g., in theorem 0.0.1, $h^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$), and we prove theorem 0.0.1 and its analogous in the other cases (torsion $\{0\}$).

In chapter 4 we write an explicit resolution for a defining ideal of Y in case ia). These are the following

THEOREM 0.0.2. *If Y is the tri-bicanonical model of a numerical Godeaux surface with $h \leq 1$, then there exist linear forms $L_0, \dots, L_{11-5h}$ and quadratic forms $Q_0, \dots, Q_{7-2h}$ in $\mathbb{P}^3$, such that, if λ_0, λ_1 are a basis for the linear forms on $\mathbb{P}^1$, the ideal sheaf of Y has a resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-5, 2h-7)^{5-3h} \xrightarrow{\beta_h} \begin{array}{c} \mathcal{O}_{\mathbb{P}}(-3, 2h-7)^{12-5h} \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-5, 2h-6)^{6-3h} \end{array} \xrightarrow{\alpha_h}$$

$$\xrightarrow{\alpha_h} \begin{array}{c} \mathcal{O}_{\mathbb{P}}(-2, 2h-7) \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-3, 2h-6)^{13-5h} \end{array} \longrightarrow I_Y \longrightarrow 0$$

where

$$\alpha_h = \begin{pmatrix} L_0 & \cdots & L_{11-5h} & 0 & \cdots & 0 \\ -\lambda_0 & & 0 & Q_{7-2h} & & 0 \\ \lambda_1 & & & \vdots & \ddots & \\ & \ddots & \vdots & & & Q_{7-2h} \\ & \ddots & & Q_0 & & \vdots \\ & & -\lambda_0 & & \ddots & \\ 0 & & \lambda_1 & 0 & & Q_0 \end{pmatrix}$$

and

$$\beta_h = \begin{pmatrix} Q_{7-2h} & & 0 \\ \vdots & \ddots & \\ Q_0 & & Q_{7-2h} \\ & \ddots & \vdots \\ 0 & & Q_0 \\ \lambda_0 & & 0 \\ -\lambda_1 & \ddots & \\ & \ddots & \lambda_0 \\ 0 & & -\lambda_1 \end{pmatrix}$$

THEOREM 0.0.3. *If Y is the tri-bicanonical model of a numerical Godeaux surface with $h = 2$, then there exist linear forms L_0, L_1, quadratic forms*

$Q_0, \dots, Q_3$ and a cubic form C in $\mathbb{P}^3$, such that, if λ_0, λ_1 are a basis for the linear forms on $\mathbb{P}^1$, the ideal sheaf of Y has a resolution

$$0 \to \begin{array}{c} \mathcal{O}_{\mathbb{P}}(-3,-3)^2 \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-5,-3) \end{array} \xrightarrow{\alpha_2} \begin{array}{c} \mathcal{O}_{\mathbb{P}}(-2,-3) \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-3,-2)^3 \end{array} \to I_Y \to 0,$$

where $\alpha_2 = \begin{pmatrix} L_0 & L_1 & C \\ -\lambda_0 & 0 & \lambda_1 Q_3 \\ \lambda_1 & -\lambda_0 & \lambda_0 Q_1 + \lambda_1 Q_2 \\ 0 & \lambda_1 & \lambda_0 Q_0 \end{pmatrix}.$

By these two theorems we can explixitely write a parameter space for the corresponding slices of the moduli space, as the relations induced by all the closed conditions (we explicit also adjunction condition).

In particular for $h = 2$ (the only case for which we can show existence) we manipulate these relations in order to simplify those equations; unfortunately at the moment the equations are still too complicated for my computer. In this chapter we also exclude the case with three distinc hyperelliptic fibres (again in case ia)) and we develop a parameter computation for case ii).

Finally in chapter 5 we study the two existing explicit constructions (Barlow and Craighero Gattazzo), showing that in both cases we are in case ia), and $h = 2$. Moreover we show that the local moduli space of the Craighero Gattazzo surface is smooth of the expected dimension (8, this holds also for the Barlow surface, as proved in [**CL**]).

Notation

For the reader's convenience, we enclose here a list of notations more often used, and of our abbreviations.

In this thesis we denote by S the minimal model of a numerical Godeaux surface and by X its canonical model.

We let moreover $\mathrm{Tors}(S)$ be the torsion subgroup of the first homology group $H_1(S,\mathbb{Z})$ (equivalently, of $H^2(S,\mathbb{Z})$).

For a Gorenstein algebraic variety Z (e.g. S, X) we denote by K_Z a Cartier divisor associated to its dualizing sheaf ω_Z. The rational map associated to a divisor D is denoted by φ_D or $\varphi_{|D|}$; similarly the rational map associated to a line bundle $\mathcal{L}$ is denoted by $\varphi_{\mathcal{L}}$.

$\tilde{S}$ is the blow up of S at the base points of the movable part $|M|$ of $|2K_S|$.$\tilde{X}$ is the surface obtained by $\tilde{S}$ contracting all the (-2)-curves. This is the blow up of X in the smooth base points of $|2K_X|$ except (possibly) in case ib) (see introduction). The induced morphism are denoted by $\beta : \tilde{S} \to S$, $\hat{\beta} : \tilde{X} \to X$.

We have already defined $\varphi : S \dashrightarrow \mathbb{P}^{g-1} \times \mathbb{P}^1$ in the introduction; let us denote by $\hat{\varphi} : X \dashrightarrow \mathbb{P}^{g-1} \times \mathbb{P}^1$ the induced map on the canonical model, and set $Y := \varphi(S)$, $\Sigma := \varphi_{3K_S}(S)$.

Moreover, we denote by $\pi_1 : \mathbb{P}^{g-1} \times \mathbb{P}^1 \to \mathbb{P}^{g-1}$, $\pi_2 : \mathbb{P}^{g-1} \times \mathbb{P}^1 \to \mathbb{P}^1$ the natural projections.

This allows us to define the morphisms $g := \varphi \circ \beta : \tilde{S} \to Y$, $\hat{g} := \hat{\varphi} \circ \hat{\beta} : \tilde{X} \to Y$

Last, we denote by $f := \pi_2 \circ \varphi \circ \beta : \tilde{S} \to \mathbb{P}^1$ the fibration associated to the bicanonical pencil, and by $\hat{f} := \pi_2 \circ \hat{\varphi} \circ \hat{\beta} : \tilde{X} \to \mathbb{P}^1$ the analogous fibration on the canonical model X.

Quite often, given a Cartier divisor D on a scheme Z, by slight abuse of notation we denote also by D the associated invertible sheaf $\mathcal{O}_Z(D)$; and we often write, as shorthand notation, $H^0(D)$ instead of $H^0(\mathcal{O}_Z(D))$.

Moreover, $\mathcal{O}_{\mathbb{P}^{g-1}\times\mathbb{P}^1}(a,b)$ is also a quite understandable notation for the tensor product $\pi_1^*\mathcal{O}_{\mathbb{P}^{g-1}}(a)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(b)$.

CHAPTER 1

Fibrations of curves of low genus

1.1. Fibrations with $p_g = q = 0$

Let S a minimal surface, M a pencil on S without fixed components. Let $f_M : S \dashrightarrow \mathbb{P}^1$ be the induced rational map.

It is well known that there exists a unique minimal sequence of blow ups $\beta : \tilde{S} \to S$, and (relatively minimal) fibration $f : \tilde{S} \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{S} & & \\ \beta \downarrow & \searrow^{f} & \\ S & \overset{f_M}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

The fibre of f is the strict transform on $\tilde{S}$ of the pencil M, so let us denote the corresponding linear system by $\tilde{M}$. For every $p \in \mathbb{P}^1$, we denote by $\tilde{M}_p$ the fibre of f corresponding to p.

We shall assume the generic fibre 1-connected, so that f is a fibration of genus g, where g is the genus of $\tilde{M}$. The linear system $|K_{\tilde{S}} + \tilde{M}|$ restricted to every such a fibre induces the dualizing sheaf $\omega_{\tilde{M}_p}$; recall that, being f a fibration, $\forall p \in \mathbb{P}^1$, $H^0(\mathcal{O}_{\tilde{M}_p}) = 1$, so $H^0(\omega_{\tilde{M}_p}) = g$.

We consider the map $g : \tilde{S} \to \mathbb{P}^{h^0(K_{\tilde{S}}+\tilde{M})-1} \times \mathbb{P}^1$, induced by the two linear systems $\omega := |K_{\tilde{S}} + \tilde{M}|$ and $\tilde{M}$. This map restricted to every fibre of f is induced by a subsystem of its canonical system, so the image Y should be contained in the total space of some fibration $\mathcal{F} \to \mathbb{P}^1$ with fibre $\mathbb{P}^l$, $l \leq g-1$: the image of every fibre is a projection of its canonical image. The key idea is to study Y instead of S, using results on canonical images of curves.

The situation becomes considerably easier in the case we are interested in.

Remark 1.1.1. *If $p_g(S) = q(S) = 0$, then $\mathcal{F} = \mathbb{P}^{g-1} \times \mathbb{P}^1$, and every fibre of f is mapped with its canonical morphism.*

Proof.

We have just to remark that, for every $p \in \mathbb{P}^1$, from the exact sequence

$$0 \to K_{\tilde{S}} \to \omega \to \omega_{\tilde{M}_p} \to 0$$

we have that the restriction map $H^0(\omega) \to H^0(\omega_{\tilde{M}_p})$ is an isomorphism.

□

We are mainly interested in numerical Godeaux surfaces. So in the following we will assume $p_g(S) = q(S) = 0$, that, by remark 1.1.1, considerably simplify the situation.

First of all, we need some lemma on canonical image of curves.

REMARK 1.1.2. *Let C be a genus 3 curve, let ω be the dualizing sheaf of C. Assume that φ_ω embeds C. Then $\varphi_\omega(C)$ is a plane quartic.*

Proof.

The image is a curve in $\mathbb{P}^{g-1} = \mathbb{P}^2$ whose degree is $deg(\omega_C) = 2g(C) - 2 = 4$.

□

The following is in fact is a slight improvement of Noether theorem (see, e.g., [**GH**]) in case $g = 4$.

LEMMA 1.1.3. *Let C be a 3−connected genus 4 Gorenstein curve, let ω be the dualizing sheaf of C. Assume that φ_ω embeds C. Then $\varphi_\omega(C)$ is a complete intersection of type* $(2, 3)$.

Proof.

C has genus 4, so $h^0(C, \omega) = 4$, $h^0(C, \omega^2) = 9$, $h^0(C, \omega^3) = 15$. Then the natural map $S^2(H^0(C, \omega)) \to H^0(C, \omega^2)$ has a non-trivial kernel.

Assume, by contradiction, that this kernel has dimension greater than 1.

Thus, there exist two distinct quadrics Q_1, Q_2 containing the degree 6 curve $\varphi_\omega(C)$. If Q_1, Q_2 have no common components, their intersection is a curve of degree 4, a contradiction.

Therefore there do exist linear forms L_0, L_1, L_2 such that $Q_1 = L_0 L_1, Q_2 = L_0 L_2$, and $\varphi_\omega(C) \subset L_0 L_1 \cap L_0 L_2 = L_0 \cup (L_1 \cap L_2)$.

But $\varphi_\omega(C)$ is non degenerate, so we can write $C = C_1 + C_2$, with $\varphi_\omega(C_1) \subset L_0$ of degree 5, $\varphi_\omega(C_2) = L_1 \cap L_2$ a line.

Now, recalling that C is assumed to be 3−connected, we can compute

$$4 = g(C) = g(C_1) + g(C_2) - 1 + C_1 C_2 \geq 6 + 0 - 1 + 3 = 8,$$

hence we derive a contradiction. Note that we don't need to embed C in a surface in order to derive $C_1 C_2$; it can be defined as $\deg_{C_1}(\omega_C)$ - $\deg_{C_1}(\omega_{C_1})$, cf. [**CFHR**].

Therefore there is only one quadric containing $\varphi_\omega(C)$, let us denote it by Q.

Now, by a dimension count, the map $S^3(H^0(C,\omega)) \to H^0(C,\omega^3)$ has a kernel of dimension at least 5. In particular we get at least one cubic surface G containing $\varphi_\omega(C)$ and not having Q as a component.

If G and Q have no common components, their intersection is a degree 6 curve containing $\varphi_\omega(C)$, so $\varphi_\omega(C) = Q \cap G$ and we are done.

Otherwise, there must exist linear forms L_0, L, and a quadratic form Q', such that $Q = L_0 L$, $G = L_0 Q'$, and $\varphi_\omega(C) \subset L_0 \cup (L \cap Q')$. Again, we can decompose C as $C_1 + C_2$, with $\varphi_\omega(C_1) \subset L_0$, $\varphi_\omega(C_2) \subset L \cap Q'$. If $\varphi_\omega(C_2) \neq L \cap Q'$, we have decomposed $\varphi_\omega(C)$ as the union of a plane quintic and of a line, and we have already excluded this case. Else, $\varphi_\omega(C_1)$ is a plane quartic, $\varphi_\omega(C_2)$ a conic, and again we get

$$4 = g(C) = g(C_1) + g(C_2) - 1 + C_1C_2 \geq 3 + 0 - 1 + 3 = 5,$$

a contradiction.

□

By remarks 1.1.1 and 1.1.2, for a genus 3 fibration whose total space has $p_g = q = 0$ we expect that Y is a divisor in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(4,d)|$ for some suitable d that we would like to compute.

Similarly, by remarks 1.1.1 and lemma 1.1.3, for a genus 4 fibrations we would expect that Y is a complete intersection of a divisor in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(2,d_2)|$ for some suitable d_2 and a divisor in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(3,d_3)|$ for some suitable d_3. Unfortunately this is not true and in general Y will not be a complete intersection.

In propositions 1.1.5 and 1.1.6 we give answers to these "expectations".

LEMMA 1.1.4. *Let $\tilde{S} \to \mathbb{P}^1$ a genus g fibration such that $p_g(\tilde{S}) = q(\tilde{S}) = 0$, and such that the generic fibre is embedded by its canonical map.*

Let ω be defined as before. Then $f_*(\omega) = \mathcal{O}^g_{\mathbb{P}^1}$.

Proof.

Let C a fibre of f. We look at the exact sequence

$$0 \to K_{\tilde{S}} \to \omega \to \omega_C \to 0.$$

For a fibration every fibre is either 1-connected or multiple. In both cases $H^0(\mathcal{O}_C) = 1$, so, by Riemann-Roch, $h^0(\omega_C) = g$.

By ([**BPV**]), theorem I.8.5, $f_*\omega$ is a rank g locally free sheaf, so splits as direct sum of line bundles $\sum_1^g \mathcal{O}(d_i)$.

By $p_g = 0$, we get that the map $H^0(\omega) \to H^0(\omega_C)$ is injective, so $f_*\omega$ is generated ($\forall i\ d_i \geq 0$).

By $q = 0$ that map is also surjective, so $g = h^0(\omega) = h^0(f_*\omega) = \sum_1^g (d_i + 1)$, then $\sum_1^g d_i = 0$, and we are done.

□

Let us recall that, in the setting of lemma 1.1.4, we denote by $\tilde{M}$ the pencil given by the fibres of f, and by $Y \subset \mathbb{P}^{g-1} \times \mathbb{P}^1$ the image of the rational map induced by the linear systems $|K_{\tilde{S}} + \tilde{M}|$ and $\tilde{M}$.

Consider the natural homomorphisms of sheaves

$$S^n(f_*(\omega)) \xrightarrow{\sigma_n} f_*(\omega^n),$$

and set $\mathcal{L}_n = \ker \sigma_n$ and $\mathcal{T}_n = \operatorname{coker} \sigma_n$.

PROPOSITION 1.1.5. *If $g = 3$, and if the generic fibre is embedded by its canonical class, Y is a divisor in $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4, d)$.*

Moreover $\mathcal{L}_4 = \mathcal{O}_{\mathbb{P}^1}(-d)$.

Proof.

If the generic fibre is embedded, Y is a surface (so a divisor) in $\mathbb{P}^2 \times \mathbb{P}^1$.

By remark 1.1.2, a generic fibre C induces a quartic in $\mathbb{P}^2$, that is a generator of the kernel of the map $S^4(H^0(\omega_C)) \to H^0(\omega_C^4)$. For a genus g fibration, $\forall l \geq 0$, $h^0(\omega_C^l)$ is constant.

So $f_*\omega^l$ is a locally free sheaf of rank $H^0(\omega_C^l)$; more precisely, if $\tilde{M}_p$ is the fibre of a point in $\mathbb{P}^1$, and $\mathcal{M}_p$ is the ideal sheaf of p, there is a natural isomorphism between $\mathcal{M}_p f_*(\omega^l)$ and $H^0(\omega_{\tilde{M}_p}^l)$.

So we can consider the map $\sigma_4 : S^4(f_*\omega) \to f_*\omega^4$, by remark 1.1.2 and Nakayama lemma we conclude that this is a generic surjective morphism of locally free sheaves, and is kernel is a rank 1 locally free sheaf, so we can write it as $\mathcal{O}_{\mathbb{P}^1}(-d)$ for some suitable d in $\mathbb{Z}$.

By lemma 1.1.4 the injection $\mathcal{O}_{\mathbb{P}^1}(-d) \to S^4(f_*\omega) = \mathcal{O}_{\mathbb{P}^1}^{15}$ describes a divisor Z in $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4, d)$ that "generically" is Y, in the sense that they coincide on a generic $p \in \mathbb{P}^1$. Y is irreducible, so $Y \subset Z$. Assume now, arguing by contradiction $Z = Y + D$, $D > 0$.

Recall that $\operatorname{Pic}(\mathbb{P}^2 \times \mathbb{P}^1) = \operatorname{Pic}(\mathbb{P}^2) \times \operatorname{Pic}(\mathbb{P}^1)$: on a generic fiber Y is a quartic, so there exist $a > 0$ such that $D \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(0, a)|$, $Y \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4, d - a)|$.

But such a divisor would induce a complex

$$\mathcal{O}_{\mathbb{P}^1}(a - d) \to S^4(f_*\omega) \xrightarrow{\sigma_4} f_*(\omega^4).$$

We already computed that the kernel of σ_4 is $\mathcal{O}_{\mathbb{P}^1}(-d)$, so, being $a > 0$, the map $\mathcal{O}_{\mathbb{P}^1}(a-d) \to S^4(f_*\omega)$ should be zero, that is a contradiction (Y would be the divisor corresponding to the zero section!).

□

PROPOSITION 1.1.6. *If $g = 4$, the generic fibre embedded by the canonical class, there exists a divisor $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, d_2)|$, and a divisor $Z \in |\mathcal{O}_{\mathcal{Q}}(3, d_3)|$, s.t. $Y \subset Z$, and $Z - Y$ is supported on the points p where the corresponding quadric and the corresponing cubic contain a common plane.*

Moreover $\mathcal{L}_2 \cong \mathcal{O}_{\mathbb{P}^1}(-d_2)$ and we have an exact sequence

$$0 \to \mathcal{L}_2^4 \to \mathcal{L}_3 \to \mathcal{O}_{\mathbb{P}^1}(-d_3) \to 0.$$

Proof.

Part of the proof is similar to the previous one and we shall skip most details.

Being the generic fibre embedded by the canonical class, by lemma 1.1.3 the canonical image of the generic fibre is a complete intersection of a quadric and a cubic.

Arguing as in the previous proof, we see that such a quadrics "glue" to a divisor $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, d_2)|$, and $\mathcal{L}_2 \cong \mathcal{O}_{\mathbb{P}^1}(-d_2)$.

If we look at the system of cubics, we get a slightly more complicated situation. In fact, σ_3 is still generically surjective, so $\mathcal{L}_3$ is a locally free sheaf of rank $\binom{h^0(\omega)+3}{3} - h^0(\omega^3) = 20 - 15 = 5$. In fact we have 5 independent cubics containing the canonical image of a generic fibre: 4 of them are the multiples of the quadric.

Now we show that the morphism of sheaves

$$\mathcal{L}_2 \otimes H^0(f_*\omega) \xrightarrow{i} \mathcal{L}_3$$

is injective, and that its cokernel is locally free.

First, the injectivity; it is enough to show the injectivity of the map $\mathcal{L}_2 \otimes H^0(f_*\omega) \to S^3(f_*\omega)$.

This is a morphism of locally free sheaves, so the kernel is still locally free, and it is enough to show that this map is injective on the stalk of point. Let us fix a point $x \in \mathbb{P}^1$, and choose a local parameter t for $\mathcal{O}_x$.

Then we can write the map as $(f_0(t), f_1(t), f_2(t), f_3(t)) \mapsto (\sum_0^3 f_i y_i) q(t, y_i)$, where q is the quadric in the variables y_i with coefficients in $\mathcal{O}_x$ defining σ_2.

Then q is not 0; being the monomials of degree 3 in the variables y_i a basis for the free $\mathcal{O}_x$-module $S^3(f_*\omega)_x$, we immediately get the injectivity.

Now we show that the cokernel is locally free, i.e. that every stalk of the cokernel is free as $\mathcal{O}_x$-module. By the fundamental theorem on finitely generated modules over a principal ideal domain, the stalk of the cokernel is a direct sum of cyclic modules.

Then, if it is not free, it contains an element $m \neq 0$, such that there exists $f \neq 0$ in $\mathcal{O}_x$, with $fm = 0$. Let g be a preimage of m in the stalk of $\mathcal{L}_3$. Then, $g \notin i(\mathcal{L}_2 \otimes H^0(f_*\omega))$, whence $fg \in i(\mathcal{L}_2 \otimes H^0(f_*\omega))$.

Then we can write $fg = q\sum_0^3 f_i y_i$, where $f_i \in \mathcal{O}_x$, $\{y_i\}$ is a basis for $H^0(f_*\omega)$, q is a generator for the rank 1 free module $(\mathcal{L}_2)_x$, whence g cannot be written in this form; in particular $\exists i$ such that $f_i/f \notin \mathcal{O}_x$.

Let t be a local parameter for $\mathcal{O}_x$. Then, if a is the minimum of the multiplicities in 0 of f, f_i, we can write $f = t^{a+1}f'$, $f_i = t^a f_i'$, with $f', f_i' \in \mathcal{O}_x$. Dividing by t^a, we get $tf'g = q\sum_0^3 f_i' y_i$, where there exists i with $f_i'(0) \neq 0$.

Evaluating both the sides of the last equation in 0, we would get a not trivial linear relation between the four cubics multiple of a given quadric in $\mathbb{P}^3$, that is clearly a contradiction.

Now, arguing as in previous proof we have two divisors $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, d_2)|$, $Z \in |\mathcal{O}_{\mathcal{Q}}(3, d_3)|$, and $Y \subset Z$. We have to show that $Z - Y$ is supported on the points p where the corresponding quadric and cubic have a common plane.

For every $p \in \mathbb{P}^1$, let us denote by Y_p, Z_p the fibers of Y, Z on p; $\forall p$, $Y_p \subset Z_p$. Y_p is a curve of degree 6, Z_p is complete intersection of a quadric and a cubic. So, if Z_p is a curve, $Y_p = Z_p$.

If Z_p is not a curve, then Z has a component that projects to the point p. Writing $Z_p = Q_p \cap G_p$, Q_p corresponding quadric, G_p corresponding cubic, they have a common component; since by construction $Q_p \not| G_p$, we have done.

$\square$

REMARK 1.1.7. *By lemma 1.1.3 (and its proof), if $p \in \mathbb{P}^1$ corresponds to a 3-connected non-hyperelliptic fibre, then $Z_p = Y_p$.*

REMARK 1.1.8. *In the setting of proposition 1.1.6, Z is complete intersection of two divisors in $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, d_2)|$ and $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(3, d_3)|$ if and only if the exact sequence*

$$0 \to \mathcal{L}_2 \otimes H^0(f_*(\omega)) \to \mathcal{L}_3 \to \mathcal{O}_{\mathbb{P}^1}(-d_3) \to 0$$

splits.

We shall see in the following study of numerical Godeaux surfaces that this exact sequence is not in general splitting.

Now we want to show how one can compute the sheaves $\mathcal{L}_l$.

We can use the exact sequences

$$0 \to \mathcal{L}_l \to S^l(f_*\omega) \to f_*\omega^l \to \mathcal{T}_l \to 0.$$

$S^l(f_*\omega)$ is a trivial bundle of suitable rank; therefore we have to compute the sheaves $f_*\omega^l$ and $\mathcal{T}_l$.

More precisely, the sheaves we are interested in (for $g = 3$ $\mathcal{L}_4$, for $g = 4$ $\mathcal{L}_2$ and $\mathcal{L}_3/\mathcal{L}_2^4$), are line bundles, so it would be enough to compute the Euler characteristics of the sheaves $f_*\omega^l$ and $\mathcal{T}_l$.

We conclude this section looking at the Euler characteristic of $f_*\omega^l$. A more precise computation of these sheaves can be done in the specific cases, as we shall do for torsion $\{0\}$ Godeaux surfaces in chapter 2, section 2.5.

PROPOSITION 1.1.9. *For a genus $g \geq 2$ fibration f, $\forall l \geq 2$, $i \leq 1$ $h^i(f_*\omega^l) = h^i(\omega^l)$.*

Proof.

By definition of direct image sheaf $h^0(f_*\omega^l) = h^0(\omega^l)$.

For what concern h^1, we have just to remark that, by Leray spectral sequence, we have an exact sequence

$$0 \to H^1(f_*\omega^l) \to H^1(\omega^l) \to H^0(R^1 f_*\omega^l).$$

For genus greater than 2, $\forall l \geq 2$ $h^1(\omega_C^l) = 0$. By [**BPV**], theorem I.8.5, we conclude that $R^1 f_*\omega^l$ is a locally free sheaf of rank 0, so is the "zero" sheaf.

□

COROLLARY 1.1.10. *If $\tilde{S}$ is the total space of a genus $g \geq 2$ fibration f, and $p_g(\tilde{S}) = q(\tilde{S}) = 0$, then $\forall l \geq 2$, $i \leq 1$, $\chi(f_*\omega^l) = \chi(\omega^l)$.*

Proof.

By proposition 1.1.9 we have to show that $\forall l \geq 2$, $h^0(K_{\tilde{S}} \otimes \omega^{-l}) = h^2(\omega^l) = 0$.

But we assumed $p_g(\tilde{S}) = 0$, so it is enough to show that $\forall l$ $h^0(\omega^l) > 0$, that is equivalent to $h^0(\omega) > 0$. But if $p_g(\tilde{S}) = q(\tilde{S}) = 0$, we have that for every fibre C, $H^0(\omega)$ is isomorphic to $H^0(\omega_C) \cong \mathbb{C}^g$; so $h^0(\omega) = g$.

□

Remark that last corollary can be stated in more general hypotheses, in fact we need only that $h^0(K_{\tilde{S}} \otimes \omega^{-l}) = h^2(\omega^l) = 0$; e.g. if $\omega > 0$ we only need that $h^0(K_{\tilde{S}} \otimes \omega^{-2}) = 0$.

1.2. The sheaves $\mathcal{T}_l$ and the multiplicity of hyp. fibres

In this section we will show how one can compute the sheaves $\mathcal{T}_l$ we have defined in the previous sections.

PROPOSITION 1.2.1. *Let $f : \tilde{S} \to B$ be a fibration whose generic fibre is embedded by its canonical class.*

Then the sheaves $\mathcal{T}_l, l \geq 2$ are supported on the points of the curve B whose corresponding fibre has canonical image not projectively normal.

Proof.

Let $\mathcal{M}_p$ be the maximal ideal sheaf of the point p in $\mathcal{O}_B$. Let us denote by $\tilde{M}_p$ the fibre of the point p.

By Grauert's base change theorem (cf. e.g. [**BPV**], theorem I.8.5.iv) $\forall p \in B$, $\frac{(f_*\omega^k)}{(\mathcal{M}_p f_*\omega^k)} \cong H^0(\tilde{M}_p, \omega^k)$; mod $\mathcal{M}_p$ the morphism σ_k acts on the stalks as $S^k(H^0(\tilde{M}_p, \omega)) \to H^0(\tilde{M}_p, \omega^k)$, whence it is surjective when the corresponding fibre has projectively normal canonical image.

Then, by Nakayama's lemma, if the corresponding fibre has projectively normal canonical image, then $(\sigma_n)_p$ is surjective, and the theorem is proved.

□

As we showed in previous section, we are mainly interested on the Euler characteristics of these sheaves. We will see in proposition 2.2.3 that in our cases the generic fibres are non hyperelliptic, so by proposition 1.2.1, they are skyscraper sheaves. So we have to compute the lengths of these sheaves in the "non projectively normal" points. Remark 1.1.2 and lemma 1.1.3 show some class of "projectively normal" points.

The first interesting case (and the easier one) in which some point fails to be "projectively normal" is when the corresponding fiber is honestly hyperelliptic.

LEMMA 1.2.2. *Let B be a smooth curve and $f : \tilde{S} \to B$ a genus 3 fibration whose generic fibre is non hyperelliptic. Let $\tilde{M}$ be the fibre of f, set $\omega = \tilde{M} + K_{\tilde{S}}$.*

Consider the homomorphisms of sheaves

$$S^n(f_*\omega) \xrightarrow{\sigma_n} f_*(\omega^n),$$

and denote by $\mathcal{L}_n = \ker \sigma_n$ and $\mathcal{T}_n = \operatorname{coker} \sigma_n$. Then

i) $\mathcal{T}_n$ is a torsion sheaf with support contained in the image of the hyperelliptic fibres.

ii) Let $p \in \mathbb{P}^1$ be the image of some 2-connected hyperelliptic fibre; then $\exists s > 0, s \in \mathbb{N}$, such that

$$\forall k \geq 2 \ \ \textit{length}\ (\mathcal{T}_k, p) = s(2k-3).$$

Proof.

i) Follows from remark 1.1.2 and proposition 1.2.1.

ii) Recall that, by [**ML**], the canonical ring of a 2-connected hyperelliptic fibre has the form

$$R = \mathbb{C}[x_1, x_2, x_3, y]/ < r_1 := Q(x_i), r_2 := y^2 - F(x_i) >,$$

where deg $x_i = 1$, deg $y = 2$, deg Q=2, deg $F = 4$.

Let us denote by R_k the homogeneous part of R of degree k. Remark that

$$(f_*\omega^k)_p \otimes_{\mathcal{O}_p} \mathbb{C} = R_k,$$

so, by flatness, $\oplus_k (f_*\omega^k)_p = \mathcal{O}_p[x_1, x_2, x_3, y]/ < \overline{r}_1, \overline{r}_2 >$, where the $\overline{r}_i$'s are lifts to $\mathcal{O}_p$ of the r_i's.

Moreover, every syzygy of R lifts to a syzygy of the $\mathcal{O}_p$-module $\oplus_k (f_*\omega^k)_p$.

Every lift of r_1 can be written as

$$\overline{r}_1 = \overline{Q}(x_i, t) + F(t)y$$

where $\overline{Q}(x_i, 0) = Q(x_i)$, and $F(0) = 0$. Remark that the assumption that the generic fibre is non hyperelliptic imposes $F \not\equiv 0$. Let s be the multiplicity of F in 0.

For a suitable change of the local parameter t in $\mathcal{O}_p$, we can write

$$\overline{r}_1 = \overline{Q}'(x_i, t) + t^s y$$

for some $s > 0$, $\overline{Q}'(x_i, 0) = Q(x_i)$.

This allows us to compute, using the lift of r_2 to eliminate the multiples of y^2, that the set $\{t^i q_j y | i < s\}$ is a basis for the stalk of $\mathcal{T}_k$ in p when the set $\{q_j\}$ is a basis for the homogeneous part of degree $k-2$ of the quotient ring $\mathbb{C}[x_1, x_2, x_3]/Q$.

□

LEMMA 1.2.3. *Let B be a smooth curve and $f : \tilde{S} \to B$ be a genus 4 fibration whose generic fibre is non hyperelliptic. Let $\tilde{M}$ be a fibre of f, set $\omega = \tilde{M} + K_{\tilde{S}}$.*

Let $p \in B$ be the image of a honestly hyperelliptic fibre; then

i) $\mathcal{T}_n$ is a torsion sheaf whose support does not contain the image of the 3-connected non-hyperelliptic fibres.

ii) there is a positive integer s such that

$$\forall k \geq 2 \ \ length\ (\mathcal{T}_k, p) = s(3k-4).$$

Proof.

i) Follows from lemma 1.1.3 and proposition 1.2.1.

ii) If $\tilde{M}_p$ is a honestly hyperelliptic fibre, then $\tilde{M}_p$ is a double cover of $\mathbb{P}^1$ branched, by Hurwitz formula, on a divisor of degree 10 . We can embed $\tilde{M}_p$ in $\mathbb{P}(5,1,1)$ as the hypersurface defined by the equation $w^2 = P(t_0, t_1)$, where P is the homogeneous polynomial of degree 10 whose divisor is the branch divisor of the canonical map.

Following the same line of [**ML**] it is easy to prove that the canonical ring of $\tilde{M}_p$ is generated in degree 2; this ring can be described as a subring of the ring $\mathcal{A} = \mathbb{C}[t_0, t_1, w]/ < w^2 = P(t_0, t_1) >$.

In fact, generators for $H^0(\omega)$ are $y_0 = t_0^3$, $y_1 = t_0^2 t_1$, $y_2 = t_0 t_1^2$, $y_3 = t_1^3$; the kernel of the map $S^2(H^0(\tilde{M}_p, \omega)) \to H^0(\tilde{M}_p, \omega^2)$ has dimension 3 (three independent quadrics through a twisted cubic), so the cokernel has dimension $9 - 10 + 3 = 2$, and we can see that it is generated by $v_0 = t_0 w$, $v_1 = t_1 w$.

It follows that, if we choose 3 degree 4 polynomials $P_{00}(y_i)$, $P_{01}(y_i)$, $P_{11}(y_i)$, s.t. in the ring $\mathcal{A}$ is $P_{00} = t_0^2 P$, $P_{01} = t_0 t_1 P$, $P_{11} = t_1^2 P$, we get the following 9 relations:

$$\begin{array}{lll} r_1 := y_1^2 - y_0 y_2 & r_2 := y_2^2 - y_1 y_3 & r_3 := y_0 y_3 - y_1 y_2 \\ r_4 := v_0 y_1 - v_1 y_0 & r_5 := v_0 y_2 - v_1 y_1 & r_6 := v_0 y_3 - v_1 y_2 \\ r_7 := v_0^2 - P_{00} & r_8 := v_0 v_1 - P_{01} & r_9 := v_1^2 - P_{11}. \end{array}$$

So we can describe the canonical ring R of $\tilde{M}_p$ as a quotient of the graded ring $\mathbb{C}[y_0, y_1, y_2, y_3, v_0, v_1]/ < r_1, \ldots, r_9 >$, where deg $y_i = 1$, deg $v_i = 2$. We have $H^0(\tilde{M}_p, \omega) = 4$, and $\forall k \geq 2$ $H^0(\tilde{M}_p, \omega^k) = 6k-3$ but on the other hand an easy calculation yields that the homogeneous part of degree k of our ring has at most the same dimension. Therefore follows that $R = \mathbb{C}[y_0, y_1, y_2, y_3, v_0, v_1]/ < r_1, \ldots, r_9 >$.

Let us denote by R_k the homogeneous part of R of degree k.

As in the previous proof, we can remark that

$$(f_* \omega^k)_p \otimes_{\mathcal{O}_p} \mathbb{C} = R_k,$$

so, by flatness, $\oplus_k (f_* \omega^k)_p = \mathcal{O}_p[y_0, y_1, y_2, y_3, v_0, v_1]/ < \overline{r}_1, \ldots, \overline{r}_9 >$, where the $\overline{r}_i$'s are lifts to $\mathcal{O}_p$ of the r_i's.

Moreover, every syzygy of R lifts to a syzygy of the $\mathcal{O}_p$-module $\oplus_k (f_* \omega^k)_p$.

Let t be a local parameter for $\mathcal{O}_p$, and write

$$\begin{pmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \overline{r}_3 \end{pmatrix} = \begin{pmatrix} \tilde{q}_1(y_j,t) \\ \tilde{q}_2(y_j,t) \\ \tilde{q}_3(y_j,t) \end{pmatrix} + \begin{pmatrix} \tilde{\alpha_{11}}(t) & \tilde{\alpha}_{12}(t) \\ \tilde{\alpha_{21}}(t) & \tilde{\alpha}_{22}(t) \\ \tilde{\alpha_{31}}(t) & \tilde{\alpha}_{32}(t) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix},$$

where $\tilde{q}_i(y_j,0) = r_i$, and $\tilde{\alpha}_{i,j}(0) = 0$. Let s be the minimum of the orders of vanishing of the $\tilde{\alpha}_{ij}(t)$'s; we can then find a new basis for the respective vectors spaces generated by v_0 and v_1, and by $\overline{r}_1, \overline{r}_2, \overline{r}_3$, and a new local parameter t, so that we can write our relations in the following simpler form:

$$\begin{pmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \overline{r}_3 \end{pmatrix} = \begin{pmatrix} q_1(y_j,t) \\ q_2(y_j,t) \\ q_3(y_j,t) \end{pmatrix} + \begin{pmatrix} t^s & t^{s+1}\alpha_{12}(t) \\ t^{s+1}\alpha_{21}(t) & t^s\alpha_{22}(t) \\ t^{s+1}\alpha_{31}(t) & t^s\alpha_{32}(t) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

Clearly then the linear space of conics generated by the $q_i(y_j,0)$'s coincides with the space generated by the r_i's $(i = 1,2,3)$.

Lifting the syzygy $y_3r_1 + y_1r_2 + y_2r_3$, by degree reasons we get a syzygy of the form $L_3(y_j,t)\overline{r}_1 + L_1(y_j,t)\overline{r}_2 + L_2(y_j,t)\overline{r}_3 + f_4(t)\overline{r}_4 + f_5(t)\overline{r}_5 + f_6(t)\overline{r}_6$, where the $L_i(y_j,0)$'s are three independent linear forms, and $\overline{r}_4$, $\overline{r}_5$, $\overline{r}_6$ are lifts of r_4, r_5, r_6.

Working modulo the ideal generated by t^{s+1} and by the monomials of degree 3 in the (y_j)'s we get

$$t^s(L_3(y_j,0)v_0 + (\alpha_{22}(0)L_1(y_j,0) + \alpha_{23}(0)L_2(y_j,0))v_1) \in (\overline{r}_4, \overline{r}_5, \overline{r}_6)$$

But in fact, there are no constant coefficients syzygies among r_4, r_5, r_6, thus we conclude that

$$L_3(y_j,0)v_0 + (\alpha_{22}(0)L_1(y_j,0) + \alpha_{23}(0)L_2(y_j,0))v_1 \in (r_4, r_5, r_6)$$

which excludes the possibility that $\alpha_{22}(0) = \alpha_{23}(0) = 0$.

Therefore, choosing new bases for the respective $\mathcal{O}_p$-modules generated by v_0 and v_1, and by $\overline{r}_1, \overline{r}_2, \overline{r}_3$, we can write our relations in the following even simpler form:

$$\begin{pmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \overline{r}_3 \end{pmatrix} = \begin{pmatrix} q_1(y_j,t) \\ q_2(y_j,t) \\ q_3(y_j,t) \end{pmatrix} + \begin{pmatrix} t^s & 0 \\ 0 & t^s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

This allows us to compute, using the lifts of r_7, r_8, r_9 to eliminate the multiples of v_0^2, v_0v_1, v_1^2, and the lifts of r_4, r_5, r_6 to eliminate the multiples of v_0

as much as possible, that there exists a nonzero linear form $L_0(y_j)$ such that the set $\{t^i v_0 L_0^{k-2}(y_j), t^i v_1 q_l(y_j) | i < s\}$ is a basis for $(\mathcal{T}_k)_p$, when $\{q_l\}$ is a basis for the homogeneous degree $k-2$ part of $\mathbb{C}[y_0, y_1, y_2, y_3]/ < r_1, r_2, r_3 >$. But this is the projective coordinate ring of a twisted cubic, whose homogeneous part of degree d has dimension $3d+1$; whence the dimension of $(\mathcal{T}_k)_p$ equals $s(1+3(k-2)+1) = s(3k-4)$.

□

The integer s arising in lemmas 1.2.2 and 1.2.3 can in fact be defined as follows:

Definition 1.2.4. *Let C be a honestly hyperelliptic curve of genus g, occurring as a fibre of a genus g fibration with generic fibre non hyperelliptic $f : \tilde{S} \to B$ where $\tilde{S}$ is smooth. Define the* **multiplicity** *of C (or mult(C)), as the multiplicity of C in the conductor ideal of f (recall that the conductor is a divisorial ideal).*

In order to prove that this integer equals the previous one, we need some local study.

Proposition 1.2.5. *Let C be a honestly hyperelliptic genus $g \leq 4$ fibre occurring as a fibre of a fibration $f : \tilde{S} \to B$ whose generic fibre is non-hyperelliptic; $\varphi_\omega : \tilde{S} \to \mathbb{P}^{g-1}$ a morphism inducing on every fibre the canonical morphism, φ the map induced by φ_ω and f on the suitable product. Let $\Gamma = \varphi(C)$, $Y = \varphi(\tilde{S})$.*

Assume that the multiplicity of C equals s: then, in the neighbourhood of a general point $p \in \Gamma$, if $g = 4$, there exist local coordinates (y_1, y_2, y_3, t), such that Y is defined by the equations $y_2 = y_1(y_1 - t^s) = 0$, Γ by $y_1 = y_2 = t = 0$, whence if $g = 3$, there exist local coordinates (y_1, y_3, t), such that Y is defined by the equations $y_1(y_1 - t^s) = 0$, Γ by $y_1 = t = 0$, and the projection π_2 is (still) given by the coordinate t.

Proof.

We prove the statement only in the case $g = 4$; the proof in case $g = 3$ is identical (just forget the coordinate y_2).

Near a the general $p \in \Gamma$ for a suitable choice of a neighbourhood U of p in Y we can see U embedded in $\mathbb{C}^3 \times \mathbb{C}$, such that $\varphi^{-1}(U)$ has two smooth connected components, and φ identifies the two smooth holomorphic curves corresponding to C.

So, for a first suitable choice of local coordinates in the source and in the target, we can assume that $\Gamma = \{y_1 = y_2 = t = 0\}$, the projection π_2 is given by the coordinate t, and the two branches of Y are parametrized as follows

$$\begin{array}{ccc} (u_1, t_1) & \to & (0, 0, u_1, t_1) \\ (u_2, t_2) & \to & (t_2\phi_1(u_2, t_2), t_2\phi_2(u_2, t_2), u_2, t_2). \end{array}$$

So, for a suitable local analytic coordinate change that fixes t, we get the simpler form

$$\begin{array}{ccc} (u_1, t_1) & \to & (0, 0, u_1, t_1) \\ (u_2, t_2) & \to & (t_2^a, 0, u_2, t_2). \end{array}$$

And Y is described by the equations $y_2 = y_1(y_1 - t^a) = 0$.

Finally, remarking that the conductor ideal is generated by y_1, t^a, we get $a = s$.

□

The following corollary is useful for the adjunction computations on Y.

COROLLARY 1.2.6. *Assume that a fibre $F = \{t = 0\}$ appears in the conductor divisor with multiplicity s. Thus, if $Q(y_i, t)$ represents a divisor in $\mathbb{P}^{g-1} \times \mathbb{P}^1$ s.t. $\varphi^* div(Q(y_i, t)) \geq 2sF$, then $t^s | Q(y_i, t)$ (mod $\mathcal{I}_Y$).*

Proof.

By our assumption div $Q(y_i, t)$ pulls back to a divisor $\geq 2sF$.

Since we are interested in Q mod $\mathcal{I}_Y$, using the local coordinates introduced above, in case $g = 4$ we can look at Q modulo y_2, and writing $Q'(y_1, y_3, t) = Q(y_1, 0, y_3, t)$ we have no more differences in the two cases.

The first condition that we get is

$$1) \; Q' \in (y_1, t^{2s}),$$

i.e. $Q' = y_1 q' + t^{2s} g$ and it suffices therefore to prove that $t^s | q'$.

The condition imposed by the second branch is that $t^{2s} | t^s q'(t^s, v, t)$, i.e. $q' \in (y_1, t^s)$. Thus, mod $\mathcal{I}_Y$, $Q' \equiv y_1^2 a + y_1 t^s b + t^{2s} g$.

But $\mathcal{I}_Y \ni y_1(y_1 - t^s)$ and thus $Q' \equiv t^s(y_1(a + b) + t^s g)$.

□

Now let us come back to the multiplicity.

PROPOSITION 1.2.7. *The integer s associated to a honestly hyperelliptic fibre C as in lemmas 1.2.2 and 1.2.3 equals the multiplicity.*

Proof.

Let $p \in B$ be a point such that C is the fibre of p and U a sufficiently small affine open neighbourhood of p. Let $Y \subset \mathbb{P}^{g-1} \times U$ be the image of map φ induced by the linear system $K + C$.

By abuse of notation let us still denote by $\tilde{S} = f^{-1}(U)$. The sheaf of double points Δ, supported on the image Γ of C is defined via the exact sequence

$$0 \to \mathcal{O}_Y \to \varphi_* \mathcal{O}_{\tilde{S}} \to \Delta \to 0.$$

Twisting the exact sequence by $\mathcal{O}_{\mathbb{P}^{g-1} \times \mathbb{P}^1}(n, 0)$, and observing that $\varphi^* \mathcal{O}_{\mathbb{P}^{g-1} \times \mathbb{P}^1}(1, 0) \cong \omega$, from the definition of $\mathcal{T}_n$ we get that

$$(\mathcal{T}_n)_p \cong H^0(\Gamma, \Delta(n)).$$

From lemmas 1.2.2 and 1.2.3 we conclude that the length of Δ at the generic point of Γ equals s. Since (as we saw in lemma 1.2.5) at the general point of Γ we have a singularity consisting of two smooth branches, we conclude immediately that s equals the multiplicity of C in the conductor divisor.

□

The geometric meaning of definition 1.2.4 and proposition 1.2.7 above is that s should be interpreted as the intersection multiplicity of the curve B with the hyperelliptic locus inside the moduli space of curves of the correct genus.

CHAPTER 2

The numerical Godeaux surfaces

2.1. The canonical ring

Let S be a numerical Godeaux surface, i.e. a minimal surface of general type with $K_S^2 = 1$, $p_g(S) = q(S) = 0$.

Recall that the canonical ring of S (cf. [**Mu1**]) is defined as the graded ring

$$R(S) := \oplus_{n \in \mathbb{N}} H^0(nK_S).$$

In our case of numerical Godeaux surfaces we have

$$h^0(K_S) = 0;\ \forall n \geq 2\ \ h^0(nK_S) = \binom{n}{2} + 1.$$

Let us look for a minimal system of generators of this ring (as a $\mathbb{C}$-algebra).

As usual we denote by 1 the identity of $R(S)$, given by the constant function equal to 1, moreover we fix a basis $\{x_0, x_1\}$ of $H^0(2K_S)$, and a basis $\{y_0, y_1, y_2, y_3\}$ of $H^0(3K_S)$.

We remark that x_0^2, x_0x_1, x_1^2 are independent in $H^0(4K_S)$, since $R(S)$ is an integral domain; whence, we can complete these elements to a basis $\{x_0^2, x_0x_1, x_1^2, v_0, v_1, v_2, v_3\}$ of $H^0(4K_S)$.

Let $X := Proj(R(S))$ be the canonical model of S, and let $\pi : S \to X$ be the natural map; X has an invertible dualizing sheaf and, as customary, we denote by K_X an associated Cartier divisor. Since $\pi^*(K_X) = K_S$, one has a natural isomorphism between the canonical rings $R(S)$ and $R(X)$.

LEMMA 2.1.1. *The fixed part F of the bicanonical pencil $|2K_S|$ is supported on the fundamental cycles of S (normal crossing configurations of smooth rational curves with self-intersection -2).*

In particular $|2K_X|$ has no fixed part.

This was already proved in [**Mi1**] with similar argument.

Proof.

We can write $|2K_S| = |M| + F$ where M is a linear pencil without fixed components; since K_S is nef and the only curves with $K_S C = 0$ are the finitely many smooth rational (-2) curves, building the so called fundamental cycles (cf. [**Bo**], [**BPV**]), we know that $K_S M > 0$, $K_S F \geq 0$.

Since $K_S M + K_S F = 2K_S^2 = 2$ we get $0 < K_S M \leq 2$, and clearly our purpose is to show that $K_S F = 0$, equivalently $K_S M = 2$.

Assume by contradiction that $K_S M = 1$. M being a pencil without fixed part, we have $M^2 \geq 0$, but $M^2 + K_S M = 0 \pmod 2$. It follows then that $M^2 = 1$, whence equality holds in the inequality given by algebraic index theorem.

Our conclusion is thus that M is numerically equivalent to K_S, and since $h^1(\mathcal{O}_S) = 0$ but $h^0(K_S) = 0 \neq h^0(M) = 2$, $M - K_S = \mu$ yields a non zero torsion element μ in Pic S.

An easy calculation ($\chi(M) = \chi(K_S) = 1$, $h^0(M) = 2 \Rightarrow 1 \leq h^1(M) = h^1(K_S + \mu) = h^1(-\mu)$) shows that the covering of S induced by μ, yields an irregular covering of S. This is a contradiction, because the equality $K_S^2 = \chi(S)$ holds for S, hence for all its unramified coverings , whereas for minimal irregular surfaces Y we have the inequality $K_Y^2 \geq 2\chi(Y)$ (cf. [**Bo**]).

□

REMARK 2.1.2. *We have seen that $KF = 0$, $KM = 2$. So, we have three possibilities for F and M, namely*

i) $M^2 = 4 \quad F = 0$
ii) $M^2 = 2 \quad MF = 2 \quad F^2 = -2$
iii) $M^2 = 0 \quad MF = 4 \quad F^2 = -4.$

In the second case F is precisely a fundamental cycle, i.e., on the canonical model X, we get in the base point scheme a reduced singular point.

LEMMA 2.1.3. *$H^0(2K) \otimes H^0(3K) \to H^0(5K)$ is injective.*

Proof.

Otherwise we would have a relation $x_0 y = x_1 y'$ for suitable elements y, y' in $H^0(3K)$. By lemma 2.1.1, on X $\min(div(x_0), div(x_1)) = 0$; whence, $div(x_0) < div(y')$ and therefore the rational section y'/x_0 of $3K_X - 2K_X = K_X$is a regular section, contradicting $p_g(X) = 0$.

□

COROLLARY 2.1.4. *We can fix a basis of $H^0(5K)$ of the form*

$$\{x_i y_j, w_1, w_2, w_3\}.$$

Let us now consider the polynomial ring $A := \mathbb{C}[y_0, y_1, y_2, y_3]$, and let us look for a set of generators of $R(S)$ as $A-$module, ($R(S)$ is an $A-$algebra via the natural homomorphism $A \to R(S)$).

Define

$$R^{(0)} = \oplus_{n\geq 0} H^0(3nK_S),$$

$$R^{(1)} = \oplus_{n\geq 0} H^0((3n+1)K_S),$$

$$R^{(2)} = \oplus_{n\geq 0} H^0((3n+2)K_S),$$

Of course, there is a splitting (as $A-$modules) $R(S) = R^{(0)} + R^{(1)} + R^{(2)}$.

THEOREM 2.1.5. *There are three resolutions*

$$0 \to A(-3)^7 \stackrel{\alpha}{\to} A \oplus A(-2)^6 \to R^{(0)} \to 0$$

$$0 \to A(-4) \oplus A(-2)^6 \stackrel{\alpha^t}{\to} A(-1)^7 \to R^{(1)} \to 0$$

$$0 \to A(-3)^2 \oplus A(-2)^3 \stackrel{\beta}{\to} A^2 \oplus A(-1)^3 \to R^{(2)} \to 0$$

where $\beta = \beta^t$.

Proof.

It is an easy exercise following the same argument of [**Cat3**].

□

COROLLARY 2.1.6. $R(S) = R(X)$ *is generated in degree* ≤ 6 *as an* $A-$*module.*

REMARK 2.1.7. *In* [**Ci**] *is shown the weaker result that* $R(S) = R(X)$ *is generated in degree* ≤ 6 *as a ring.*

From now on, let us assume $\mathrm{Tors}(S) = 0$ or $\mathbb{Z}/2\mathbb{Z}$. In this last case, denote by μ the nonzero torsion element in Pic (S).

Under this assumption, we can prove

PROPOSITION 2.1.8. $R(S) = R(X)$ *is generated as a ring in degree* ≤ 5.

Proof.

By corollary 2.1.6, we must only prove that every section of $H^0(6K_X)$ can be written as sum of products of sections of degree ≤ 5.

Take an effective divisor $C \in |2K_X|$, and let $C = div(c)$; then

$$H^0(6K_X) \supset W_C := cH^0(4K_X) + S^2(H^0(3K_X)).$$

We will prove that there exists some C s.t. this inclusion is an equality. Since $H^1(4K) = 0$ we get the following exact sequence

$$0 \to H^0(\mathcal{O}_X(4K)) \stackrel{C\cdot}{\to} H^0(\mathcal{O}_X(6K)) \stackrel{\pi}{\to} H^0(\omega_C^2) \to 0.$$

By definition W_C contains Ker π; whence, it suffices to show that there exists C s.t. $\pi : S^2(H^0(3K_X)) \to H^0(\omega_C^2)$ is surjective; this is equivalent to the surjectivity of $\pi|_C : S^2(H^0(\omega_C)) \to H^0(\omega_C^2)$, that is verified for every C irreducible and non hyperelliptic by Noether's theorem.

It is clear that the general C is irreducible (since we have no fixed part, and $|M|$ is a linear pencil with $h^0(\mathcal{O}_S(M)) = 2$). That the general C is non-hyperelliptic follows from the forthcoming proposition 2.2.3.

□

2.2. The bicanonical fibration

In this section we study the fibration on a numerical Godeaux surface induced by the bicanonical pencil. We have already denoted the movable part of the bicanonical system by $|M|$.

Let $\beta : \tilde{S} \to S$ be a minimal sequence of ordinary blow ups such that $f := \varphi_{2K} \circ \beta$ is a morphism; the fibre of f is $\tilde{M}$, the strict transform of $|M|$ on $\tilde{S}$.

As we have already shown, by $p_g(\tilde{S}) = q(\tilde{S}) = 0$ we can consider the map $g : \tilde{S} \to \mathbb{P}^{g(\tilde{M})-1} \times \mathbb{P}^1$ induced by $|K_{\tilde{S}} + \tilde{M}|$ and $\tilde{M}$, and define $Y := g(\tilde{S})$.

Denote by π_1, π_2 the two respective projections of Y on $\mathbb{P}^{g(\tilde{M})-1}$ and $\mathbb{P}^1$; $f = \pi_2 \circ \varphi \circ \beta : \tilde{S} \to \mathbb{P}^1$.

By lemma 2.1.1 and remark 2.1.2 we immediately see that we have to study 4 cases:

ia)	$M^2 = 4$	$F = 0$	$\tilde{M}$ has genus 4	$\lvert M\rvert$ has 4 base points	
ib)	$M^2 = 4$	$F = 0$	$\tilde{M}$ has genus 3	$\lvert M\rvert$ has 1 double base point	
ii)	$M^2 = 2$	$MF = 2$	$F^2 = -2$	$\tilde{M}$ has genus 3	
iii)	$M^2 = 0$	$MF = 4$	$F^2 = -4$	$\tilde{M}$ has genus 2.	

In fact, while in cases ii) and iii) (by $M^2 \leq 3$) the generic divisor in M is smooth in the fixed points, in case i) ($M^2 = 4$) we can have one single base point where every divisor of M has a double point. In the last case the genus of $\tilde{M}$ drops by one.

The situation can be summarized in the following diagram:

$$\begin{array}{ccccc} \tilde{S} & & \Sigma & \subset & \mathbb{P}^{g(\tilde{M})-1} \\ \hat{\beta}\downarrow & \searrow^{g} & \uparrow\pi_1 & & \\ S & \dashrightarrow & Y & \subset & \mathbb{P}^{g(\tilde{M})-1}\times\mathbb{P}^1 \\ & & \downarrow\pi_2 & & \\ & & \mathbb{P}^1 & & \end{array}$$

LEMMA 2.2.1. *The only possible decompositions of a bicanonical divisor as sum of divisors A and B with $AB < 3$ are of the following form:*

1) C is 2−connected, $C = D_1 + D_2$, $K_S D_1 = 0$, $D_1^2 = -2$, $D_1 D_2 = 2$

2) Tors$(S) = \mathbb{Z}/2\mathbb{Z}$, $C = 2D$, $D \equiv K_S$.

Proof.

If C is not 3−connected we have a decomposition $C = D_1 + D_2$ with $D_1 D_2 \leq 2$ and either $\forall i\ K_S D_i = 1$, or we can assume $K_S D_1 = 0$, $K_S D_2 = 2$.

In the last case $D_1^2 = -D_1 D_2 = -2$ and we get case ii).

Otherwise we get then $D_1^2 + D_2^2 = (2K_S)^2 - 2D_1 D_2 \geq 0$, so we can assume D_1^2 non negative, whence positive because it must be odd; by the algebraic index theorem $D_1^2 = 1$, and $D_1 = K_S + \varepsilon$, $D_2 = K_S - \varepsilon$, $\varepsilon \in$ Tors(S).

□

PROPOSITION 2.2.2. *In every case* (ia),ib),ii),iii)) *f is a fibration*

Proof.

By the Stein factorisation, if f is not a fibration, then it is composed with a (rational or irrational) pencil. By $q(S) = 0$, there are not irrational pencil on S; then if f is not a fibration, its fibres describe a pencil in $|nD|$, with $n \geq 2$, $h^0(\mathcal{O}(D)) \geq 2$, then $h^0(\mathcal{O}(nD)) \geq 3$ whence we know that the fibres of f describe a complete linear system.

□

PROPOSITION 2.2.3. *In cases* ia), ib) *and* ii) *the generic fibre of f is not hyperelliptic.*

Remark that in the case iii) the genus of the fibration is 2, so every fibre is hyperelliptic.

Proof.

In the case ia) the restriction of β to every element in the linear system $|\tilde{M}|$ is an isomorphism onto the corresponding element of $|2K_S|$. So we need to show that there exists $C \in |2K_S|$ on which the dualizing sheaf ω_C is very ample. But $\omega_C = 3K_{|C}$ and $p_g = q = 0$ implies that the rational map induced by ω_C is exactly the restriction to C of the tricanonical map. So, if every fibre would be hyperelliptic, the tricanonical map would be not birational, contradicting [**Cat1**].

In cases ib) and ii) the morphism induced by ω is given composing the tricanonical morphism with the projection on $\mathbb{P}^2$ with center respectively the image of the unique base point (case ib)) or the (singular) point image of the fundamental cycle that gives the fixed part of $|2K_S|$ (case ii)).

Let us restrict now to the case ib). If every divisor in $\tilde{M}$ were hyperelliptic, we would have an involution $\tilde{\imath}$ on $\tilde{S}$ fixing every divisor in $|\tilde{M}|$. If E is the exceptional divisor of β, being the only (-1)-curves in $\tilde{S}$, we can conclude $\tilde{\imath}(E) = E$, and $\tilde{\imath}$ would induce an involution $i : S \to S$ that acts trivially on the bicanonical pencil, still contradicting the birationality of the tricanonical morphism.

Finally, in case ii), if every divisor in $\tilde{M}$ were hyperelliptic, we would have an involution i on S fixing every divisor in $|2K_S - F|$. F is the only fundamental cycle in S with strictly positive intersection with $|2K_S - F|$, so $i(F) = F$, and again we get an involution of S that acts trivially on the bicanonical pencil.

□

2.3. The fibres

In this section we will have a more careful look to what kind of canonical images we can obtain when the genus of the fibration is at least 3. Case iii) will be excluded in section 2.5.

Let us recall (cf. [**Cat2**]) that a honestly hyperelliptic curve is a finite covering of degree 2 of $\mathbb{P}^1$.

Let us start with the "harder" case.

LEMMA 2.3.1. *Let $C \in |2K_S|$, S Godeaux surface of type* ia) *and Tors(S)= $\{0\}$ or $\mathbb{Z}/2\mathbb{Z}$. Then one of the following holds:*

a) *C is embedded by ω_C, $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is the complete intersection of a quadric and a cubic; moreover, if $\varphi_{\omega_C}(C)$ is reducible, it decomposes as the union of two plane cubics intersecting (with multiplicity) in three points;*

b) *C is honestly hyperelliptic, $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is a double twisted cubic curve;*

c) *$C = 2D$; in this case Tors(S)=$\mathbb{Z}/2\mathbb{Z}$, $D \in |K + \mu|$, and $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is a sextuple line.*

Case a) *is the general one.*

Proof.

Let us consider first the case where the image of C in the pluricanonical model is not 3-connected. Then, by [**CFHR**], lemma 4.2 and its proof, C is not 3−connected, and we have a decomposition $C = D_1 + D_2$ with $D_1 D_2 \leq 2$ and with $K_S D_i = 1$.

As we showed in lemma 2.2.1, we get then $D_1^2 = 1$, and $D_1 = K_S + \varepsilon,\ D_2 = K_S - \varepsilon,\ \varepsilon \in \mathrm{Tors}(S)$.

Since $H^0(K_S) = 0$, by our hypothesis on the torsion group follows that $\mathrm{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$, and that $\varepsilon = \mu = -\mu$.

Remark that $h^0(K + \mu) = 1$ (cf. [**R3**]) , whence $D_1 = D_2 \in |K_S + \mu|$.

Since $h^0(3K - (K + \mu)) = h^0(2K - \mu) = h^0(2K + \mu) = 2$, then $\varphi_{|3K|}(D_1)$ is a curve of degree $(K_S + \mu)3K_S = 3$ contained in a line, thus it is a triple line. This gives case c).

Now we can assume C 3−connected (this is true at least in the canonical model); if C is 3−connected, by [**CFHR**], theorem 3.6, either $\omega_C = 3K_{|C}$ is very ample or C is honestly hyperelliptic. Note that if C is honestly hyperelliptic and reducible, then C consists of two smooth rational curves intersecting (with multiplicity) in 5 points. In this case $\varphi_{\omega_C}(C)$ is an irreducible non degenerate curve of degree 3 in $\mathbb{P}^3$, so its schematic image is a double structure on a twisted cubic curve.

Assume now that C is canonically embedded: by lemma 1.1.3 $\varphi_{\omega_C}(C)$ is a complete intersection of type $(2, 3)$.

Finally, if C is reducible, $C = C_1 + C_2$, where C_1, C_2 are irreducible and $C_1 \neq C_2$ by the hypothesis of 3-connectedness (else, $C_1 C_2 = 1$). Since $K_S \pi^*(C_i) = 1$, $\varphi_{\omega_{\pi^*(C)}}(C_i)$ is a plane curve of degree $3K_S \pi^*(C_i) = 3$, and we get thus two distinct irreducible plane cubics intersecting in three points.

Remark that case a) is the general by proposition 2.2.3.

□

LEMMA 2.3.2. *Let S be a Godeaux surface of type* ib) *or* ii), $\beta : \tilde{S} \to S$ *the sequence of blow ups we have already defined, $C \in |\tilde{M}|$ strict transform of the bicanonical system, Tors(S)= $\{0\}$ or $\mathbb{Z}/2\mathbb{Z}$.*

Then one of the following holds:

a) *C is embedded by ω_C, $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is a plane quartic.*

b) *C is hyperelliptic, $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ is a (double) plane conic;*

c) $C = 2D$; *in this case Tors(S)=*$\mathbb{Z}/2\mathbb{Z}$, $D \in |K + \mu|$, *and* $\varphi_{|3K|}(C) = \varphi_{\omega_C}(C)$ *is a (quadruple) line.*

Case a) *is the general one.*

Proof.

First, we consider the case when C is the strict transform of the double bicanonical divisor.

In case ib), $C = 2D$, $D = \beta^*(K_{\tilde{S}} + \mu) - E$, E exceptional divisor for β. D maps on a curve of degree 4, and $h^0(K_{\tilde{S}} + \tilde{M} - D) = h^0(\beta^*(2K_{\tilde{S}} + \mu) - E) = 1$, so the image of D a line.

In case ii), the blow up is in smooth points, so we can consider C as a double divisor in S, and one can easily show $h^0(2K_S + \mu - F) = 1$.

Now, by lemma 2.2.1, we can assume that, if the strict transform of C can be decomposed as $A + B$ with $AB \leq 2$, then $AB = 2$ and A is supported on the fundamental cycles.

If C is 2− connected, we have done by classification of genus 3 fibres (see [**ML**]).

In the case ib), every divisor in $|\tilde{M}|$ induced by a 3−connected divisor in $|2K_S|$ is 2−connected. In fact, consider $C \in \tilde{M}$ and D the corresponding divisor in $|2K_S|$; $C = \beta^*D - 2E$, with E exceptional divisor for β. For a decomposition $C = \tilde{A} + \tilde{B}$, we can find a decomposition $D = A + B$ such that either $\tilde{A} = \beta^*A$, $\tilde{B} = \beta^*B - 2E$ or $\tilde{A} = \beta^*A - E$, $\tilde{B} = \beta^*B - E$. In first case $\tilde{A}\tilde{B} = AB$, whence in the second $\tilde{A}\tilde{B} = AB - 1$, so the 3−connectedness of D implies the 2−connectedness of C.

We are left, by lemma 2.2.1, with the case in which the bicanonical divisor associated to C is of the form $A + Z$ with Z supported on the fundamental cycle. But by the same argument, in this case C fails to be 2−connected only if, if we denote by p the point blown up from β, p is contained in both A and Z. This would induce a singular base point for the bicanonical system on the canonical model, a contradiction.

In the case ii), recall that β is a sequence of two blow ups and $E = K_{\tilde{S}} - \beta^*(K_S)$. One can easily show, by intersection arguments and Zariski lemma, that the only exceptional divisor for β contained in a divisor in $\tilde{M} = \beta^*(2K_S - F) - E$, is the exceptional divisor with self-intersection -2 (this case holds only if β is a blow up in two points infinitely near).

Then, assume $C \in \tilde{M}$ not 2−connected, $C = \tilde{A} + \tilde{B}$. $\tilde{A}\tilde{B} \leq 1$. By previous remark neither $\tilde{A}$ nor $\tilde{B}$ are supported on the exceptional locus. Then we can find two (nonzero) divisor A and B on S, such that $A+B \in |2K_{\tilde{S}} - F|$, $\tilde{A} = \beta^*A - E_1$, $\tilde{B} = \beta^*B - E_2$, $E_1 + E_2 = E$, E_i possibly zero, and with negative part supported on the eventual -2-curve.

Now one can easily show that then $E_1E_2 \geq 0$, so $\tilde{A}\tilde{B} = AB + E_1E_2 \geq AB$.

Then, if C is not 2-connected, its image is a not 2-connected divisor $D \in |2K_S - F|$. Write $D = A + B$, $AB \leq 1$. Recall that, by lemma 2.2.1, $D + F$ is 2-connected, then $A(B+F) \geq 2$, so $AF \geq 1$, and, of course, also $BF \geq 1$. But $AF + BF = (A+B)F = -F^2 = 2$, then $AF = BF = 1$, and, again by $A(B+F) \geq 2$, $AB = 1$.

So, $A(B+F) = 2$, and, again by lemma 2.2.1, either $KA = 0$ or $KB = K(B+F) = 0$; but F is a fundamental cycle, so it has negative self-intersection with every (-2)-curve, and again we get a contradiction.

□

Remark now that, using propositions 1.2.3 and 1.2.2, when the fibration has genus at least 3, we are able to compute the length of the sheaves $\mathcal{T}_l$ in all cases except for curves of type c, that occurs only in case $\mathrm{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$. Moreover, looking at the proof of proposition 1.1.6 we have that in case of torsion $\{0\}$, $Z = Y$, whence in case of torsion $\mathbb{Z}/2\mathbb{Z}$, $Z - Y$ is supported on the special fibre corresponding (again) to the curve of type c.

For such a curve, in cases ib) and ii), we can explicitely describe the canonical ring. Let us denote by $\overline{M}$ this special fibre.

PROPOSITION 2.3.3. *In case* ib) $\overline{M}$ *is a double fibre.*

Proof.

This is an easy remark. The special bicanonical divisor is $2C_\mu$, where $C_\mu \in |K_S + \mu|$, μ non trivial element in the torsion group of S.

Recall that $\beta : \tilde{S} \to S$ is a blow up in a single point p of S, double for every bicanonical divisor. So, p must be smooth for C_μ (otherwise we woul get $(2K_S)^2 > 4$), we can write $\beta^*(C_\mu) = \tilde{C}_\mu + E$, and $\overline{M} = \tilde{C}_\mu$.

□

So, by the classification of genus 3 fibrations in [**ML**], we can explicitely write the canonical ring of $\overline{M}$.

COROLLARY 2.3.4. *In case* ib) *the canonical ring of* $\overline{M}$ *is*

$$\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2, z_1, z_2]/I,$$

where deg $x_i = 1$, deg $y_i = 2$, deg $z_i = 3$, and I is generated by the following list of polynomials

$$x_0^2$$

$$x_0x_1$$

$$x_1^2$$

$$x_0y_0$$

$$x_1y_0$$

$$x_0(y_1 - \lambda y_2) - \delta x_1 x_2^2$$

$$x_1y_1 - x_0y_2$$

$$y_0^2$$

$$x_0(z_1 - \lambda z_2) - \delta x_2^2 y_0$$

$$x_1z_1 - x_0z_2$$

$$y_0y_1 - x_0z_2$$

$$y_0y_2 - x_1z_2$$

$$y_1(y_1 - \lambda y_2) - \delta x_2^2 y_2 - P_0$$

$$y_2z_1 - y_1z_2 - P_1$$

$$y_1(z_1 - \lambda z_2) - \delta x_2^2 z_2 - P_2$$

$$y_0z_1 - x_0Q$$

$$y_0z_2 - x_1Q$$

$$z_1(z_1 - \lambda z_2) - \delta x_2^2 Q - P_3$$

$$z_1z_2 - y_1Q - P_4$$

$$z_2^2 - y_2Q - P_5$$

where $Q = \delta_0 y_2^2 + \delta_1 y_1 y_2 + \delta_2 x_2^2 y_2 + \delta_3 x_2^2 y_1 + \delta_4 x_2^4$, with $\delta_0 \neq 0$, and $\forall i$ $P_i \in (x_2^{n-2}y_0) + x_1\mathbb{C}[x_2, y_2, z_2] + x_0\mathbb{C}[x_2, y_2, z_2]$, where in every case n is the degree of P_i, that is 5 for $i \leq 2$, and 6 otherwise.

Moreover $\delta \neq 0$, $\lambda \neq 0$.

Proof.

By proposition 2.3.3 and classification of genus 3 fibres in [**ML**], we have only to show that $\overline{M}/2$ is $2-$ connected.

But $\overline{M}/2$ is the strict trasform of the divisor C_μ in $H^0(K+\mu)$ and we are blowing up a smooth point of this, so is enough to show thay C_μ is $2-$connected. For every decomposition $C_\mu = A + B$ we can assume $KA = 1$, $KB = 0$, so that B is supported on the fundamental cycles.

Then $AB = (K - B)B = -B^2 \geq 2$.

□

Case ii) is slighty harder.

PROPOSITION 2.3.5. *In case* ii) $\overline{M}$ *is a 1-connected fibre with an analytic decomposition of type I with* $n = 1$.

From the definition of 1-connected fibre with an analytic decomposition of type I with $n = 1$ see [**ML**].

Proof.

In this case $\beta : \tilde{S} \to S$ is a sequence of 2 blow ups. Moreover, the fixed part of $2K_S$ must be a fundamental cycle. $F \subset 2C_\mu$ (where as usual C_μ is the only divisor numerical equivalent to the canonical class), so $F \subset C_\mu$.

Let us define $C := C_\mu - F$; recall that by [**Lan**] F is of type A_n (with $n \leq 3$). So $H^0(2K_S - 2F) = 1$ and the restriction map $H^0(\mathcal{O}_S(2K_S - F)) \to H^0(\mathcal{O}_F(2K_S - F))$ have image of dimension 1, that implies that the fixed points of M are supported on F, and they are obviously smooth for F.

Viceversa, let p base point for $|2K_S - F|$. We know that $2K_S - F = 2C + F$. So, if p were in C, we would have a divisor in $|2K_S - F|$ with multiplicity higher than 3 in p, a contradiction because $(2K_S - F)^2 = 2$. So no base point for $2K_S - F$ are supported on C. In particular $F \not\subset C$.

This allows us to write the special fibre as $\overline{M} = \tilde{C} + (\beta^* F - E)$, where $\tilde{C} := \beta^* C$, E exceptional divisor for β, so $E^2 = K_{\tilde{S}} E = -2$.

Now remark that $\tilde{C}^2 = C_\mu^2 + F^2 = -1$ is an elliptic cycle. We want to prove that $\tilde{C} + (\tilde{C} + (\varepsilon^* F - E))$, is a decomposition as in [**ML**] II.4.2. So, as proved there, is enough to show that:

a) $\tilde{C} + (\varepsilon^* F - E)$ 2-connected

b) for every $(-2)-$curve $\theta \subset \tilde{C}$, $\theta\tilde{C} \leq 0$,

c) If $E_{\tilde{C}}$ is the elliptic tail contained in $\tilde{C}$, then for every $(-2)-$curve $\theta \subset E_{\tilde{C}}$, $\theta\tilde{C} = 0$, The easiest part is part c); if $\theta \subset E_{\tilde{C}}$ than $\theta \subset \tilde{C}$. We have seen that the exceptional divisors are "far" from $\tilde{C}$, so θ is the pull-back of a $(-2)-$curve θ' on S contained in C. So, $\theta\tilde{C} = \theta' K_S - \theta' F = 0$ because θ' is in C while F is a fundamental cycle of type A_1 not in C.

For part b), we have few cases to consider.

If θ is exceptional for β, we have done because $\tilde{C}$ is a pull-back, so $\theta\tilde{C} = 0$

So θ is the strict transform of some curve on S. If θ is a pull-back of some $(-2)-$curve on S not contained in F, then again $\theta\tilde{C} = \theta F = 0$.

Otherwise, θ is the strict trasform of some rational curve θ' on S through the base locus of $2K_S - F$; so θ' had self intersection greater than -2, so it had negative intersection with the canonical class, a contradiction because K_S is nef.

We are left with part a). Remark that C_μ is 2-connected, because if $C_\mu = A + B$, then I can assume $K_S A = 1$, $K_S B = 0$, that implies by index therem $A^2 \leq -1$, $B^2 \leq -2$, so $AB = \frac{1}{2}C_\mu^2 - A^2 - B^2 \geq 2$.

Now, a decomposition for $\tilde{C} + (\beta^* F - E)$ can be written as $(\varepsilon^* A - E') + (\varepsilon^* B - E'')$, with $E' + E'' = E$, and $(\varepsilon^* A - E')(\varepsilon^* B - E'') = AB + E'E''$. So is enough to show that E is 0-connected. But E is contractible, so every divisor supported on that has (strictly) negative self-intersection, and we can conclude, by $E^2 = -2$, $E'^2 \leq -2$, $E''^2 \leq -1$, that $E'E'' \geq 0$.

□

Now, as in the previous case, we can get the canonical ring by [**ML**].

COROLLARY 2.3.6. *In case* ib) *the canonical ring of* $\overline{M}$ *can be written as* $\mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/I$, *where* I *is generated by the* 2×2 *minors of the matrix*

$$\begin{pmatrix} 0 & x_0 & x_1 & y_1 \\ x_0 & A & y_2 & z \end{pmatrix}$$

and by the polinomials

$$y_1(y_1 - x_1(\alpha_0 x_1 + \alpha_1 x_2) - \alpha_2 x_0 x_2) - \lambda x_1(x_1 P_1 + x_2 P_2) - x_0 x_2 Q$$

$$z(y_1 - x_1(\alpha_0 x_1 + \alpha_1 x_2) - \alpha_2 x_0 x_2) - \lambda y_2(x_1 P_1 + x_2 P_2) - x_2 A Q - x_0 H$$

$$z(z - y_2(\alpha_0 x_1 + \alpha_1 x_2) - \alpha_2 x_2 A) - \lambda y_2^2 P_1 - x_2 G Q - A H - x_0 x_2 M$$

where

$$A = x_1^2 + \delta x_2^2 + \gamma y_2 + a_0 x_1 x_2 + a_1 y_1$$

$$G = x_1 y_2 + a_0 x_2 y_2 + a_1 z,$$

with $\lambda, \delta \neq 0$, $\gamma a_0 = \gamma a_1 = a_0 a_1 = 0$, $P_1, P_2, Q \in \mathbb{C}[x_2, y_2]_2$, $M, H \in \mathbb{C}[x_2, y_2]_4$, *and* P_2, Q *satisfy*

$$\lambda y_2 P_2 = -(\delta x_2^2 + \gamma y_2) Q.$$

By $\delta \neq 0$, *last relation induces that there exists* k *s.t.* $Q = k\lambda y_2$, $P_2 = -k(\delta x_2^2 + \gamma y_2)$.

2.4. The numerical invariants

Finally, we are able to compute the fundamental numerical invariants of Y, that means the d_i we defined in lemmas 1.1.5 and 1.1.6, when $\mathrm{Tors}(S) = \{0\}$ and the genus of the fibration is at least 3.

THEOREM 2.4.1. *Assume that* S *is a numerical Godeaux surfaces with torsion* $\{0\}$, *s.t.* f *is a genus* 4 *fibration. Let* $h = \sum_{C \text{ hyperelliptic}} mult(C)$.

Then $\exists \mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$, *s.t.* $Y := \varphi(S)$ *is a divisor in* $|\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$.

Proof.

Recall that the map β we had already defined at the beginning of this section, is a sequence of blow-ups of smooth points of the generic bicanonical divisor.

Let E_i, $i = 1, \ldots, 4$ be the corresponding exceptional divisors of the first kind, and set $E = \sum E_i$. Then $K_{\tilde{S}} = \beta^*(K_S) + E$, and if $\tilde{M}$ is a generic fibre of f (the strict transform of a generic bicanonical divisor on S by β), $\tilde{M} = \beta^*(2K_S) - E = 2K_{\tilde{S}} - 3E$ is a genus 4 curve.

The pull-back of the tricanonical system is given by $\omega := \beta^*(3K_S) = 3K_{\tilde{S}} - 3E = K_{\tilde{S}} + \tilde{M}$.

First we show that the surface Z in proposition 1.1.6 is exactly Y.

We have to show that for every fibre of f, say $\tilde{M}_p$, the quadric and the cubic (never multiple to the quadric) defining the corresponding fibre of $Z \to \mathbb{P}^1$, say Z_p, containing the corresponding fibre of $Y \to \mathbb{P}^1$, say Y_p, have a common component. Remember that Y_p is the canonical image of $\tilde{M}_p$.

But by lemma 2.3.1, either Y_p is complete intersection of a quadric and a cubic or Y_p is a twisted cubic curve; in the first case it is obvious that do not exist a quadric and a cubic not multiple of the quadric containing Y_p and having a common component, and in the second there are not reducible quadrics containing Y_p at all so $Z = Y$.

Now, in order to induce the thesis, by proposition 1.1.6 we have to consider the exact sequences

$$0 \to \mathcal{L}_2 \to \mathcal{S}^2(f_*\omega) \xrightarrow{\sigma_2} f_*\omega^2 \to \mathcal{T}_2 \to 0; \tag{2.4.1}$$

$$0 \to \mathcal{L}_3 \to \mathcal{S}^3(f_*\omega) \xrightarrow{\sigma_3} f_*\omega^3 \to \mathcal{T}_3 \to 0. \tag{2.4.2}$$

$$0 \to \mathcal{L}_2^4 \to \mathcal{L}_3 \to \mathcal{L}' \to 0. \tag{2.4.3}$$

and compute that the line bundles $\mathcal{L}_2 = \mathcal{O}_{\mathbb{P}^1}(-d_2)$, $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^1}(-d_3)$ are exactly $\mathcal{O}_{\mathbb{P}^1}(2h-7)$ and $\mathcal{O}_{\mathbb{P}^1}(6-3h)$.

In view of lemma 2.3.1 the hypotheses of lemma 1.2.3 are satisfied.

So, by lemma 1.1.4, corollary 1.1.10, and lemma 1.2.3, we can compute the Euler characteristic of the sheaves involved in the exact sequence (2.4.1).

We get

$$\chi(\mathcal{L}_2) = 1 - d_2;$$

$$\chi(S^2(f_*\omega)) = \chi(\mathcal{O}^{10}) = 10;$$

$$\chi(f_*\omega^2) = \chi(\omega^2) = 16;$$

$$\chi(\mathcal{T}_2) = \text{length } (\mathcal{T}_2) = 2h;$$

so $1 - d_2 + 16 = 10 + 2h$, i.e. $d_2 = 7 - 2h$.

Moreover, again by lemmas 1.1.4, corollary 1.1.10, and lemma 1.2.3, we can compute the Euler characteristics of the sheaves involved in the other two exact sequences, and get

But then

$$\chi(\mathcal{L}_3) = 8h - d_3 - 23;$$

$$\chi(S^3(f_*\omega)) = \chi(\mathcal{O}^{20}) = 20;$$

$$\chi(f_*\omega^3) = \chi(\omega^3) = 37;$$

$$\chi(\mathcal{T}_3) = \text{length } (\mathcal{T}_3) = 5h;$$

so, by the exact sequence (2.4.2) we get $8h - d_3 - 23 + 37 = 20 + 5h$, i.e. $d_3 = 3h - 6$.

□

Consider now the case where $F = 0$, but f is not a genus 4 fibration. In this case β is the blow up of S in the single base point P of $|2K_S|$. If E is the exceptional divisor of β', the strict transform of the bicanonical system is given by $\beta'^* 2K_S - 2E$.

As in the previous case, let $g : \tilde{S} \to \mathbb{P}^2 \times \mathbb{P}^1$ be the morphism obtained from $|\beta^* 3K_S - E| \times |\beta^* 2K_S - 2E|$, let $\pi_2 : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the second projection, set $f = \pi_2 \circ g$.

THEOREM 2.4.2. *For a numerical Godeaux surface with torsion* $\{0\}$, *and of type ib) (bicanonical system without fixed part possessing a double base point)* f *is a genus* 3 *fibration, and* g *yields fibrewise the canonical map of the fibres. Moreover,* f *has exactly 7 hyperelliptic fibres (counted with multiplicity according to 1.2.2) and the image of* g *is a divisor in* $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4, 8)|$.

Proof.

By proposition 1.1.5, setting $\omega := \tilde{M} + K_{\tilde{S}}$, we have to consider the exact sequences

$$0 \to \mathcal{L}_2 \to \mathcal{S}^2(f_*\omega) \xrightarrow{\sigma_2} f'_*\omega^2 \to \mathcal{T}_2 \to 0; \tag{2.4.4}$$

$$0 \to \mathcal{L}_3 \to \mathcal{S}^3(f_*\omega) \xrightarrow{\sigma_3} f'_*\omega^3 \to \mathcal{T}_3 \to 0; \tag{2.4.5}$$

$$0 \to \mathcal{L}_4 \to \mathcal{S}^4(f_*\omega) \xrightarrow{\sigma_4} f'_*\omega^4 \to \mathcal{T}_4 \to 0; \tag{2.4.6}$$

and show that the line bundle $\mathcal{L}_4 = \mathcal{O}_{\mathbb{P}^1}(-d_4)$ is in fact $\mathcal{O}_{\mathbb{P}^1}(-8)$.

Remark that the maps σ_2 and σ_3 are injective, so $\mathcal{L}_2$ and $\mathcal{L}_3$ are in fact the 0 sheaves.

By lemmas 2.3.2 and 1.2.2 the sheaves $\mathcal{T}_j$ are torsion sheaves supported on the points corresponding to the hyperelliptic fibers and for every such point $p \in \mathbb{P}^1$ there is a multiplicity s_p s.t. $\forall i \geq 2$

$$\text{length } (\mathcal{T}_i, p) = (2i - 3)s_p.$$

So let us write $h = \sum s_p$. Using lemmas 1.1.4 and 1.2.2 (and Riemann-Roch) we can compute that the Euler characteristics in (2.4.4) are

$$\chi(\mathcal{S}^2(f_*\omega)) = \chi(\mathcal{O}^6) = 6;$$

$$\chi(f_*\omega^2) = \chi(\omega^2) = 13;$$

$$\chi(\mathcal{T}_2) = h;$$

then h=7.

Similar computation on (2.4.5) give the same result.

In (2.4.6) we have

$$\chi(\mathcal{L}_4) = 1 - d_4;$$

$$\chi(\mathcal{S}^4(f_*\omega)) = \chi(\mathcal{O}^{15}) = 15;$$

$$\chi(f_*\omega^4) = \chi(\omega^4) = 57;$$

$$\chi(\mathcal{T}_4) = 5h = 35;$$

so that $1 - d_4 + 57 = 15 + 35 \Leftrightarrow d_4 = 8$.

□

Finally, consider the case where F is a fundamental cycle, that we denoted by case ii). In this case β is the blow up of S in the two base points of $|2K_S - F|$. If E is the exceptional divisor of β, the strict transform of the bicanonical system is given by $\beta^*2K_S - 2E$.

As in the previous case, let $g : \tilde{S} \to \mathbb{P}^2 \times \mathbb{P}^1$ be the morphism obtained from $|\beta^*3K_S - E| \times |\beta^*2K_S - 2E|$, let $\pi_2 : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the second projection, set $f = \pi_2 \circ g$.

THEOREM 2.4.3. *For a numerical Godeaux surface with torsion* $\{0\}$*, and of type ii) (bicanonical system with a fundamental cycle as fixed part)* f *is a genus* 3 *fibration, and* g *yields fibrewise the canonical map of the fibres. Moreover,* f *has exactly 6 hyperelliptic fibres (counted with multiplicity according to 1.2.2) and the image of* g *is a divisor in* $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(4,7)|$.

Proof.

This case is identical to the previous one, except for the Euler characteristics of the sheaves $f_*\omega^l$.

More precisely, $\chi(f_*\omega^2) = 12$, so by (2.4.4) we get $h = 6$.

Again, the exact sequence 2.4.5 give the same result, whence for equation (2.4.6) we get $\chi(f_*\omega^4) = 51$, that implies $d_4 = 7$.

□

2.5. Computations of the direct image sheaves

In this section we want to have a careful look at the sheaves involved in the computation of the maps σ_n. At the end of the section we'll show that the case iii) cannot occur.

Let us recall once again our notation. In all cases we denote by S the minimal model of a Godeaux surface, M the movable part of the bicanonical system, $\beta : \tilde{S} \to S$ the blow-up of S at the base locus of M, $f : \tilde{S} \to \mathbb{P}^1$ the morphism $|M|$ induces on $\tilde{S}$. Finally, we denote by $\tilde{M}$ the strict transform of M on $\tilde{S}$, i.e. the fibres of f.

As we showed in previous section, the genus of $\tilde{M}$ is 4 in case ia), 3 in cases ib) and ii), and 2 in case iii).

We also define $\omega = \tilde{M} + K_{\tilde{S}}$, so that $\forall C \in \tilde{M}$, $\omega_C = \omega_{|C}$. We want to study the sheaves $f_*\omega^l$ for small values of l.

Recall that in lemma 1.1.4 we showed that (in all the cases) $f_*\omega = \mathcal{O}^g$, where g is the genus of the fibration.

Now we want to compute $f_*\omega^l$ for some $l > 1$.

PROPOSITION 2.5.1. *In case* ia)

$$f_*(\omega^2) = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^4 \oplus \mathcal{O}^4_{\mathbb{P}^1}$$

$$f_*(\omega^3) = \mathcal{O}_{\mathbb{P}^1}(3)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)^4 \oplus \mathcal{O}^4_{\mathbb{P}^1}$$

Proof.

The exceptional divisor for β, say $E := K_{\tilde{S}} - \beta^*(K_S)$ has $E^2 = K_{\tilde{S}}E = -4$. Moreover, $\omega := \beta^*(3K_S) = 3K_{\tilde{S}} - 3E$, whence $\tilde{M} = 2K_{\tilde{S}} - 3E$.

Remark that $f_*\omega^2 = f_*(2K_{\tilde{S}}) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$.

Consider (for every $C \in \tilde{M}$) the exact sequence

$$0 \to 3E \to 2K_{\tilde{S}} \to \omega_C^2 \to 0.$$

By standard result on fibrations (e.g., [**BPV**], theorem I.8.5) we get that $f_*(2K_{\tilde{S}})$ is a locally free sheaf of rank $h^0(\omega_C^2) = 12 - 4 + 1 = 9$.

Moreover, $H^0(f_*(2K_{\tilde{S}})) = H^0(2K_{\tilde{S}}) = 2$, and these 2 sections generate a subbundle of dimension $1 = \dim \mathrm{Im}(H^0(2K_{\tilde{S}}) \to H^0(\omega_C^2))$.

So $f_*(2K_{\tilde{S}}) = \mathcal{O}(1) \oplus \oplus_{i=1}^8 \mathcal{O}(a_i)$ with $\forall i\ a_i < 0$.

Moreover, by lemma 2.3.1, the map $\sigma_2 : S^2(f_*\omega) \to f_*\omega^2$ is generically surjective; but, by proposition 1.1.4, $S^2(f_*\omega)$ is a trivial bundle, so $\forall i\ a_i \geq -2$, and we can write $f_*(2K_{\tilde{S}}) = \mathcal{O}(1) \oplus \mathcal{O}(-1)^a \oplus \mathcal{O}(-2)^b$ with $a + b = 8$.

Finally, we remark that $3 + a = h^0(f_*(2K_{\tilde{S}}) \otimes \mathcal{O}(1)) = h^0(4K_{\tilde{S}} - 3E) = P_4(S) = 7$, and we have done (remark that, in general, nE is in the fixed part of $|nK_{\tilde{S}}|$, so $\forall i \leq n$, $h^0(nK_{\tilde{S}} - iE) = P_n(S)$.

Similar computation allows us to compute $f_*\omega^3$; this is a locally free sheaf of rank 15, so let us write $f_*\omega^3 = \oplus_{i=1}^{15} \mathcal{O}(l_i)$, with (still by proposition 1.1.4 and lemma 2.3.1) $\forall i\ l_i \geq 0$.

Let us define $n_j = \#\{l_i | l_i = j\}$.

The first remark that $h^0(K_{\tilde{S}}+3E) = 0$. In fact, E is the exceptional divisor of a sequence of 4 blow up. Let us assume by sake of simplicity β blow up in 4 distinct points, so that E is the union of 4 $(-1)-$curves, E_i one of these curves. Then $\forall i$ $(K_{\tilde{S}} + 3E)E_i) = -4$, so E is in the fixed part and we get a non trivial section of $K_{\tilde{S}} + 2E$. Iterating this argument we can induce a non trivial section in $K_{\tilde{S}}$, contradicting $p_g = 0$.

If β is not a blow up in 4 distinct points, we have few distinct cases to consider, and similar argument can be carried out in all the cases.

Arguing as before, by the exact sequence

$$0 \to K_{\tilde{S}} + 3E \to 3K_{\tilde{S}} \to \omega_C^3 \to 0$$

we get that $f_*\omega^3 \otimes \mathcal{O}_{\mathbb{P}^1}(-3)$ has 4 independent global sections that span a dimension 4 subbundle, so $n_3 = 4$, $\forall j \geq 4$ $n_j = 0$.

Then, by $f_*\omega^3 \otimes \mathcal{O}_{\mathbb{P}^1}(-2) = f_*(5K_{\tilde{S}} - 3E)$, computing global sections, we get $n_2 = 3$.

Finally, by $f_*\omega^3 \otimes \mathcal{O}_{\mathbb{P}^1}(-2) = f_*(7K_{\tilde{S}} - 6E)$ we get $n_1 = 4$, so $n_0 = 15 - 4 - 3 - 4 = 4$.

□

PROPOSITION 2.5.2. *In case* ib)

$$f_*(\omega^2) = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^4 \oplus \mathcal{O}_{\mathbb{P}^1}$$

$$f_*(\omega^3) = \mathcal{O}_{\mathbb{P}^1}(3)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)^3$$

$$f_*(\omega^4) = \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(2)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Proof.

In this case we have that E is a single exceptional curve (so $E^2 = K_{\tilde{S}}E = -1$), $\tilde{M} = 2K_{\tilde{S}} - 4E$, $\omega = 3K_{\tilde{S}} - 4E$. Arguing as in previous proposition we immediately get that $f_*\omega^2$ is locally free of rank 6 and "semipositive", i.e. $f_*\omega^2 = \oplus_{i=1}^6 \mathcal{O}(l_i)$ with $\forall i$ $l_i \geq 0$.

By

$$0 \to 4E \to 2K_{\tilde{S}} \to \omega_C^2 \to 0$$

we have $f_*\omega^2 = \mathcal{O}(3) \oplus \oplus_{i=2}^6 \mathcal{O}(l_i)$ with $0 \leq l_i \leq 1$, and by $h^0(f_*\omega^2 \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = h^0(f_*(4K_{\tilde{S}} - 4E)) = 7$, there are exactly 4 l_i equal to 1.

Similarly we can write $f_*\omega^3 = \oplus_{i=1}^{10} \mathcal{O}(l_i)$ with $l_i \geq 0$.

Remark again $h^0(K_{\tilde S}+4E)=0$. In fact, in this case, E is a (-1)−curve, and its intersection with some effective divisor $D\in|K_{\tilde S}+nE|$ should be $-(n+1)$, so E should be contained in D inducing an effective divisor in $|K_{\tilde S}+(n-1)E|$; iterating we contradict $p_g=0$.

Then, by

$$0\to K_{\tilde S}+4E\to 3K_{\tilde S}\to\omega_C^3\to 0,$$

we get $f_*\omega^3=\mathcal{O}(3)^4\oplus\oplus_{i=5}^{10}\mathcal{O}(l_i)$ with $2\geq l_i\geq 0$.

Then, by $h^0(f_*\omega^3\otimes\mathcal{O}(-2))=h^0(5K_{\tilde S}-4E)=P_5=11$ we find that there are exactly 3 l_i equal to 2.

Moreover, by $h^1(\omega_C^3)=0$ (and, as usual, theorem I.8.5 in [**BPV**]), we know that $h^1(f_*(5K_{\tilde S}-4E))=h^1(5K_{\tilde S}-4E)$ and the last one vanishes by Riemann-Roch, so there are no l_i equal to 0, and the computation of $f_*\omega^3$ is complete.

Finally let us $f_*\omega^4=\oplus_{i=1}^{14}\mathcal{O}(l_i)$ with $l_i\geq 0$.

It is not difficult to show, by intersection arguments, that $h^0(2K_{\tilde S}+4E)=2$.

By

$$0\to 8E\to 2K_{\tilde S}+4E\to\omega_C^4\to 0$$

we have $f_*\omega^4=\mathcal{O}(6)\oplus\oplus_{i=2}^{14}\mathcal{O}(l_i)$ with $4\geq l_i\geq 0$.

So,computing the h^0 of the sheaves

$$f_*\omega^4\otimes\mathcal{O}_{\mathbb{P}^1}(-4)=f_*(4K_{\tilde S});$$

$$f_*\omega^4\otimes\mathcal{O}_{\mathbb{P}^1}(-3)=f_*(6K_{\tilde S}-4E);$$

$$f_*\omega^4\otimes\mathcal{O}_{\mathbb{P}^1}(-2)=f_*(8K_{\tilde S}-8E);$$

we have $f_*\omega^4=\mathcal{O}(6)\oplus\mathcal{O}^4(4)\oplus\mathcal{O}^4(3)\oplus\mathcal{O}^4(2)\oplus\mathcal{O}(l_{14})$ with $1\geq l_{14}\geq 0$.

Finally by Riemann-Roch $h^1(f_*(8K_{\tilde S}-8E))=0$, so $l_{14}=1$.

□

PROPOSITION 2.5.3. *In case* ii)

$$f_*(\omega^2)=\mathcal{O}_{\mathbb{P}^1}(3)\oplus\mathcal{O}_{\mathbb{P}^1}(1)^3\oplus\mathcal{O}_{\mathbb{P}^1}^2$$

$$f_*(\omega^3)=\mathcal{O}_{\mathbb{P}^1}(3)^4\oplus\mathcal{O}_{\mathbb{P}^1}(2)^2\oplus\mathcal{O}_{\mathbb{P}^1}(1)^2\oplus\mathcal{O}_{\mathbb{P}^1}^2$$

whence either

$$f_*(\omega^4)=\mathcal{O}_{\mathbb{P}^1}(6)\oplus\mathcal{O}_{\mathbb{P}^1}(4)^4\oplus\mathcal{O}_{\mathbb{P}^1}(3)^3\oplus\mathcal{O}_{\mathbb{P}^1}(2)^2\oplus\mathcal{O}_{\mathbb{P}^1}(1)^2\oplus\mathcal{O}_{\mathbb{P}^1}^2$$

or

$$f_*(\omega^4) = \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}^3$$

Proof.

Here the situation is slightly more complicated that in the previous case. We have the exceptional locus (for β) E s.t. $E^2 = K_{\tilde{S}}E = -2$, $\tilde{M} = 2K_{\tilde{S}} - 3E - \beta^*F$, $\omega = 3K_{\tilde{S}} - 3E - \beta^*F$.

As in the previous propositions, we start looking at $f_*(2K_{\tilde{S}}) = f_*\omega^2 \otimes \mathcal{O}_{\mathbb{P}^1}(-2)$.

By the exact sequence

$$0 \to 3E + \beta^*F \to 2K_{\tilde{S}} \to \omega_C^2 \to 0$$

we get that this is a rank 6 locally free sheaf with 2 global section that span a dimension 1 subsheaf, and $h^1 = 2$, that, with proposition 1.1.4, give us the statement for $f_*\omega^2$.

Now we want to compute $f_*\omega^3$.

First remark that $h^0(K_{\tilde{S}} + 3E + \beta^*F) = 0$.

In fact, by intersection arguments as in the previous cases, an effective divisor in $|K_{\tilde{S}} + 3E + \beta^*F|$ would induce an effective divisor in $|\beta^*(K_S + F)|$. Now the same argument can be applied to F (in general, if you have an effective divisor $D \in |nK + \sum a_iE_i|$, $a_i > 0$, E_i (-2)-curves, then, by $(\sum a_iE_i)^2 < 0$, one get E_i s.t. $DE_i < 0$; iterating the argument one can easily show $\sum a_iE_i$ is in the fixed part of $|nK + \sum a_iE_i|$), and again we contradict $p_g = 0$.

Moreover, having $5K_S$ no base points (and being F a fundamental cycle), $h^0(5K_{\tilde{S}}) - 3E - \beta^*F) = P_5(S) - 1 = 10$.

So, by exact sequences

$$0 \to K_{\tilde{S}} + 3E + \beta^*F \to 3K_{\tilde{S}} \to \omega_C^3 \to 0$$

and

$$0 \to 3K_{\tilde{S}} \to 5K_{\tilde{S}} - 3E - \beta^*F \to \omega_C^3 \to 0$$

remarking by Riemann-Roch that $h^1(5K_{\tilde{S}} - 3E - \beta^*F) = 2$ we get the expression for $f_*\omega^3$ in the statement.

Last, we can do the same computation for $f_*(\omega^4)$. Is easy to show that $h^0(6E + 2\beta^*F) = 1$, $h^0(2K_{\tilde{S}} + 3E + \beta^*F) = 2$; by exact sequences

$$0 \to 6E + 2\beta^*F \to 2K_{\tilde{S}} + 3E + \beta^*F \to \omega_C^4 \to 0$$

$$0 \to 2K_{\tilde{S}} + 3E + \beta^*F \to 4K_{\tilde{S}} \to \omega_C^4 \to 0$$

$$0 \to 4K_{\tilde{S}} \to 6K_{\tilde{S}} - 3E - \beta^*F \to \omega_C^4 \to 0$$

we compute

$$f_*(\omega^4) = \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(2)^a \oplus \mathcal{O}_{\mathbb{P}^1}(1)^b \oplus \mathcal{O}_{\mathbb{P}^1}^a$$

with $2a + b = 6$.

Moreover by exact sequence

$$0 \to 6K_{\tilde{S}} - 3E - \beta^*F \to 8K_{\tilde{S}} - 6E - 2\beta^*F \to \omega_C^4 \to 0$$

we get $a = h^0(8K_{\tilde{S}} - 6E - 2\beta^*F) - 23$. It is clear that $h^0(8K_{\tilde{S}} - 6E - \beta^*F) = P_8(S) - 1 = 28$.

We are looking for section in this line bundle containing β^*F, i.e. for sections in $h^0(8K_S - F)$ containing F; $5 - a$ is the dimension of the cokernel of the map $H^0(\mathcal{O}(8K_S - F)) \to H^0(\mathcal{O}_F(-F)) \cong \mathbb{C}^3$; so $a \geq 2$ (and by $b \geq 0$ we get $a \leq 3$).

Remark the $a = 2$ if and only if the restriction $H^0(\mathcal{O}(8K_S - F)) \to H^0(\mathcal{O}_F(-F))$ is surjective.

□

PROPOSITION 2.5.4. *In case* iii)

$$f_*(\omega^2) = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2$$

$$f_*(\omega^3) = \mathcal{O}_{\mathbb{P}^1}(3)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

Proof.

Let us recall the situation.

$f : S \to \mathbb{P}^1$ is induced by the pencil $|M| = |2K_S - F|$, $K_SF = 0$, $F^2 = -4$. There is (at least) a fundamental cycle $Z \subset F$. Let us write $F' = F - Z$, so $F'^2 = -2$, $F'Z = 0$. Remark that both Z and F' are $1-$connected, and either F' is supported on Z or is a different fundamental cycle.

We need the following:

LEMMA 2.5.5. $h^0(K_S + F) = 0$.

Proof.

We consider the invertible sheaf $\mathcal{O}_{F'}(-Z)$. This is a sheaf on a genus 0 1-connected divisor, with degree 0, and, by properties of fundamental cycles, of degree ≥ 0 on every component, so 0 on every component.

By [**ML**], corollary I.3.4, $\mathcal{O}_{F'}(-Z) = \mathcal{O}_{F'}$.

So, we have the exact sequence

$$0 \to \mathcal{O}_{F'} \to \mathcal{O}_F \to \mathcal{O}_Z \to 0$$

$h^0(\mathcal{O}_F) = 2$. Using

$$0 \to \mathcal{O}_S(-F) \to \mathcal{O}_S \to \mathcal{O}_F \to 0$$

we can conclude (by $q = 0$ and the trivial $h^0(\mathcal{O}_S(-F)) = 0$), $h^1(\mathcal{O}_S(-F)) = 1$, and, by Riemann-Roch $h^2(\mathcal{O}_S(-F)) = 0$, that, by Serre duality, give us the result.

□

Proof of proposition 2.5.4.

We try to develop in case iii) the same computation we did in the previous cases. As usual, $\omega = K + M$.

By

$$0 \to F \to 2K_S \to \omega_M^2 \to 0$$

we easily get that $f_*(2K_S) = f_*\omega^2 \otimes \mathcal{O}_{\mathbb{P}^1}(-2)$ is a locally free rank 3 line bundle with $h^0(f_*(2K_S)) = 2$, but whose global sections generate a subsheaf of rank 1, and, by Leray spectral sequence, $h^1(f_*(2K_S)) = 0$. So $f_*\omega^2 = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2$.

By

$$0 \to K_S + F \to 3K_S \to \omega_M^3 \to 0,$$

using lemma 2.5.5 we easily get that $f_*(3K_S) = f_*\omega^3 \otimes \mathcal{O}_{\mathbb{P}^1}(-3)$ is a locally free rank 5 line bundle with $h^0(f_*(3K_S)) = 3$, and whose global sections generates a subsheaf of rank 4, and, again by Leray spectral sequence, $h^1(f_*(2K_S)) = 0$. So $f_*\omega^3 = \mathcal{O}_{\mathbb{P}^1}(3)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.

Corollary 2.5.6. *The case* iii) *does not occur.*

Proof.

We need to take a careful look to the maps $\sigma_l : S^l(f_*(\omega)) \to f_*(\omega^l)$.

By lemma 2.2.2 the generic fibre is smooth. Remark that, by [**ML**], the canonical ring of a 2-connected genus 2 fibre is of the form $R(M, \omega_M) = \mathbb{C}[x_0, x_1, z]/F_6$, with $\deg(x_i) = 1$, $\deg(z) = 3$, $\deg(F_6) = 6$, $F_6 = Z^2 + \dots$.

For a 1-connected (but non 2-connected) genus 2 fibre the canonical ring has the form $R(M, \omega_M) = \mathbb{C}[x_0, x_1, y, z]/(Q_2(x_i), Q_6)$, with $\deg(x_i) = 1$, $\deg(y) = 2$, $\deg(z) = 3$, $\deg Q_j = j$, $Q_6 = z^2 - y^3 + \dots$.

So we can easily see that σ_2 is injective, that its cokernel $\mathcal{T}_2$ is a torsion sheaf supported on the points corresponding to the not 2-connected fibres, and that the length of such a sheaf in such a point is exactly the multiplicity of the coefficient of y in the deformation of the equation Q_2.

Computing the Euler characteristics in the exact sequence

$$0 \to S^2(f_*(\omega)) \to f_*\omega^2 \to \mathcal{T}_2 \to 0$$

we get that the sum of these multiplicities (that equals the Euler characteristic of $\mathcal{T}_2$) is 5.

Now remark that we have an involution on S induced by hyperelliptic involution on every element of $|M|$.

Let us define the two subsheaves $f_*\omega^3_+$ and $f_*\omega^3_-$ given by the invariant and anti-invariant parts of $f_*\omega^3$ respect to our involution; so $f_*\omega^3$splits $f_*\omega^3_+ \oplus f_*\omega^3_-$.

Ii is a trivial check to remark that the involution acts on the canonical rings we described before (so on every stalk), fixing x_0, x_1 and y, and acting as $z \mapsto -z$, so $f_*\omega^3_+$ has rank 4, whence $f_*\omega^3_-$ has rank 1.

Arguing as before, the map $\sigma_3^+ : S^3(f_*\omega) \to f_*\omega^3_+$ is well defined and injective, and its cokernel is a torsion sheaf of length (locally is given by the monomials of type $x_i y$) $2 \cdot 5 = 10$.

So $\chi(f_*\omega^3_+) = 14$, $\chi(f_*\omega^3_-) = \chi(f_*\omega^3) - \chi(f_*\omega^3_+)$.

Finally, $\chi(f_*\omega^3) = 19$, then $\chi(f_*\omega^3_-) = 5$, so $f_*\omega^3_- = \mathcal{O}_{\mathbb{P}^1}(4)$ that contradicts proposition 2.5.4.

□

CHAPTER 3

Constructing numerical Godeaux surfaces

3.1. The adjunction condition

PROPOSITION 3.1.1. *Let Y be the tri-bicanonical image of a numerical Godeaux surface with torsion group $\{0\}$ and type ia) with h hyperelliptic fibres (counted with multiplicity).*

Then there exists a non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(2,h-3))$, where $\mathcal{C}$ is the conductor ideal of the normalization of Y.

We shall denote by Q'' the divisor associated to this section. As usual we shall denote in the same way the analogous in the other cases (in propositions 3.1.2 and 3.1.3).

Proof.

Let X be the canonical model of S, and let $\hat{\beta}:\tilde{X}\to X$ be the blow up in the base points of $|2K_X|$ (they are smooth points of X by our hypothesis).

Let ψ be given by the relative canonical map of $\hat{f}:\tilde{X}\to\mathbb{P}^1$.

We have the following diagram

$$\begin{array}{ccccc} \tilde{X} & & \Sigma & \subset & \mathbb{P}^3 \\ \hat{\beta}\downarrow & \searrow^{\hat{g}} & \uparrow\pi_1 & & \\ X & \overset{\hat{\varphi}}{\dashrightarrow} & Y & \subset & \mathbb{P}^3\times\mathbb{P}^1 \\ & & \downarrow\pi_2 & & \\ & & \mathbb{P}^1 & & \end{array}$$

Recall that $\hat{g}=\hat{\varphi}\circ\hat{\beta}$ is a birational morphism factoring through a possible contraction $\epsilon:\tilde{X}\to\hat{X}$ of strings of (-2) curves (to rational double point singularities), and a finite birational map $\tilde{g}:\hat{X}\to Y$.

By [**H**], ex. III.6.10 and III.7.2, $\hat{g}_*(K_{\tilde{X}})=\mathcal{H}om_{\mathcal{O}_Y}(\hat{g}_*\mathcal{O}_{\tilde{X}},K_Y)$. Moreover $\omega_Y=\mathcal{O}_Y(2+3-4,7-2h+3h-6-2)=\mathcal{O}_Y(1,h-1)$, whence $\hat{g}_*(K_{\tilde{X}})=\mathcal{C}\otimes\mathcal{O}_Y(1,h-1)$, $\mathcal{C}$ being the conductor ideal of $\tilde{g}$.

The pull back to $\tilde{X}$ of the conductor ideal $\mathcal{C}$ is an invertible sheaf $\mathcal{O}_{\tilde{X}}(-D)$, D is here the adjunction divisor. We have $K_{\tilde{X}} + D = \hat{g}^*(\mathcal{O}_Y(1, h-1))$, so D is given by h fibres of ψ (as we already know). More generally, since $\mathcal{C} = \hat{g}_*\mathcal{O}_{\tilde{X}}(-D)$, the n^{th} adjoint ideal $\hat{g}_*\mathcal{O}_{\tilde{X}}(-nD)$ equals $\mathcal{C}^n$.

Whence

$$h^0(S, nK_S) = h^0(\tilde{X}, nK_{\tilde{X}}) = h^0(Y, \hat{g}_*(nK_{\tilde{X}})) = h^0(Y, \mathcal{C}^n\mathcal{O}_Y(n, n(h-1))).$$

In particular, a global section of $g_*(K_{\tilde{S}})$ is a global section of $\mathcal{O}_Y(1, h-1)$, whose divisor pulls back to an effective divisor containing the honestly hyperelliptic fibres with their multiplicity; in particular its divisor contains the special twisted cubics. Since no plane contains a twisted cubic curve, we recover the basic assumption $h^0(K_S) = 0$.

Moreover, letting E be defined as $K_{\tilde{S}} - \beta^* K_S$ (in the generic case the sum of the four (-1) divisors of the blow-up), since $|\varphi^*(\mathcal{O}_Y(0,1))| = |\beta^* 2K_S - E| = |2K_{\tilde{S}} - 3E| = |\varphi^*(\mathcal{O}_Y(2, 2h-2)) - 2D - 3E|$, there exists $Q' \in |\mathcal{O}_Y(2, 2h-3)|$ whose pull-back on $\tilde{S}$ is a divisor consisting of $3E$ plus the sum of the honestly hyperelliptic fibres counted each $2s$ times (s being their respective multiplicity).

It is easy to show that, for every $p \in \mathbb{P}^1$, for a suitable neighbourhood Δ of p, the restriction of Q' to $Y \cap (\mathbb{P}^3 \times \Delta)$, is restriction of a divisor on $\mathbb{P}^3 \times \Delta$. Then we can apply corollary 1.2.6, and write $Q' = Q''P$, with $P \in H^0(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(0, h))$.

Since the pull-back of Q' contains the adjunction divisor D doubly, while P pulls-back to D, follows that the pull back of div Q'' is at least D.

□

PROPOSITION 3.1.2. *Let Y be the tri-bicanonical image of a numerical Godeaux surface with torsion group $\{0\}$ and type ib) (so we have 7 hyperelliptic fibres, counted with multiplicity).*

Then there exists a non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(2,4))$, where $\mathcal{C}$ is the conductor ideal of the normalization of Y.

Proof.

The proof is similar to the previous one. Let us recall what the notation means in this case.

Let X be the canonical model of S, and let $\hat{\beta} : \tilde{X} \to X$ be the blow up in the base point of $|2K_X|$ (it is smooth for X by our hypothesis).

Let ψ be given by the relative canonical map of $\hat{f} : \tilde{X} \to \mathbb{P}^1$.

As before, we have the following diagram

$$\begin{array}{ccccc}
\tilde{X} & & \mathbb{P}^2 & & \\
\hat{\beta}\downarrow & \searrow^{\hat{g}} & \uparrow\pi_1 & & \\
X & \overset{\hat{\varphi}}{\dashrightarrow} & Y & \subset & \mathbb{P}^2 \times \mathbb{P}^1 \\
 & & \downarrow\pi_2 & & \\
 & & \mathbb{P}^1 & &
\end{array}$$

and recall that $\hat{g} = \hat{\varphi} \circ \hat{\beta}$ is a birational morphism factoring through a possible contraction $\epsilon : \tilde{X} \to \hat{X}$ of strings of (-2) curves (to rational double point singularities), and a finite birational map $\tilde{g} : \hat{X} \to Y$.

Now, $\omega_Y = \mathcal{O}_Y(4-3, 8-2) = \mathcal{O}_Y(1,6)$, whence $\hat{g}_*(K_{\tilde{X}}) = \mathcal{C} \otimes \mathcal{O}_Y(1,6)$, $\mathcal{C}$ being the conductor ideal of $\tilde{g}$.

The pull back to $\tilde{X}$ of the conductor ideal $\mathcal{C}$ is an invertible sheaf $\mathcal{O}_{\tilde{X}}(-D)$; more generally, $\hat{g}_*\mathcal{O}_{\tilde{X}}(-nD)$ equals $\mathcal{C}^n$. We have $K_{\tilde{X}} + D = \hat{g}^*(\mathcal{O}_Y(1,6))$, so D is given by 7 fibres of ψ.

Whence

$$h^0(S, nK_S) = h^0(Y, \mathcal{C}^n \mathcal{O}_Y(n, 6n)).$$

and a global section of $g_*(K_{\tilde{S}})$ is a global section of $\mathcal{O}_Y(1,6)$, whose divisor pulls back to an effective divisor containing the honestly hyperelliptic fibres with their multiplicity; in particular its divisor contains the special (irreducible) conics. No line contains a irreducible conic, so $h^0(K_S) = 0$.

Moreover, letting E be the exceptional divisor of the blow-up, since $|\varphi^*(\mathcal{O}_Y(0,1))| = |2\beta^* K_S - 2E| = |2K_{\tilde{S}} - 4E| = |\varphi^*(\mathcal{O}_Y(2,12)) - 2D - 4E|$, there exists $Q' \in |\mathcal{O}_Y(2,11)|$ whose pull-back on $\tilde{S}$ is a divisor consisting of $4E$ plus the sum of the honestly hyperelliptic fibres counted each $2s$ times (s being their respective multiplicities).

By corollary 1.2.6 Q' belongs to the sheaf of ideals $(\mathcal{Q}, P)$, where $\mathcal{Q}$ is the polynomial defining Y, P is a polynomial of degree 7 on $\mathbb{P}^1$ such that its divisor pulls back to the adjunction divisor on $\tilde{S}$. Since $(\mathcal{Q}, P)$ form a regular sequence, it follows easily that there exists Q'' such that $Q' = Q''P$, that obviously has the properties in the statement.

Since the pull-back of Q' contains the adjunction divisor D doubly, while P pulls-back to D, follows that the pull back of div Q'' is at least D.

□

PROPOSITION 3.1.3. *Let Y the tri-bicanonical image of a numerical Godeaux surface with torsion group $\{0\}$ and type ii) (so we have 6 hyperelliptic fibres, counted with multiplicity).*

Then there exists a non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(2,3))$, where $\mathcal{C}$ is the conductor ideal of the normalization of Y.

Proof.

In this case we have a singular base point for the bicanonical system on the canonical model.

Then in this case we define $\tilde{X}$ as the surface obtained by $\tilde{S}$ contracting all its $(-2)-$curves; this could not coincide with the blow up of X in the smooth base points of $|2K_X|$ (when on S there is a fixed points of $|M|$ contained in F).

From now on the proof is almost identical to the previous one, so let us skip most details.

In this case the dualizing sheaf of Y is $\omega_Y = \mathcal{O}_Y(1,5)$, and the adjunction divisor D of the normalization of Y is given by 6 fibres (as usual counted with multiplicity).

We know that $h^0(S, nK_S) = h^0(Y, \mathcal{C}^n\mathcal{O}_Y(n,5n))$.

But $\omega_Y^2 = \mathcal{O}_Y(2,10)$, so, arguing as in previous proof, there exists a non trivial section $Q' \in H^0(\mathcal{C}^2\mathcal{O}_Y(2,9))$, and again by corollary 1.2.6, we can divide Q' by the polynomial $P \in H^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(0,6))$ whose pull back on the normalization of Y is D, and get a non trivial section Q'' in $H^0(\mathcal{C}\mathcal{O}_Y(2,3))$.

$\square$

3.2. The classification theorems

THEOREM 3.2.1. *Assume that S is a numerical Godeaux surface with torsion $\{0\}$ and of type ia), i.e., such that the bicanonical pencil yields a genus 4 fibration f.*

Let $h = \sum_{C \text{ hyperelliptic}} mult(C)$: then $0 \le h \le 3$.

Moreover $\exists \mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, 7-2h)|$, s.t. $Y := \varphi(S)$ is a divisor in $|\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ whose singular curves are exactly the twisted cubic curves image of the (honestly) hyperelliptic bicanonical divisors. Moreover, if $\mathcal{C}$ is the conductor ideal, $h^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$.

Viceversa, assume that $0 \le h \le 3$ and that $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, 7-2h)|$ is an irreducible divisor, and that in turn $Y \in |\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$ is an irreducible divisor whose normalization is a surface $\tilde{X}$ with rational double points as the only singularities. Suppose moreover that the conductor ideal $\mathcal{C}$ defines a divisor on $\tilde{X}$ equal to h fibres (counted with multiplicity). Assume moreover that the singular curves of Y are (irreducible) twisted cubics, and that $h^0(\mathcal{C}\mathcal{O}_Y(2, h-$

$3)) > 0$. *Then Y is the tri-bicanonical model of a numerical Godeaux surface with torsion $\{0\}$ and of type ia).*

Proof.

The first part of the statement is given by theorem 2.4.1, lemma 2.3.1 (part b) and proposition 3.1.1; we have still to show that a surface in $\mathbb{P}^3 \times \mathbb{P}^1$ enjoying these properties is a tri-bicanonical model of a numerical Godeaux surface of type ia).

First, in order to let the proof be more clear for the reader, let us recall what exactly, for a numerical Godeaux surface of type 1a), the divisor of the not trivial section $Q'' \in H^0(\mathcal{C}\mathcal{O}_Y(2, h-3))$ is.

We already computed $\omega_Y = \mathcal{O}_Y(2+3-4, 7-2h+3h-6-2) = \mathcal{O}_Y(1, h-1)$; $g^*\mathcal{O}_Y(0,1)$ gives the movable part of the bicanonical system, and $g_*\omega_{\tilde{S}}^2 = \mathcal{C}^2\omega_Y^2$. So we have a non trivial section Q' in $H^0(\mathcal{C}^2\mathcal{O}_Y(2, 2h-3))$, which, by proposition 1.2.5, induces a non trivial section Q'' in $H^0(\mathcal{C}\mathcal{O}_Y(2, h-3))$, removing "once" (with multiplicity) every special fibre.

Let us denote by H_1 the class of a divisor in $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,0)|$ and by H_2 the class of a divisor in $|\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(0,1)|$.

The divisor associated to the non trivial section Q' of $H^0(\mathcal{C}^2\mathcal{O}_Y(2, 2h-3))$ gives a curve in $\mathbb{P}^3\times\mathbb{P}^1$ of class $(2, 7-2h)(3, 3h-6)(2, h-3) = 12H_1^3 + 12hH_1^2H_2$.

This divisor induces the immersion $\mathcal{O}_Y(0,1) \to g_*\omega_{\tilde{S}}^2$. This contains 4 times the h singular twisted cubic curves (with multiplicity), because $g_*\omega_{\tilde{S}}^2 = \mathcal{C}^2\omega_Y^2$; we found a subdivisor of class $12hH_1^2H_2$.

Let E'' be the residual curve of class $12H_1^3$; E'' is the union of the images of curves in the fixed part of the bicanonical system of $\tilde{S}$ and eventually of some more curve coming from the conductor ideal. But, by construction, the bicanonical system has $3(K_{\tilde{S}} - \beta^*K_S)$ in the fixed part.

Easy intersection arguments show that every $(-1)-$curve in $K_{\tilde{S}} - \beta^*K_S$ maps on a curve of class H_1^3; moreover for a sequence of n blow ups $\tilde{S} \to S$, the exceptional divisor $K_{\tilde{S}} - \beta^*K_S$ can have many different configurations, but one can easily prove that in every case it contains (with multiplicity) at least n (4 in our case) (-1)-curves; again by intersection arguments one see that every other curve in the exceptional divisor for our sequence of blow ups contracts to a point on Y, so E'' is exactly the image of the fixed part of $2K_{\tilde{S}}$.

Vice versa, let $Y \subset \mathbb{P}^3 \times \mathbb{P}^1$ be as described. Let us consider the normalization $\varepsilon : \tilde{X} \to Y$, and a minimal resolution of singularities $\delta : \tilde{S} \to \tilde{X}$; we have $\varepsilon_*\omega_{\tilde{X}} \cong \mathcal{C}\omega_Y$. By our assumptions, $\omega_Y = \mathcal{O}_Y(1, h-1)$.

First, we claim that $p_g(\tilde{X}) = 0$.

In fact, an easy computation shows that the restriction maps

$$H^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,h-1)) \to H^0(\mathcal{O}_Q(1,h-1)) \to H^0(\mathcal{O}_Y(1,h-1))$$

are isomorphisms.

So, if $h = 0$, $p_g(\tilde{X}) = h^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,-1)) = 0$, while, if $h > 0$, a not trivial section of $H^0(\varepsilon_*\omega_{\tilde{X}})$ induces a not trivial section of $H^0(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1,h-1))$ containing some of the singular twisted cubic curves; since a plane in $\mathbb{P}^3$ cannot contain a twisted cubic curve, we derive a contradiction.

Let us now denote by Q'' a non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(2,h-3))$; let moreover F' be a non trivial section (unique up to scalar multiplication) in $H^0(\mathcal{O}_Y(0,h))$ whose pull back in $\tilde{X}$ gives the conductor divisor. Let us set $Q' = F'Q'' \in H^0(\mathcal{C}^2\mathcal{O}_Y(2,2h-3))$, $\overline{Q}' = F'Q' \in H^0(\mathcal{C}^3\mathcal{O}_Y(2,3h-3))$.

The sections Q' and $\overline{Q}'$ define two injective homomorphisms of sheaves

$$\mathcal{O}_Y(0,1) \hookrightarrow \mathcal{C}^2\mathcal{O}_Y(2,2h-2) \cong \varepsilon_*\omega^2_{\tilde{X}}$$

$$\mathcal{O}_Y(1,0) \hookrightarrow \mathcal{C}^3\mathcal{O}_Y(3,3h-3) \cong \varepsilon_*\omega^3_{\tilde{X}}.$$

In particular we can conclude that the morphisms $\pi_2 \circ \varepsilon : \tilde{X} \to \mathbb{P}^1$ and $\pi_1 \circ \varepsilon : \tilde{X} \to \mathbb{P}^3$ are induced by some subsystem of the bicanonical, respectively of the tricanonical system. It follows that $\tilde{X}$ is of general type.

Since $\tilde{X}$ has only R.D.P.'s as singularities, $\tilde{S}$ is a surface of general type with geometric genus $p_g = 0$; in particular $q = 0$ and $\chi = 1$.

Let us denote by S the minimal model of $\tilde{S}$; then $K_S^2 \geq 1$. In order to prove that S is a numerical Godeaux surface, we need only to prove that $K_S^2 = 1$.

Observe that the divisor associated to Q'' gives a curve in $\mathbb{P}^3 \times \mathbb{P}^1$ of class $12H_1^3 + 6hH_1^2H_2$.

The assumption $Q'' \in H^0(\mathcal{C}\mathcal{O}_Y(2,h-3))$ ensures that such a divisor contains h fibres; so we can consider the residual curve E'' of class $12H_1^3$ (thus consisting with multiplicity of exactly 12 fibres of the projection over $\mathbb{P}^3$). Let us denote by E' and by E the respective divisors in $\tilde{X}$ and $\tilde{S}$ given by the difference between the pull back of $div(Q'')$ and the h fibres corresponding to the conductor divisor.

We have

$$(2K_{\tilde{S}}^2) = ((\varepsilon \circ \delta)^*\mathcal{O}_Y(0,1) + E)^2 = 24 + E^2$$

$$(3K_{\tilde{S}}^2) = ((\varepsilon \circ \delta)^*\mathcal{O}_Y(1,0) + E)^2 = 9 + E^2.$$

In particular $K_{\tilde{S}}^2 = (9 + E^2 - 24 - E^2)/5 = -3$.

The morphism $\beta : \tilde{S} \to S$ is a sequence of n blow ups. Since $\tilde{S}$ is of general type and $K_{\tilde{S}}^2 = -3$, it follows that $n = K_{\tilde{S}}^2 - K_S^2 \geq 4$.

We already noticed that, if we denote by $\overline{E}$ the difference $K_{\tilde{S}} - \beta^* K_S$, $\overline{E}$ contains, with multiplicity, at least n $(-1)-$curves. Remark that the morphism $\tilde{S} \to Y$ is composition of a finite map $(\tilde{X} \to Y)$ and of the minimal resolution of the singularities of $\tilde{X}$. By hypotheses, $\tilde{X}$ has only R.D.P., so the only curves contracted are $(-2)-$curves, and our $(-1)-$curves cannot be contracted to Y.

Now we only need to remark that the fixed part of $3K_{\tilde{S}}$ contains $3\overline{E}$, whence at least $3n$ (-1)-curves; and the corresponding divisor maps on Y to E'', which has 12 components.

Since $n \geq 4$, $3\overline{E}$ is exactly the fixed part of $3K_{\tilde{S}}$; in particular $n = 4$, $K_S^2 = 1$ and S is a numerical Godeaux surface.

Thus $3\overline{E}$ is the fixed part of both $2K_{\tilde{S}}$ and $3K_{\tilde{S}}$; the rational map $S \dashrightarrow Y$ is the tri-bicanonical morphism, $3K_S$ has no base points, whence (as shown in [**Cat1**], [**Mi1**]) the torsion group of S is either 0 or $\mathbb{Z}/2\mathbb{Z}$. But if the torsion were $\mathbb{Z}/2\mathbb{Z}$, by lemma 2.3.1, part c), in the singular locus we would obtain a fibre consisting of a line with multiplicity 6, a contradiction.

Since the bicanonical system yields a genus 4 fibration, we are in case 1a).

□

THEOREM 3.2.2. *Assume that S is a numerical Godeaux surface with torsion $\{0\}$ and of type ib).*

$Y := \varphi(S)$ is a divisor in $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4, 8)|$ whose singular curves are exactly the 7 (with multiplicity) double conics image of the hyperelliptic bicanonical divisors. Moreover, if $\mathcal{C}$ is the conductor ideal, $h^0(\mathcal{C}\mathcal{O}_Y(2, 4)) > 0$, and no fibres of the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ are contained in Y.

Conversely, assume that $Y \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4, 8)|$ is an irreducible divisor whose normalization is a surface $\tilde{X}$ with rational double points as the only singularities. Suppose moreover that the conductor ideal $\mathcal{C}$ defines a divisor on $\tilde{X}$ equal to 7 fibres (counted with multiplicity) whose image is supported on irreducible conics, and that $h^0(\mathcal{C}\mathcal{O}_Y(2, 4)) > 0$.

Assume moreover that Y does not contain fibres of the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$; then Y is the tri-bicanonical model of a numerical Godeaux surface with torsion $\{0\}$ and of type ib).

Remark that in this case no fibres of the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ are contained in Y, whence in the other cases, the images of the exceptional divisor for β are exactly of this form.

In order to construct the tri-bicanonical model of a surface of this class, this condition should be interpreted as the open condition that there are no fixed points for the system of quartics.

Proof.

The first part is already proved in theorem 2.4.2 and proposition 3.1.2; the first statement we have still to prove is the remark about the fibres of the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$.

The strict transform on $\tilde{S}$ of such a (rational) curve would have intersection -1 with the canonical class, so it should be a exceptional divisor of the first kind.

On $\tilde{S}$ we have only one exceptional divisor of the first kind corresponding to the single base point of the bicanonical system, that has intersection 2 with every fibre of the projection on $\mathbb{P}^1$; on the contrary a fibre of the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ is of class H_1^2 (so intersect the generic fibre of the projection on $\mathbb{P}^1$ transversally in one point).

Conversely, assume $Y := \varphi(S)$ in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(4,8)|$, so that $\omega_Y = \mathcal{O}_Y(1,6)$. If Y is as described in the statement, considering $\varepsilon : \tilde{X} \to Y$ normalization, we have $\varepsilon_* \omega_{\tilde{X}} \cong \mathcal{C}\omega_Y$. This immediately implies (no line contains an irreducible plane conic) $p_g(\tilde{X}) = 0$.

Again, let Q'' be a non trivial section of $H^0(\mathcal{C}\mathcal{O}_Y(2,4))$, F' the natural non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(0,7))$, and remark that $Q''F'$ and $Q''F'^2$ define two injective morphisms

$$\mathcal{O}_Y(0,1) \hookrightarrow \varepsilon_* \omega_{\tilde{X}}^2$$

$$\mathcal{O}_Y(1,0) \hookrightarrow \varepsilon_* \omega_{\tilde{X}}^3$$

that implies that the the two projections are induced by subsystems of the bicanonical and the tricanonical system. In particular, $\tilde{X}$ is of general type.

By assumption $\tilde{X}$ has only R.D.P., so, if $\delta : \tilde{S} \to \tilde{X}$ is a minimal desingularization of $\tilde{X}$, $\tilde{S}$ has geometric genus 0 (so $\chi = 1$, $p_g = q = 0$). We want to compute K_S^2 where S is a minimal model for $\tilde{S}$.

The divisor of Q'' is a curve in $\mathbb{P}^2 \times \mathbb{P}^1$ of class $8H_1^2 + 32H_1H_2$, containing 7 fibres, so we can consider the residual part E'' of type $8H_1^2 + 4H_1H_2$ that contains both the images of the fixed parts of $2K_{\tilde{S}}$ and $3K_{\tilde{S}}$. Let us denote by E the divisor on $\tilde{S}$ whose image is E''. E contains all the (-1) curves in $\tilde{S}$.

We have

$$(2K_{\tilde{S}}^2) = ((\varepsilon \circ \delta)^* \mathcal{O}_Y(0,1) + E)^2 = 16 + E^2$$

$$(3K_{\tilde{S}}^2) = ((\varepsilon \circ \delta)^* \mathcal{O}_Y(1,0) + E)^2 = 8 + 8 + E^2,$$

that gives us $K_{\tilde{S}}^2 = 0, E^2 = -16$. Every (-1) curve in E has multiplicity greater than 3 (because E contains the fixed part of $3K_{\tilde{S}}$), and remark that the image on Y of a (-1)-curve should be of type $aH_1^2 + (a-1)H_1H_2$ (intersecting with the canonical class), with $a \geq 2$ by our assumption on the fibres of the projection on $\mathbb{P}^3$, so it contains at most 4 such a curve (with multiplicity).

Then, $P_2 = 2$, and $\tilde{S} \to S$ is exactly a blow up in a point of S. In fact, $\tilde{S}$ is not minimal because it is of general type but $K_{\tilde{S}} = 0$; but if $\tilde{S} \to S$ would be a sequence of 2 or more blow ups, as we remarked in the previous proofs, we would get at least 6 (with multiplicity) $(-1)-$curves in the fixed part of the tricanonical system, whence we proved that there are at most 4 of them. Then $K_S^2 = 1$ and S is a numerical Godeaux surface.

Then the projection of Y on $\mathbb{P}^1$ is given by the complete bicanonical system, and by our classification on the bicanonical system of a numerical Godeaux surface, we conclude that we are in case ib).

We are left with the torsion group. The natural rational map $S \dashrightarrow \mathbb{P}^2$ is induced by a subsystem of the tricanonical system and has degree 8.

We know that the rational map $S \dashrightarrow Y$ is given by $2K$, $3K - p$ with p base point for $2K$ but not for $3K$, so the tricanonical model has degree 9, that implies torsion $\{0\}$ or $\mathbb{Z}/2\mathbb{Z}$, and we can exclude last case because, by lemma 2.3.2, this would induce a multiple line as fibre of $Y \to \mathbb{P}^1$.

□

THEOREM 3.2.3. *Assume that S is a numerical Godeaux surface with torsion $\{0\}$ and of type ii).*

$Y := \varphi(S)$ is a divisor in $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4,7)|$ whose singular curves are exactly the 6 (with multiplicity) double conics image of the hyperelliptic bicanonical divisors. Moreover, if $\mathcal{C}$ is the conductor ideal, $h^0(\mathcal{C}\mathcal{O}_Y(2,3)) > 0$.

Conversely, assume that $Y \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(4,7)|$ is an irreducible divisor whose normalization is a surface $\tilde{X}$ with rational double points as the only singularities. Suppose moreover that the conductor ideal $\mathcal{C}$ defines a divisor on $\tilde{X}$ equal to 6 fibres (counted with multiplicity), and that $h^0(\mathcal{C}\mathcal{O}_Y(2,3)) > 0$. Then Y is the tri-bicanonical model of a numerical Godeaux surface with torsion $\{0\}$ and of type ii).

Proof.

The proof is similar to the previous one, so let us skip most details. The first part of the statement is given by theorem 2.4.3 and proposition 3.1.3.

Exactly with the same arguments used in proof of theorem 3.2.2, we immediately get $\tilde{X}$ of general type and with $p_g = 0$.

Now the divisor of Q'' is a curve in $\mathbb{P}^2 \times \mathbb{P}^1$ of class $8H_1^2 + 26H_1H_2$, containing 6 fibres, so we can consider the residual part E'' of type $8H_1^2 + 2H_1H_2$ that contains both the images of the fixed parts of $2K_{\tilde{S}}$ and $3K_{\tilde{S}}$. The corresponding divisor E on $\tilde{S}$ contains all the (-1) curves in $\tilde{S}$.

We have

$$(2K_{\tilde{S}}^2) = ((\varepsilon \circ \delta)^* \mathcal{O}_Y(0,1) + E)^2 = 16 + E^2$$

$$(3K_{\tilde{S}}^2) = ((\varepsilon \circ \delta)^* \mathcal{O}_Y(1,0) + E)^2 = 7 + 8 + E^2,$$

that gives us $K_{\tilde{S}}^2 = -1, E^2 = -18$.

Being S of general type, we have then that $\tilde{S} \to S$ is given by a sequence of at least 2 blow ups.

If $\tilde{S} \to S$ were given by a sequence of 3 or more blow ups, then we would have at least 3 $(-1)-$curves (with multiplicity) in the relative canonical divisor; this would induce at least 9 $(-1)-$curves in the fixed part of the tricanonical system. We showed that the image of this fixed part is contained in a curve of class $8H_1^2 + 2H_1H_2$; but the image of every $(-1)-$curve is of class $aH_1^2 + (a-1)H_1H_2$, and this gives a contradiction.

Then the surface is a numerical Godeaux surface, $P_2 = 2$, and looking at our classification of the bicanonical fibration for a numerical Godeaux surface we can conclude that we are in the case ii).

In particular the rational map $X \dashrightarrow Y$ is given by $2K$, $3K - p$ where p is a singular base point for $2K$ but not for $3K$.

The map $Y \to \mathbb{P}^2$ has degree 7, and, being both the tricanonical map of a numerical Godeaux surface and the map $X \to Y$, birational, we get that the induced rational map from the tricanonical model (in $\mathbb{P}^3$) to $\mathbb{P}^2$ has still degree 7. But this map is given by the projection from a singular point in the tricanonical model, so the tricanonical model has degree (at least) 9, that implies torsion $\{0\}$ or $\mathbb{Z}/2\mathbb{Z}$.

Finally, we can exclude that the torsion is $\mathbb{Z}/2\mathbb{Z}$, because, by lemma 2.3.2, this would induce a multiple line as fibre of $Y \to \mathbb{P}^1$.

□

CHAPTER 4

Some explicit computation

4.1. Resolution of the ideal sheaf of Y in case ia)

In this section, we look for the resolution of the ideal sheaf of a numerical Godeaux surface of type ia), for $h \leq 2$.

First, we need an easy technical lemma. Recall that, by theorem 3.2.1, the bitricanonical image of a numerical Godeaux surface is a divisor $Y \in |\mathcal{O}_{\mathcal{Q}}(3, 3h-6)|$, where $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 7-2h)|$. By sake of simplicity, let us write $\mathbb{P} := \mathbb{P}^3 \times \mathbb{P}^1$.

LEMMA 4.1.1. *For every $n \in \mathbb{Z}$, the natural restriction maps induce isomorphisms*

$$H^*(\mathcal{O}_{\mathbb{P}}(n, 6-2h)) \cong H^*(\mathcal{O}_{\mathcal{Q}}(n, 6-2h))$$

and

$$H^*(\mathcal{O}_{\mathcal{Q}}(2, n)) \cong H^*(\mathcal{O}_Y(2, n)).$$

Proof.

The first isomorphism comes from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(n-2, -1) \to \mathcal{O}_{\mathbb{P}}(n, 6-2h) \to \mathcal{O}_{\mathcal{Q}}(n, 6-2h) \to 0$$

remarking that $\forall n \in \mathbb{Z}\ H^*(\mathcal{O}_{\mathbb{P}}(n-2, -1)) = 0$.

For every $n \in \mathbb{Z}$, $H^*(\mathcal{O}_{\mathbb{P}}(-1, n)) = H^*(\mathcal{O}_{\mathbb{P}}(-3, n)) = 0$; this allows us to conclude, by exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-3, n+2h-7) \to \mathcal{O}_{\mathbb{P}}(-1, n) \to \mathcal{O}_{\mathcal{Q}}(-1, n) \to 0$$

that $\forall n \in \mathbb{Z}\ H^*(\mathcal{O}_{\mathcal{Q}}(-1, n)) = 0$.

Finally, exact sequence

$$0 \to \mathcal{O}_{\mathcal{Q}}(-1, n-3h+6) \to \mathcal{O}_{\mathcal{Q}}(2, n) \to \mathcal{O}_Y(2, n) \to 0$$

gives us the result.

□

Let us write

$$\mathcal{Q} = \sum_{i=0}^{7-2h} \lambda_0^{7-2h-i} \lambda_1^i Q_i,$$

where λ_0 and λ_1 form a basis for $H^0(\mathcal{O}_{\mathbb{P}}(0,1))$.

Moreover, let us denote by D a section in $H^0(\mathcal{O}_{\mathcal{Q}}(3,3h-6))$ whose associated divisor is Y.

By lemma 4.1.1, there exists a unique equation

$$G = \sum_{i=0}^{6-2h} \lambda_0^{6-2h-i} \lambda_1^i G_i,$$

$G_i \in H^0(\mathcal{O}_P(3,0))$, s.t. the restriction of G to $\mathcal{Q}$ gives exactly $\lambda_1^{12-5h} D \in H^0(\mathcal{O}_{\mathcal{Q}}(3,6-2h))$.

Then set $Q_i = 0$ if $i \geq 8-2h$, $G_i = 0$ if $i \geq 7-2h$.

LEMMA 4.1.2. *There exist linear forms* $L_0, \dots, L_{11-5h}$ *s.t.*

$$\begin{pmatrix} L_0 & & & & & \\ L_1 & L_0 & & & 0 & \\ L_2 & L_1 & L_0 & & & \\ & & \ddots & \ddots & & \\ \vdots & & & L_0 & & \\ & & & L_1 & L_0 & \\ L_{11-5h} & & \cdots & L_2 & L_1 & L_0 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \\ \vdots \\ \\ Q_{11-5h} \end{pmatrix} = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ \\ \vdots \\ \\ G_{11-5h} \end{pmatrix}$$

Remark that, if $11-5h > 6-2h$ ($h \leq 1$), this matrix induces relations with linear coefficients among the quadrics defining $\mathcal{Q}$.

Proof. Let us define $G^{(k)} = \lambda_0^k \lambda_1^{12-5h-k} D \in H^0(\mathcal{O}_{\mathcal{Q}}(3,6-2h))$, and write, by lemma 4.1.1, $G^{(k)} = \sum_{i=0}^{6-2h} \lambda_0^{6-2h-i} \lambda_1^i G_i^{(k)}$. Then $G^{(0)} = G$.

We know that, mod $\mathcal{Q}$, $\forall 0 \leq k < 12-5h$, $G^{(k+1)} = \frac{\lambda_0}{\lambda_1} G^{(k)}$.

So, $\forall 0 \leq k < 12-5h$, there are linear forms L_k such that

$$\lambda_0 G^{(k)} = \lambda_1 G^{(k+1)} + L_k \mathcal{Q}. \tag{4.1.1}$$

In particular $G_0^{(k)} = L_k Q_0$.

Finally, equation 4.1.1 allows us to compute

$$G_j = G_j^{(0)} = L_0 Q_j + G_{j-1}^{(1)} = \cdots = L_0 Q_j + L_1 Q_{j-1} + \cdots + L_j Q_0.$$

□

The ideal sheaf of Y is generated by $\mathcal{Q}$ and the equations $G^{(i)}$ defined in the proof of lemma 4.1.2.

Remark that, if $h \leq 1$, then $11-5h > 6-2h$; then in this case lemma 4.1.2 explicit $G^{(0)}$ as function of $\mathcal{Q}$ and of the linear forms. The same argument used in the proof of the lemma can be applied to $G^{(k)}$ for every k, reducing the problem of the construction of a numerical Godeax surface of this type to the problem of the construction of $12-5h$ linear forms and $8-2h$ quadratic forms in $\mathbb{P}^3$ with suitable properties.

This is no more true in the case $h = 2$ (lemma 4.1.2 gives explicit expressions for G_0 and G_1 but not for G_2). This difference forces us to treat separately the two cases.

THEOREM 4.1.3. *If Y is the tri-bicanonical model of a numerical Godeaux surface with $h = 2$, then there exist linear forms L_0, L_1, quadratic forms $Q_0, \dots, Q_3$ and a cubic form C in $\mathbb{P}^3$, such that, if λ_0, λ_1 are a basis for the linear forms on $\mathbb{P}^1$, the ideal sheaf of Y has a resolution*

$$0 \to \begin{matrix} \mathcal{O}_{\mathbb{P}}(-3,-3)^2 \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-5,-3) \end{matrix} \overset{\alpha_2}{\to} \begin{matrix} \mathcal{O}_{\mathbb{P}}(-2,-3) \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-3,-2)^3 \end{matrix} \to I_Y \to 0,$$

where $\alpha_2 = \begin{pmatrix} L_0 & L_1 & C \\ -\lambda_0 & 0 & \lambda_1 Q_3 \\ \lambda_1 & -\lambda_0 & \lambda_0 Q_1 + \lambda_1 Q_2 \\ 0 & \lambda_1 & \lambda_0 Q_0 \end{pmatrix}$.

Proof.

Consider, in the notation of the proof of the lemma 4.1.2, the equation $\lambda_0 Q_0 G^{(2)} + (\lambda_0 Q_1 + \lambda_1 Q_2) G^{(1)} + \lambda_1 Q_3 G^{(0)}$.

On $\mathcal{Q}$, this equation restricts to $\lambda_0^3 Q_0 D + \lambda_0\lambda_1(\lambda_0 Q_1 + \lambda_1 Q_2) D + \lambda_1^3 Q_3 D = 0$, then there exists a cubic C s.t. $-C\mathcal{Q} = \lambda_0 Q_0 G^{(2)} + (\lambda_0 Q_1 + \lambda_1 Q_2) G^{(1)} + \lambda_1 Q_3 G^{(0)}$.

This is a relation between the generators of our ideal sheaf. Other two relations come from equation (4.1.1): this induces a complex

$$\mathcal{O}_{\mathbb{P}}(-3,-3)^2 \oplus \mathcal{O}_{\mathbb{P}}(-5,-3) \overset{\alpha_2}{\to} \mathcal{O}_{\mathbb{P}}(-2,-3) \oplus \mathcal{O}_{\mathbb{P}}(-3,-2)^3 \to I \to 0,$$

where $\alpha_2 = \begin{pmatrix} L_0 & L_1 & C \\ -\lambda_0 & 0 & \lambda_1 Q_3 \\ \lambda_1 & -\lambda_0 & \lambda_0 Q_1 + \lambda_1 Q_2 \\ 0 & \lambda_1 & \lambda_0 Q_0 \end{pmatrix}$.

Remark that the 3×3 minors of α_2 are exactly $(\mathcal{Q}, G^{(0)}, G^{(1)}, G^{(2)})$. So, by Hilbert-Burch theorem, this complex is exact, and, in particular, a minimal resolution, if and only if these minors describe a surface.

□

In case $h \leq 1$, by lemma 4.1.2, we get $5-3h$ relations with linear coefficients between the quadrics defining $\mathcal{Q}$, that we can write in the following form:

COROLLARY 4.1.4. *If $h \leq 1$, there exist linear forms $L_0, \dots, L_{11-5h}$ s.t.*

$$\begin{pmatrix} L_{4-3h} & L_{3-3h} & & L_1 & L_0 \\ L_{5-3h} & L_{4-3h} & & L_2 & L_1 \\ & & & & \\ & & & & \\ L_{10-5h} & L_{9-5h} & & L_{7-2h} & L_{6-2h} \\ L_{11-5h} & L_{10-5h} & & L_{8-2h} & L_{7-2h} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{6-2h} \\ Q_{7-2h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Finally, we can compute the resolution of the ideal sheaf of Y.

THEOREM 4.1.5. *If Y is the tri-bicanonical model of a numerical Godeaux surface with $h \leq 1$, then there exist linear forms $L_0, \dots, L_{11-5h}$ and quadratic forms $Q_0, \dots, Q_{7-2h}$ in $\mathbb{P}^3$, such that, if λ_0, λ_1 are a basis for the linear forms on $\mathbb{P}^1$, the ideal sheaf of Y has a resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-5, 2h-7)^{5-3h} \xrightarrow{\beta_h} \begin{matrix} \mathcal{O}_{\mathbb{P}}(-3, 2h-7)^{12-5h} \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-5, 2h-6)^{6-3h} \end{matrix} \xrightarrow{\alpha_h}$$

$$\xrightarrow{\alpha_h} \begin{matrix} \mathcal{O}_{\mathbb{P}}(-2, 2h-7) \\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-3, 2h-6)^{13-5h} \end{matrix} \longrightarrow I_Y \longrightarrow 0$$

where

$$\alpha_h = \begin{pmatrix} L_0 & \cdots & L_{11-5h} & 0 & \cdots & 0 \\ -\lambda_0 & & 0 & Q_{7-2h} & & 0 \\ \lambda_1 & & & \vdots & \ddots & \\ & \ddots & \vdots & & & Q_{7-2h} \\ & \ddots & & Q_0 & & \vdots \\ & & -\lambda_0 & & \ddots & \\ 0 & & \lambda_1 & 0 & & Q_0 \end{pmatrix}$$

and

$$\beta_h = \begin{pmatrix} Q_{7-2h} & & 0 \\ \vdots & \ddots & \\ Q_0 & & Q_{7-2h} \\ & \ddots & \vdots \\ 0 & & Q_0 \\ \lambda_0 & & 0 \\ -\lambda_1 & \ddots & \\ & \ddots & \lambda_0 \\ 0 & & -\lambda_1 \end{pmatrix}$$

Proof.

Looking for relations between the generators of the ideal sheaf of Y, we have the relations coming from the equation 4.1.1, and the following $6-3h$ new relations:

$$Q_0 G^{(12-5h)} + \ldots + Q_{7-2h} G^{(5-3h)} = 0,$$

$$\vdots$$

$$Q_0 G^{(7-2h)} + \ldots + Q_{7-2h} G^{(0)} = 0.$$

In fact, mod $\mathcal{Q}$, we have, e.g., $Q_0 G^{(12-5h)} + \ldots + Q_{7-2h} G^{(5-3h)} = Q_0 \lambda_0^{12-5h} D + \ldots + Q_{7-2h} \lambda_0^{5-3h} \lambda_1^{7-2h} D = \lambda_0^{5-3h} D \mathcal{Q} = 0$, then $Q_0 G^{(12-5h)} + \ldots + Q_{7-2h} G^{(5-3h)} = A\mathcal{Q}$ with $A \in H^0(\mathcal{O}_{\mathbb{P}}(3,-1))$, then $A = 0$.

Moreover, corollary 4.1.4 shows $5-3h$ syzygies. Summing up we conclude that there is a complex as in the statement.

Locally (writing either $\lambda_0 = 1$ or $\lambda_1 = 1$), we can eliminate the syzygies (and $5-3h$ relations) and conclude again by Hilbert-Burch theorem.

□

COROLLARY 4.1.6. $\chi(\mathcal{O}_Y) = 1 - 4h$.

The linear and quadratic forms (and a single cubic for $h = 2$) induce a natural "ambient" (product of projective space) in which we can embed (as locally closed subvariety) the stratum of the moduli space of numerical Godeaux surfaces; the resolutions we have computed give us some of the closed conditions.

The close conditions missing are the conditions coming from the adjunction condition in theorem 3.2.1 (or in proposition 3.1.1).

Let us compute also these.

Assume that the fibre over $(0,1)$ for the fibration $Y \to \mathbb{P}^1$ is non hyperelliptic; let Q'' be a non trivial section in $H^0(\mathcal{C}\mathcal{O}_Y(2, h-3)) > 0$.

By lemma 4.1.1 we can consider Q'' as a section in $H^0(\mathcal{O}_{\mathbb{Q}}(2, h-3))$, and there exists a unique $\overline{Q}'' \in H^0(\mathcal{O}_{\mathbb{P}}(2, 6-2h))$ whose restriction at Y gives $\lambda_0^{9-3h} Q''$. Let us write

$$\overline{Q}'' = \sum_0^{6-2h} \lambda_0^{6-2h-i} \lambda_1^i Q_i''$$

and, as usual, set $Q_i'' = 0$ if $i > 6 - 2h$.

LEMMA 4.1.7. *There are constant* $a_1, \dots, a_{9-3h}$ *s.t.*

$$\begin{pmatrix} a_1 & & & & & & \\ a_2 & a_1 & & & & 0 & \\ a_3 & a_2 & a_1 & & & & \\ & & \ddots & \ddots & & & \\ \vdots & & & & a_1 & & \\ & & & & a_2 & a_1 & \\ a_{9-3h} & & \cdots & & a_3 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \\ \vdots \\ \\ Q_{8-3h} \end{pmatrix} = \begin{pmatrix} Q_0'' \\ Q_1'' \\ Q_2'' \\ \\ \vdots \\ \\ Q_{8-3h}'' \end{pmatrix}$$

Proof.

Let us skip this proof: it is exactly the same argument used in the proof of the lemma 4.1.2.

Lemma 4.1.7 and theorems 4.1.5 and 4.1.3 give a natural system of parameters and equations that define in every case a subvariety of a product of projective spaces in which the corresponding stratum of the moduli space of numerical Godeaux surfaces can be embedded as a Zariski open set.

4.2. Some more computation for the case $h = 2$

In this section we try to understand the equations we found in last section in case $h = 2$.

Why $h = 2$? Two are the principal motivations.

First, this case looks easier than the cases with $h \leq 1$; in the other cases the number of parameters is considerably higher, so we expect higher codimension.

Second, only in this case we can show existence (see next chapter). A computation of the dimension of this stratum of the moduli space (expected to be between 6 and 8) would give interesting indications on the answer to the existence problem of the other two (a priori more general) cases.

We want to construct an open set of this stratum of the moduli space of numerical Godeaux surfaces. We had to do some open assumption; in order to ensure that we are not working on the empty set, we checked (using the program Macaulay2) that all the open conditions we imposed hold for the Craighero Gattazzo surface (we will show in last chapter that the Craighero Gattazzo surface is in fact of type ia) with $h = 2$).

First of all we restrict to surfaces with exactly two distinct hyperelliptic bicanonical divisor with multiplicity one.

In the notation of the previous section, the equation of $\mathcal{Q}$ is

$$Q_0\lambda_0^3 + Q_1\lambda_0^2\lambda_1 + Q_2\lambda_0\lambda_1^2 + Q_3\lambda_1^3 = 0 \tag{4.2.1}$$

We can assume (acting by automorphisms of $\mathbb{P}^1$) that the two honestly hyperelliptic bicanonical divisors are preimages of the coordinates points in $\mathbb{P}^1$; so the equations $\lambda_0 = 0$ and $\lambda_1 = 0$ cut on Y two double twisted cubic curves, that are the divisorial part of the singular locus of Y.

Remark now that Y is not complete intersection of two hypersurfaces of respective bidegrees $(2, 3)$ and $(3, 0)$); in fact otherwise the image of the projection of Y in $\mathbb{P}^3$ would be supported on a cubic, whereas for a numerical Godeaux surface with torsion 0, the tricanonical morphism is birational on a surface of degree 9 ([**Cat1**]).

LEMMA 4.2.1. *Q_0, Q_3 are quadric cones. If we call V_0, V_3 the respective vertices, $V_0 \in Q_1$, $V_1 \in Q_2$.*

Proof.

We prove the statement for Q_0, V_0, Q_1.

Q_0 contains a twisted cubic curve; then has rank greater than 3. Assume by contradiction Q_0 smooth. By theorem 4.1.3, lemma 4.1.2 and its proof, we see that the cubic $C - L_0Q_1$ cuts on Q_0 a double twisted cubic curve. But a smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and a cubic cuts on it a divisor of type $(3,3)$ that cannot be double, a contradiction.

So Q_0 has rank 3. Let us call V_0 its vertex.

We know that Y is singular on a twisted cubic curve (say Γ_0) contained in Q_0. So, again in the notation of the previous section, computing derivatives, we get that $Q_1(C+L_1Q_1) - Q_0(L_0Q_3+L_1Q_2)$ is singular in Γ_0. Remark that $C+L_1Q_1$ vanishes in V_0, but is smooth there, otherwise these points would appear with higher multiplicity in the intersection.

Finally, Q_0 is singular in V_0 while $C+L_1Q_1$ is smooth, $Q_1(C+L_1Q_1) - Q_0(L_0Q_3+L_1Q_2)$ singular, so $V_0 \in Q_1$.

□

Now we assume $V_0 \notin Q_1$ and $V_1 \notin Q_3$.

Let us fix (by a coordinate change in $\mathbb{P}^3$), $V_0 = (0,1,0,0)$, $V_1 = (1,0,0,0)$: then $Q_0 = Q_0(y_0, y_2, y_3)$ and $Q_3 = Q_3(y_1, y_2, y_3)$.

This assumption guarantees that the coefficients of y_0^2 in Q_0 and y_1^2 in Q_1 are nonzero. Completing the squares, we can assume $Q_0 = y_0^2 + q_0(y_2, y_3)$, $Q_3 = y_1^2 + q_3(y_2, y_3)$.

Assume now that q_0q_3 vanishes in 4 distinct points (as polynomial in $\mathbb{P}^1_{y_2,y_3}$); up to a coordinate change involving only the variables y_2, y_3 the quadrics have the form

$$Q_0 = y_0^2 - y_2y_3$$

$$Q_3 = y_1^2 - (y_2 - y_3)(\lambda y_2 + y_3)$$

for some $\lambda \in \mathbb{C}$.

Remark that, fixed a twisted cubic curve and a quadric cone containing it, the movable part of the system of quadrics defining the t.c.c., restricted to the quadric cone itself, is given the pencil of lines through the vertex.

So, fixed a line in Q_0, there is exactly one (mod Q_0) quadric cutting Γ_0 and the line.

Let us fix the line $y_0 = y_2 = 0$; a quadric trough this line can be uniquely written as $\alpha_0(y_0, y_1, y_2, y_3)y_0 + \alpha_1(y_1, y_2, y_3)y_2$; acting mod Q_0 we can further assume $\alpha_0 = \alpha_0(y_1, y_2, y_3)$ and conclude that we can describe every twisted cubic curve Γ_0 in Q_0 in unique way by the 2×2 minors of a matrix of the form

$$\begin{pmatrix} y_2 & y_0 & \alpha_0(y_1, y_2, y_3) \\ y_0 & y_3 & \alpha_1(y_1, y_2, y_3) \end{pmatrix}$$

.

Let us denote

$$Q_{0a} = y_0\alpha_0 - y_2\alpha_1$$

$$Q_{0b} = y_0\alpha_1 - y_3\alpha_0$$

By the description of the matrix α_2, we get that the surface Y in a neighborhood of the double Γ_0 is complete intersection of $\mathcal{Q}$ and the cubic $\lambda_0^2(L_1Q_1 + C) + \lambda_0\lambda_1(L_0Q_3 + L_1Q_2) + \lambda_1^2 L_1Q_3$, while in a neighborhood of the double Γ_1 is complete intersection of $\mathcal{Q}$ and the cubic $\lambda_0^2 L_0Q_0 + \lambda_0\lambda_1(L_0Q_1 + L_1Q_0) + \lambda_1^2(L_0Q_2 - C)$.

Every cubic G_0 s.t. $G_0 \cap Q_0 = 2\Gamma_0$ can be uniquely written as determinant of the matrix

$$\begin{pmatrix} y_2 & y_0 & \alpha_0 \\ y_0 & y_3 & \alpha_1 \\ \alpha_0 & \alpha_1 & l_0 \end{pmatrix}$$

with l_0 linear form.

So a first condition is that there exists (up to multiply the coefficients of α_i, l_0 for some suitable constant) l_0 such that $C + L_1Q_1 = G_0$.

Doing the same computation for Q_3, we get that Γ_1 is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} y_2 - y_3 & y_1 & \beta_0(y_0, y_2, y_3) \\ y_1 & \lambda y_2 + y_3 & \beta_1(y_0, y_2, y_3) \end{pmatrix}.$$

Again we denote

$$Q_{3a} = y_1\beta_0 - (y_2 - y_3)\beta_1$$

$$Q_{3b} = y_1\beta_1 - (\lambda y_2 + y_3)\beta_0$$

The generic cubic G_1 s.t. $G_1 \cap Q_3 = 2\Gamma_1$ is the determinant of the matrix

$$\begin{pmatrix} y_2 - y_3 & y_1 & \beta_0 \\ y_1 & \lambda y_2 + y_3 & \beta_1 \\ \beta_0 & \beta_1 & l_1 \end{pmatrix}.$$

So there exists some linear form l_1 such that $C - L_0Q_2 = G_1$. This equation suggests us to add to our system of parameters the coefficients of the 2 linear forms l_0 and l_1, and "remove" the coefficients of the cubic C (i.e., use the last equation to give explicit espression of its coefficients as function of the other parameters); in the new system of parameters the "mirror" condition

$C = G_1 + L_0Q_2$ gives us the (20) relations induced by the vanishing of the cubic $(G_0 - G_1) - (L_0Q_2 + L_1Q_1)$.

Imposing singularity of Y in Γ_0 is a slightly stronger condition. This can be written as (just compute derivatives)

$$y_0(L_0Q_3 + L_1Q_2) - (\alpha_0\alpha_1 - y_0l_0)Q_1 \in \mathcal{I}_{\Gamma_0},$$

and the singularity of Y in Γ_1 give the equation

$$y_1(L_0Q_1 + L_1Q_0) + (\beta_0\beta_1 - y_1l_1)Q_2 \in \mathcal{I}_{\Gamma_1}.$$

We are left with the adjunction condition: we have just to write lemma 4.1.7; we get that the adjunction condition is the existence of 3 non zero constants a_1, a_2, a_3 s.t. s.t. $a_1Q_1 + a_2Q_2 + a_3Q_3 \in \mathcal{I}_{\Gamma_0}$ and $a_1Q_0 + a_2Q_1 + a_3Q_2 \in \mathcal{I}_{\Gamma_1}$.

Recalling that $V_0 \in Q_1 \cap \Gamma_0$, we can easily conclude by the first equation, looking at the coefficient of y_1^2 that (up to a constant), if $Q_i = \sum q_{ijk}y_jy_k$, $a_2 = -1$ $a_3 = q_{211}$, and analogously the second equation gives $a_2 = -1$ $a_1 = q_{100}$. Let us simplify the notation with the following definitions: $q_0 := q_{100}$; $q_1 := q_{211}$

Summing up we have the following system of relations:

$$q_0Q_1 - Q_2 + q_1Q_3 \in \mathcal{I}_{\Gamma_0}$$

$$q_0Q_0 - Q_1 + q_1Q_2 \in \mathcal{I}_{\Gamma_1}$$

$$y_0(L_0Q_3 + L_1Q_2) - (\alpha_0\alpha_1 - y_0l_0)Q_1 \in \mathcal{I}_{\Gamma_0}$$

$$y_1(L_0Q_1 + L_1Q_0) + (\beta_0\beta_1 - y_1l_1)Q_2 \in \mathcal{I}_{\Gamma_1}$$

$$G_0 - G_1 = L_0Q_2 + L_1Q_1.$$

The first 2 equations can be written as

$$q_0Q_1 - Q_2 + q_1Q_3 = q_0^2Q_0 + \overline{h}_1Q_{0a} + \overline{h}_2Q_{0b}$$

$$q_0Q_0 - Q_1 + q_1Q_2 = q_1^2Q_3 + \overline{k}_1Q_{3a} + \overline{k}_2Q_{3b}$$

that in the generical assumption $q_0q_1 \neq 1$ are equivalent to ($h_i := (q_0q_1 - 1)^{-1}\overline{h}_i$, $k_i := (q_0q_1 - 1)^{-1}\overline{k}_i$)

(4.2.2) $$Q_1 = q_0Q_0 + h_1q_1Q_{0a} + h_2q_1Q_{0b} + k_1Q_{3a} + k_2Q_{3b}$$

(4.2.3) $$Q_2 = h_1Q_{0a} + h_2Q_{0b} + q_1Q_3 + k_1q_0Q_{3a} + k_2q_0Q_{3b}$$

This two equations allows us to transform the other 3 relations in the following:

$$y_0(L_0+L_1q_1)Q_3+(y_0(l_0+q_0L_1)-\alpha_0\alpha_1)(k_1Q_{3a}+k_2Q_{3b})\in\mathcal{I}_{\Gamma_0}$$

$$y_1(L_1+L_0q_0)Q_0+(y_1(-l_1+q_1L_0)+\beta_0\beta_1)(h_1Q_{0a}+h_2Q_{0b})\in\mathcal{I}_{\Gamma_1}$$

$$\begin{aligned}(-l_0Q_0+\alpha_0Q_{0b}+\alpha_1Q_{0a})-(-l_1Q_3+\beta_0Q_{3b}+\beta_1Q_{3a})=\\ =L_0(h_1Q_{0a}+h_2Q_{0b}+q_1Q_3+k_1q_0Q_{3a}+k_2q_0Q_{3b})+\\ +L_1(q_0Q_0+h_1q_1Q_{0a}+h_2q_1Q_{0b}+k_1Q_{3a}+k_2Q_{3b}).\end{aligned}$$

and the last one can be written as

$$\begin{aligned}-(l_0+q_0L_1)Q_0+(\alpha_1-h_1(L_0+q_1L_1))Q_{0a}+(\alpha_0-h_2(L_0+q_1L_1))Q_{0b}=\\ =(-l_1+q_0L_1)Q_3+(\beta_1+k_1(L_1+q_0L_0))Q_{3a}+(\beta_0+k_2(L_1+q_0L_0))Q_{3b}.\end{aligned}$$

So we can define $\bar{l}_0=-l_0-q_0L_1;\bar{l}_1=-l_1+q_1L_0$; $\overline{L}_0=L_0+q_1L_1$; $\overline{L}_1=L_1+q_0L_0$; and we have the simpler equations (where for extetic reasons we have changed the signs of the k_i)

(4.2.4) $$y_0\overline{L}_0Q_3+(y_0\bar{l}_0+\alpha_0\alpha_1)(k_1Q_{3a}+k_2Q_{3b})\in\mathcal{I}_{\Gamma_0}$$

(4.2.5) $$y_1\overline{L}_1Q_0+(y_1\bar{l}_1+\beta_0\beta_1)(h_1Q_{0a}+h_2Q_{0b})\in\mathcal{I}_{\Gamma_1}$$

(4.2.6) $$\begin{aligned}\bar{l}_0Q_0+(\alpha_1-h_1\overline{L}_0)Q_{0a}+(\alpha_0-h_2\overline{L}_0)Q_{0b}=\\ \bar{l}_1Q_3+(\beta_1-k_1\overline{L}_1)Q_{3a}+(\beta_0-k_2\overline{L}_1)Q_{3b}.\end{aligned}$$

Remark now that for every solution of this system of equation we got a dimension 2 space of solution of the original one, parametrized by q_0 and q_1 (with $q_0q_1\neq 1$).

We add a new assumption: $h_1\neq 0$, $k_1\neq 0$. This allows to define $h=h_2/h_1$, $k=k_2/k_1$, $c=1/h_1k_1$.

Remark that we can reconstruct h_1,h_2,k_2 by h,k,c,k_1 by the equations $h_1=1/ck_1$, $h_2=h/ck_1$, $k_2=kk_1$.

Let us define with a little abuse of notation $L_0=h_1\overline{L}_0$, $L_1=k_1\overline{L}_1$; our equations become

$$(4.2.7) \qquad cy_0L_0Q_3 + (y_0\bar{l}_0 + \alpha_0\alpha_1)(Q_{3a} + kQ_{3b}) \in \mathcal{I}_{\Gamma_0}$$

$$(4.2.8) \qquad cy_1L_1Q_0 + (y_1\bar{l}_1 + \beta_0\beta_1)(Q_{0a} + hQ_{0b}) \in \mathcal{I}_{\Gamma_1}$$

$$(4.2.9) \quad \bar{l}_0Q_0 + (\alpha_1 - L_0)Q_{0a} + (\alpha_0 - hL_0)Q_{0b} = \\ = \bar{l}_1Q_3 + (\beta_1 - L_1)Q_{3a} + (\beta_0 - kL_1)Q_{3b}.$$

and we got one more (k_1) free parameter.

The last system of equations define a subset of $\mathbf{P} \times \mathbb{C}^3$, where λ, h, k are the coordinate on the second factor, while $\mathbf{P}$ is a 28-dimensional weighted projective space with 9 coordinates of degree 2 (c and the coefficients of $\bar{l}_0, \bar{l}_1$), 20 of degree 1 (the coefficients of $\alpha_0, \alpha_1, \beta_0, \beta_1, L_0, L_1$).

Remarking that in our assumptions $c \neq 0$, it looks quite natural to set $c = 1$ and get simpler equations.

$$(4.2.10) \qquad y_0L_0Q_3 + (y_0\bar{l}_0 + \alpha_0\alpha_1)(Q_{3a} + kQ_{3b}) \in \mathcal{I}_{\Gamma_0}$$

$$(4.2.11) \qquad y_1L_1Q_0 + (y_1\bar{l}_1 + \beta_0\beta_1)(Q_{0a} + hQ_{0b}) \in \mathcal{I}_{\Gamma_1}$$

$$(4.2.12) \quad \bar{l}_0Q_0 + (\alpha_1 - L_0)Q_{0a} + (\alpha_0 - hL_0)Q_{0b} = \\ = \bar{l}_1Q_3 + (\beta_1 - L_1)Q_{3a} + (\beta_0 - kL_1)Q_{3b}.$$

Remark that, looking at the coefficients of y_0^3 and y_1^3 in equation 4.2.12, $\bar{l}_0 = \bar{l}_0(y_1, y_2, y_3)$, $\bar{l}_1 = \bar{l}_1(y_0, y_2, y_3)$.

More precisely, the coefficients of all the monomials multiple of y_0^2 and y_1^2 in equation 4.2.12, allows us to determine l_0 and l_1 as functions of the other parameters.

So equations 4.2.10, 4.2.11 and 4.2.12 define a subvariety of an affine 23−dimensional space, a generic point of which defines a 3-dimensional family of numerical Godeaux surfaces of type ia).

Let us conclude with some parameter computation.

Remark that dim $S^2(H^0(3K_S)) \cap (H^0(2K_S) \otimes H^0(4K_S))$ is (since $|2K_S|$ has no fixed part) by Grassman formula $10 + (2 \times 7 - 2) - 16 = 6$.

This space contains Q_i, Q_{ia}, Q_{ib}, and Q_1 and Q_2 that we have computed as functions of the other quadrics. So generically we expect that the two twisted cubics have no common quadrics.

Similar computation shows that there are at least 4 common cubics between the two t.c.c. (that in some sense are the "mirror" of the 4 common points we expect by the base locus of $|2K_S|$). So in our equations the existence of this 4 common cubics is "hidden" somewhere, and we can add this 4 condition to our list of equations; under this condition 4.2.10 and 4.2.11 give only 6 conditions, while equation 4.2.12 give 16 conditions, and we already used 8 of these equations (eliminating $\bar{l}_i$), so we have at most $4 + 6 + 6 + 8 = 24$ conditions in a 23-dimensional space.

We know that a solution exist: remembering that a generic solution gives a 3-dimensional family, we expect a family of solutions of dimension 3 or 5.

4.3. The genus 4 fibration cannot have three distinct hyperelliptic fibres

In this section we prove the following

PROPOSITION 4.3.1. *Let S be a numerical Godeaux surface with torsion $\{0\}$ such that $|2K_S|$ has 4 base points possibly infinitely near (equivalently, s.t. f is a genus 4 fibration). Then the bicanonical pencil cannot contain three distinct honestly hyperelliptic fibres.*

We shall argue by contradiction and assume by theorem 2.4.1 that $h = 3$ and that Y is a divisor in $|\mathcal{O}_{\mathcal{Q}}(3,3)|$, with $\mathcal{Q} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2,1)|$.

REMARK 4.3.2. *Y is the complete intersection of $\mathcal{Q}$ with a hypersurface $\mathcal{G}$, where $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(3,3)|$.*

This is an immediate consequence of remark 1.1.8 because $\mathrm{Ext}^1(\mathcal{O}(-3), \mathcal{O}(-1)^4) = H^1(\mathcal{O}(2)^4) = 0$.

Let us write down explicitly the equations of the two divisors whose complete intersection in this case gives our image surface Y.

Let $Q_\lambda = \lambda_0 Q_0 + \lambda_1 Q_1$; $G_\lambda = \lambda_0^3 G_{000} + \lambda_0^2 \lambda_1 G_{100} + \lambda_0 \lambda_1^2 G_{011} + \lambda_1^3 G_{111}$.

We can assume that for $\lambda = (1,0)$, $(0,1)$ or $\mu = (\mu_0, \mu_1)$ (fixed), $Q_\lambda \cap G_\lambda$ is a double twisted cubic.

An easy computation shows that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2,0)) \to H^0(\mathcal{O}_Y(2,0))$ is an isomorphism.

So, the adjunction condition in theorem 3.2.1, is equivalent to the existence of a quadric Q'' containing the three twisted cubic curves.

LEMMA 4.3.3. *Q_0, Q_1, Q_μ are quadric cones of rank 3.*

Proof.

If one of these quadrics, say Q_0, were smooth then Q_0 would be isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1)$. Then the cubic G_{000} would cut on Q_0 a divisor in the linear system $(3,3)$, while we know that this intersection must be twice an irreducible twisted cubic curve (t.c.c. for short), a contradiction (observe that a t.c.c. lies in a linear system of type $(2,1)$ or $(1,2)$). Moreover, since the t.c.c. is irreducible and non degenerate, rank $Q_0 =$ rank $Q_1 =$ rank $Q_\mu = 3$.

□

Then Q_0, Q_1, Q_μ, are quadric cones. Let V_0, V_1, V_μ be their respective vertices, $\Gamma_0, \Gamma_1, \Gamma_\mu$ the corresponding twisted cubic curves. The tricanonical image Σ of S is the hypersurface of $\mathbb{P}^3$ defined by $\Sigma = \{Q_1^3 G_{000} - Q_1^2 Q_0 G_{100} + Q_1 Q_0^2 G_{011} - Q_0^3 G_{111} = 0\}$.

LEMMA 4.3.4. *If* $V_0 = V_1 \Rightarrow Q_0, Q_1, Q''$ *have a common line* L.

Proof.

Let us consider the lines l_0, l_1, l_μ residual to the twisted cubics in the respective intersections of the three quadratic cones with the "adjoint" quadric Q''. I.e., we have $Q_0 \cap Q'' = \Gamma_0 \cup l_0$, $Q_1 \cap Q'' = \Gamma_1 \cup l_1$, $Q_\mu \cap Q'' = \Gamma_\mu \cup l_\mu$.

Observe that, since $V_0 = V_1 = V_\mu$, then clearly $V_0 \in l_0 \cap l_1 \cap l_\mu$.

$Q'' \supset \Gamma_0 \Rightarrow$ rank $Q'' \geq 3$.

If Q'' is smooth, then $Q'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and every line in Q'' is contained in one of the two rulings. So, at least two of the above lines are in the same ruling, and since they intersect, they do coincide.

This line is in the base locus of the pencil Q_λ hence our assertion follows.

If Q'' is a quadric cone, denote by V'' its vertex. Every t.c.c. in a quadric cone passes trough the vertex, so $\forall i \; V'' \in \Gamma_i \subset Q_i$; let $V = V_0 = V_1$, and observe that $V \neq V''$ (else the two quadric cones would intersect in 4 lines), whence the line $\overline{VV''}$ is contained in all these quadrics.

□

LEMMA 4.3.5. $V_0 \neq V_1$

Proof.

Observe preliminarly that the previous lemma implies that the three twisted cubics $\Gamma_0, \Gamma_1, \Gamma_\mu$ are distinct (otherwise there would be a twisted cubic Γ contained in each Q_λ: but then $Q_0 = Q_1$, since they have the same vertex V and they are the join of V with Γ).

Observe now that Σ must be singular in our three twisted cubics: in fact Σ is the image of Y under the birational morphism given by the first projection, and Y is singular along the three twisted cubics.

Thus, $Q'' \cap \Sigma \geq 2\Gamma_0 + 2\Gamma_1 + 2\Gamma_\mu$.

Both Q'' and Σ are irreducible, so their intersection must be a curve of degree 18 and equality must hold.

But, by lemma 4.3.4, we have a line in $Q'' \cap Q_0 \cap Q_1$, which is a fortiori also in $Q'' \cap \Sigma$, whence a contradiction.

□

So, we can assume $V_0 \neq V_1$.

LEMMA 4.3.6. *$\forall \lambda \in \mathbb{P}^1$, Q_λ is a quadric cone and the line $\overline{V_O V_1}$ is contained in Q_λ.*

Proof.

Recall that $Q_0 \cap G_{000} = 2\Gamma_0$. But Q_0 is singular in V_0, so G_{000} must be smooth in V_0, thus also in a general point of Γ_0.

Observe that $V_0 \in \Gamma_0 \subset \text{Sing } \Sigma$, so, by inspecting the equation of Σ, we infer that $V_0 \in Q_1$. Similarly, $V_1 \in Q_0$, and the line $\overline{V_0 V_1} \subset Q_\lambda \; \forall \lambda$.

Let us now fix coordinates s.t. $V_0 = (0,0,0,1)$, $V_1 = (0,0,1,0)$; $\overline{V_0 V_1} = \{x_0 = x_1 = 0\}$.

Thus the matrix of the quadric Q_λ has the following form:

$$Q_\lambda = \begin{pmatrix} * & * & \lambda_0 * & \lambda_1 * \\ * & * & \lambda_0 * & \lambda_1 * \\ \lambda_0 * & \lambda_0 * & 0 & 0 \\ \lambda_1 * & \lambda_1 * & 0 & 0 \end{pmatrix},$$

whence the determinant of the matrix of $|Q_\lambda|$ equals the square p_2^2, of a homogeneous polynomial p_2 of degree 2 in the λ_i's; since we know that it has at least three distinct roots, we conclude that $p_2 = 0$, therefore Q_λ is a pencil of quadric cones.

□

So, after a suitable change of coordinates in $\mathbb{P}^1$ and in $\mathbb{P}^3$, we may assume that

$$Q_\lambda = \begin{pmatrix} 0 & 0 & -\frac{\lambda_0}{2} & -\frac{\lambda_1}{2} \\ 0 & \lambda_0+\lambda_1 & 0 & 0 \\ -\frac{\lambda_0}{2} & 0 & 0 & 0 \\ -\frac{\lambda_1}{2} & 0 & 0 & 0 \end{pmatrix},$$

i.e., $Q_0 = x_1^2 - x_0x_2$, $Q_1 = x_1^2 - x_0x_3$.

Remark that this choice imposes that it cannot be $\mu_0 = -\mu_1$, because otherwise we get a t.c.c. Γ_μ contained in a reducible quadric Q_μ.

End of the proof. The vertices V_0, V_1 and V_μ of the quadric cones Q_0, Q_1 and Q_μ must be respectively contained in the twisted cubics Γ_0, Γ_1 and Γ_μ, therefore also in Q''. However, these three points lie on the same line $\overline{V_0V_1}$; in particular, we get a line intersecting a quadric in three distinct points. The conclusion is that $\overline{V_0V_1} \subset Q''$.

Recall that Σ must be singular in our three twisted cubics, and that by inspecting its equation, it follows easily that Σ is triple on the complete intersection of the two quadrics Q_0, Q_1, which contains the line $\overline{V_0V_1}$.

Let us write the complete intersection $Q_0 \cap Q_1$ as $\overline{V_0V_1} + T$, where T is thus a 1-cycle of degree 3.

Only two cases can occur:

- Γ_0, Γ_1 and Γ_μ are distinct
- $\Gamma_0 = \Gamma_1 = \Gamma_\mu = T$

In the first case, the schematic intersection $\Sigma \cap Q''$ has degree 18, however it contains Γ_0, Γ_1 and Γ_μ with multiplicity two and $\overline{V_0V_1}$ with multiplicity three: this is clearly a contradiction, since $18 + 3 = 21 > 18$.

In the second case, the irreducible twisted cubic T would intersect the line $\overline{V_0V_1}$ in the three distinct points V_0, V_1, V_μ, which is well known not to be possible.

□

4.4. A computation of parameters and conditions in case ii)

In theorem 2.4.3 we describe completely the tri-bicanonical image of a numerical Godeaux surface of type ii).

Remark that a divisor in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(4,7)|$ can be seen as a rational curve γ of degree 7 in the space of quartics of $\mathbb{P}^3$.

Moreover, by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(-2,-4) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(2,3) \to \mathcal{O}_Y(2,3) \to 0$$

the adjunction conditions (proposition 3.1.3) induce a rational curve γ' of degree 3 in the space of quadrics.

Recall that in case ii) $M^2 = 2$, so we have two base points (possibly infinitely near). Assuming the two points distinct, they induce two (-1)-curves in $\tilde{S}$ that have intersection 1 with the the fibres of f, 0 with ω, so their image are two distinct lines that are sections of the projection on $\mathbb{P}^1$ and map to two distinct points in $\mathbb{P}^2$.

Let us denote by p_1 and p_2 their image in $\mathbb{P}^2$, and let us moreover assume the six hyperelliptic fibres in proposition 3.1.3 distinct: they are defined by double quadrics, let us say $q_1^2, \ldots, q_6^2$.

So γ is contained in the 12-dimensional projective space of the quartics through p_1 and p_2, and γ' is contained in the 3$-$dimensional projective space of the quadrics through p_1 and p_2.

Recall that the space of rational curve of degree d in $\mathbb{P}^n$ has dimension $(d+1)(n+1)-4$. Then, we have $8 \cdot 13 - 4 = 100$ parameters for γ. Moreover $q_1, \ldots, q_6$ gives 6 points of $\mathbb{P}^3$ so we have other 18 parameters. But we can still act by automorphisms of $\mathbb{P}^2$ that fix two given points; the last space has dimension 4 so we found a space of parameters of dimension 114 (given by the 6 special quadrics and γ).

First of all we want to impose that γ contains $q_1^2, \ldots, q_6^2$; this gives $6 \cdot 11 = 66$ conditions.

Then we want that Y is singular in the fibres corresponding to those points. Locally, Y has equation $q_i^2 + tc + t^2 \ldots$. A derivative computation shows that the singular locus of Y contains the locus defined by the ideal (t, q_1) if and only if $q_i | c$, then if and only if the tangent vector to γ in the point q_i^2 lies in the 5$-$dimensional linear subspace of the space $(\mathbb{P}^1 2)$ of quartics we are considering, defined by the quartics multiple of q_i.

Finally, we can conclude that every singular fibre imposes $12 - 5 = 7$ conditions.

So we have found 114 parameters and $66 + 42 = 108$ conditions, but we have still to impose the adjunction condition.

As we already remarked, we are looking for a rational curve of degree 3 in $\mathbb{P}^3$ through 6 given points, so (for a generic choice of the six quadrics) we have exactly one curve. But we need that, for a suitable parametrization of γ and γ', $\gamma(t) = q_i^2 \Leftrightarrow \gamma'(t) = q_i$.

Then the adjunction gives (we can act with automorphisms of $\mathbb{P}^1$) $6-3=3$ more conditions. Finally, we have 114 parameters and 111 conditions, so we expect a space of solutions of dimension at least 3; we can sum up this computation in the following

PROPOSITION 4.4.1. *If there exists a numerical Godeaux surface with torsion* $\{0\}$, *of type* ii), *with* 2 *distinct base point for the movable part of the bicanonical system, and such that all the singular fibres have multiplicity* 1, *then the moduli space of numerical Godeaux surfaces of type ii) has dimension greater than 3.*

At the moment, we are not able to show if there exist a numerical Godeaux surface of this class.

Remark that we have not a similar computation in case ib). In that case, in fact, the image of the exceptional divisor has intersection 2 with the fibre of f and 1 with ω, so we cannot restrict to subspaces of the spaces of the quadrics and of the quartics of $\mathbb{P}^2$ as we did in this case.

CHAPTER 5

Examples

5.1. The Barlow surface

Up to now there are only two known explicit constructions of numerical Godeaux surfaces with torsion $\{0\}$ (and indeed simply connected), respectively due to Barlow ([**Ba2**]), and Craighero and Gattazzo ([**CG**]): let us consider first Barlow's example .

For the Barlow surface, we can study the bicanonical and tricanonical system according to the manuscript [**R3**], where Reid describes the canonical ring of the Barlow surface as follows.

Let A the symmetric matrix

$$A = \begin{pmatrix} -2x_4 & x_2 - x_0 - x_4 & x_0 - x_1 - x_4 & x_3 - x_2 - x_4 & x_1 - x_3 - x_4 \\ & -2x_0 & x_3 - x_1 - x_0 & x_1 - x_2 - x_0 & x_4 - x_3 - x_0 \\ & & -2x_1 & x_4 - x_2 - x_1 & x_2 - x_3 - x_1 \\ & & & -2x_2 & x_0 - x_3 - x_2 \\ & & & & -2x_3 \end{pmatrix}.$$

Let A_{ij} the ij-th entry of A, B_{ij} the ij-th entry of the adjoint matrix B of A.

Let us consider the authomorphism β of $\mathbb{C}[x_0, \ldots, x_4, y_0, \ldots, y_4]$ that acts as $\beta(x_i) = x_{i+1}$, $\beta(y_i) = y_{i+1}$, and the automorphism α that acts as $\alpha(x_i) = x_{a(i)}$ ($a = (25)(34)$ in $\mathcal{S}_5$), $\alpha(y_i) = -y_{4-i}$, where all indices are to be taken in $\mathbb{Z}/5\mathbb{Z}$. They generate a subgroup G of the group of automorphisms of $\mathbb{C}[x_0, \ldots, x_4, y_0, \ldots, y_4]$. One can indeed check that $G \cong D_{10}$.

Let $R = \mathbb{C}[x_0, \ldots, x_4, y_0, \ldots, y_4]/I$, where the ideal I is generated by

$$\sum x_i = 0,$$

$$\forall 1 \leq i \leq 5, \quad \sum_1^5 A_{ij} y_{j-1} = 0,$$

$$\forall 1 \leq i, j \leq 5, \quad y_{i-1} y_{j-1} - B_{ij} = 0.$$

We consider the ring R as a graded ring via the following grading which makes I a homogeneus ideal: deg x_i=1, deg y_i=2.

One can check that the ideal I is G-invariant, whence G acts on R. Since the action acts only with isolated fixed points, it follows (cf. [**R3**]) that the canonical ring of the Barlow surface can be described as the ring of the invariants of R for the action of G.

In order to simplify the computations, one can choose as generators for G, β and $\alpha' = \beta\alpha$; $\alpha'(x_i) = x_{a'(i)}$, with $a' = (12)(35)$, and $\alpha'(y_i) = -y_{-i}$.

So we can easily compute that there are no nontrivial invariants in R_1, while the subspace of invariants in R_2 is generated by

$$\xi_0 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_0 + x_0x_1,$$

$$\xi_1 = x_1x_3 + x_2x_4 + x_3x_0 + x_4x_1 + x_0x_2,$$

$$\xi_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_0^2.$$

Moreover, the relation $\sum x_i = 0$ induces the relation

$$2\xi_0 + 2\xi_1 + \xi_2 = 0.$$

So we can take ξ_0, ξ_1 as generators of the bicanonical system.

The tricanonical system needs more computations.

We know that R_3 is generated by $x_ix_jx_k$ and x_iy_j; the invariants must have the same decomposition.

The subspace of invariants in the span of the monomials $x_ix_jx_k$ is generated by the invariants:

$$\eta_0 = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_0 + x_4x_0x_1 + x_0x_1x_2,$$

$$\eta_1 = x_1x_2x_4 + x_2x_3x_0 + x_3x_4x_1 + x_4x_0x_2 + x_0x_1x_3,$$

$$\eta_2 = x_1^2(x_2 + x_0) + x_2^2(x_3 + x_1) + x_3^2(x_4 + x_2) + x_4^2(x_0 + x_3) + x_0^2(x_1 + x_4),$$

$$\eta_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_0^3,$$

$$\eta_4 = x_1^2(x_3 + x_4) + x_2^2(x_4 + x_0) + x_3^2(x_0 + x_1) + x_4^2(x_1 + x_2) + x_0^2(x_2 + x_3).$$

The relation $\sum x_i = 0$ induces the three linear relations

$$\begin{cases} 2\eta_0 + \eta_1 + \eta_2 = 0 \\ \eta_0 + 2\eta_1 + \eta_4 = 0 \\ \eta_3 + \eta_2 + \eta_4 = 0. \end{cases}$$

Thus the above subspace is generated by two independent generators, say η_0, η_1.

Now we have to find two more independent generators for the subspace of invariants in the span of the monomials $\{x_i y_j\}$.

Here the β invariants are generated by $\zeta_j = \sum_i x_i y_{i+j}$, where the indices $0 \leq j \leq 4$ are again to be understood as elements of $\mathbb{Z}/5\mathbb{Z}$.

The ζ_j verify the trivial relation $\sum \zeta_j = 0$, and the sum of the five linear relations $\forall 1 \leq i \leq 5$ $\sum_1^5 A_{ij} y_{j-1} = 0$. An easy calculation shows that this sum yields exactly $(-6)\zeta_1$. Whence, we have only the other relation $\zeta_1 = 0$.

Another easy calculation shows that $\alpha'(\zeta_0) = -\zeta_2$, $\alpha'(\zeta_1) = -\zeta_1$, $\alpha'(\zeta_3) = -\zeta_4$, and we can easily conclude that a system of independent generators for the tricanonical system of the Barlow surface is given by $\eta_0, \eta_1, \zeta_0 - \zeta_2, \zeta_3 - \zeta_4$.

In order to understand how many hyperelliptic divisors (with multiplicity) there are in the bicanonical system of the Barlow surface, we have only to check what is the minimal m s.t. there exists a non trivial element in

$$(S^m(< \xi_0, \xi_1 >) \otimes S^2(< \eta_0, \eta_1, \zeta_0 - \zeta_2, \zeta_3 - \zeta_4 >)) \cap I.$$

We are indebted to F.-O. Schreyer who wrote a Macaulay script that verifies that this minimal number m is indeed equal to 3, and that the relation is given by the following polynomial

$$\begin{aligned}
&1728\xi_0^3\eta_0^2 + 1872\xi_0^2\xi_1\eta_0^2 - 1296\xi_0\xi_1^2\eta_0^2 - 1584\xi_1^3\eta_0^2 + 5472\xi_0^3\eta_0\eta_1 + \\
&\quad + 5184\xi_0^2\xi_1\eta_0\eta_1 - 5184\xi_0\xi_1^2\eta_0\eta_1 - 5472\xi_1^3\eta_0\eta_1 + 1584\xi_0^3\eta_1^2 + \\
&\quad + 1296\xi_0^2\xi_1\eta_1^2 - 1872\xi_0\xi_1^2\eta_1^2 - 1728\xi_1^3\eta_1^2 - 13\xi_0^3(\zeta_0 - \zeta_2)^2 + \\
&\quad - 22\xi_0^2\xi_1(\zeta_0 - \zeta_2)^2 - 10\xi_0\xi_1^2(\zeta_0 - \zeta_2)^2 + \xi_1^3(\zeta_0 - \zeta_2)^2 + \\
&\quad + 14\xi_0^3(\zeta_0 - \zeta_2)(\zeta_3 - \zeta_4) + 24\xi_0^2\xi_1(\zeta_0 - \zeta_2)(\zeta_3 - \zeta_4) + \\
&\quad + 24\xi_0\xi_1^2(\zeta_0 - \zeta_2)(\zeta_3 - \zeta_4) + 14\xi_1^3(\zeta_0 - \zeta_2)(\zeta_3 - \zeta_4) - \xi_0^3(\zeta_3 - \zeta_4)^2 + \\
&\quad + 10\xi_0^2\xi_1(\zeta_3 - \zeta_4)^2 + 22\xi_0\xi_1^2(\zeta_3 - \zeta_4)^2 + 13\xi_1^3(\zeta_3 - \zeta_4)^2.
\end{aligned}$$

Afterwards we wrote a Macaulay2 script (available upon request) that obtains the same result in characteristic 0.

We can therefore summarize the main result of the foregoing section in the following

THEOREM 5.1.1. *The bicanonical system of the Barlow surface has exactly 4 distinct base points and contains two hyperelliptic fibres (counted with multiplicity).*

The same result was obtained independently by [**Lee**].

5.2. The Craighero Gattazzo surface

Let us now compute what happens for the Craighero Gattazzo surface. As the Barlow surface, this is a numerical Godeaux surface with torsion $\{0\}$ (and indeed simply connected, as shown in [**DW**]).

The Craighero Gattazzo surface S is constructed in [**CG**] as the minimal resolution of the quintic $X \in \mathbb{P}^3$ defined by the equation F_5

$$\begin{aligned} F_5 = (x+my+az)^2t^3 + [a^2x^3 + xy(bx+cy) + m^2y^3 + (ex^2+fxy+cy^2)z + \\ + (bx+ey)z^2 + z^3]t^2 + [2ax^3y + ex^2y^2 + 2amxy^3 + \\ + (2amx^3 + fx^2y + fxy^2 + 2my^3)z + (cx^2+fxy+by^2)z^2 + 2(mx+ay)z^3]t + \\ + x^3y^2 + a^2x^2y^3 + xy(2mx^2+bxy+2ay^2)z + \\ + (m^2x^3 + cx^2y + exy^2 + y^3)z^2 + (mx+ay)^2z^3 = 0. \end{aligned}$$

where r is a root of the polynomial t^3+t^2-1 and where the various coefficients are defined as follows :

$$\begin{array}{ll} a = r^2 & b = -\frac{1}{7}(2r^2 - 13r - 18) \\ c = \frac{1}{49}(73r^2 + 75r + 92) & e = -\frac{1}{7}(r^2 - 24r - 9) \\ f = \frac{1}{49}(181r^2 + 241r + 163) & m = \frac{1}{7}(3r^2 + 5r + 1). \end{array}$$

In [**CG**] are given different expressions for the coefficients a, e, b, m, f, c, expressed as rational functions of r; we have computed the equivalent expression as $\mathbb{Q}$-linear combinations of $1, r, r^2$ in order to simplify the calculations (we have done this both by hand and via a calculation using MAPLE).

This quintic surface X is invariant for the $\mathbb{Z}/4\mathbb{Z}-$action on $\mathbb{P}^3$ induced by the cyclical permutation of the coordinates $x \mapsto y \mapsto z \mapsto t$; the singular locus of X is the set of coordinate points $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

It is possible to show, as we shall do shortly, that in the neighbourhood of every singular point the singularity can be represented as a double cover of the plane branched on a curve with a singularity of type (3,3) (a triple point that has an infinitely near ordinary triple point). Therefore our singular points are simple elliptic (-1)-singularities (for which the exceptional curve in the minimal resolution is a smooth elliptic curve with self-intersection -1).

It follows that the adjoint divisor on the resolution is precisely the elliptic exceptional curve counted with multiplicity one, whence the bicanonical system of S is cut by the quadrics in $\mathbb{P}^3$ whose pull-back on S yields a divisor containing the exceptional locus twice, and the tricanonical system is cut by the cubics in $\mathbb{P}^3$ whose pull-back on S contains the exceptional locus with multiplicity three.

Craighero and Gattazzo compute explicitly both systems, but we found that their computation is different (and non-equivalent) to ours. It is possible that some misprint occurred, so let us sketch our calculation.

Let us look at the equation of X in a neighbourhood of $(0,0,0,1)$. Setting $w = (x + my + az)$ we can write the Taylor development of the equation F_5 in affine coordinates as follows:

$$w^2 + m^2y^3 + wf_2(w,y,z) + f_4(w,y,z) + f_5(w,y,z) = 0$$

with f_i homogeneus of degree i.

In local analytic coordinates (u,y,z), where $u = w + 1/2f_2(w,y,z)$, the equation takes the form

$$u^2 + m^2y^3 + g_4(u,y,z) + ... = 0.$$

Whence $y = 0$ is the equation (in the plane of coordinates (y,z)) of the direction of the tangent cone of the branching locus, and therefore the pull-back on S of the divisor $div(y)$ is easily shown to contain the exceptional curve E at least twice.

Of course the multiplicity in the exceptional curve of w and z is at least one. But, since w^2 belongs to the cube of the maximal ideal, it follows that $div(w) \geq 2E$.

Again, writing

$$f_2(w,y,z) = \alpha z^2 + wF_1(w,y,z) + yG_1(w,y,z)$$

$$f_4(w,y,z) = \beta z^4 + wF_3(w,y,z) + yG_3(w,y,z)$$

we are able to rewrite our equation in a slightly different way as follows:

$$w^2 + m^2y^3 + [\alpha wz^2 + w^2F_1(w,y,z) + wyG_1(w,y,z)] + \\ + [\beta z^4 + wF_3(w,y,z) + yG_3(w,y,z)] + f_5(w,y,z) = 0.$$

From the above remarks follows that the function

$$w^2 + \alpha wz^2 + \beta z^4$$

has a divisor which is greater than $5E$.

But a tedious calculation shows that $w^2 + \alpha wz^2 + \beta z^4 = (w - \frac{1}{7}(6r^2 + 3r - 5)z^2)^2$.

Whence the multiplicity of $w - \frac{1}{7}(6r^2+3r-5)z^2$ is at least 3.

It is in fact obvious that $p_g(S) = 0$; moreover it is also clear that $|2K_S|$ contains the divisors corresponding to the quadrics $Q_0 = xz$ and $Q_1 = yt$: on the other hand these two quadrics generate a fixed part free pencil on X, therefore the corresponding pencil in $|2K_S|$ has no rational curve in its fixed part; whence S is minimal. Since $K_S^2 = 1$, it follows that the bigenus $P_2(S) = 2$, hence the bicanonical system is precisely the above pencil.

We can proceed further by using the $\mathbb{Z}/4\mathbb{Z}$–invariance of F_5, since then $|3K_S|$ is generated by the $\mathbb{Z}/4\mathbb{Z}$ orbit of the cubic $C_0 = a(x+my+az)t^2 + txy + a^3yzt + \frac{1}{7}(6r^2+3r-5)xzt$.

If σ is the generator of the $\mathbb{Z}/4\mathbb{Z}$ action such that $\sigma(x) = y$, let us set $C_1 = \sigma(C_0)$, $C_2 = \sigma^2(C_0)$, $C_3 = \sigma^3(C_0)$.

In this way, also for the Craighero Gattazzo surface, we can calculate using the computer algebra program Macaulay2 what is the minimal number m such that the kernel of the map $S^m(H^0(2K)) \otimes S^2(H^0(3K)) \to H^0((2m+6)K)$ is not trivial; and again the answer we get is $m = 3$.

At the moment we cannot yet determine whether there do exist numerical Godeaux surfaces with bigger values of $m = 5$ or 7.

The explicit equation of this polynomial is

$$\begin{aligned}
&-(3r^2+5r+1)Q_0^3C_0C_2+\\
&+Q_0^2Q_1[-7r(C_0^2+C_2^2)-14(r+1)(C_0C_1+C_2C_3)-7(r+1)(C_1^2+C_3^2)+\\
&+(r^2+4r-9)C_0C_2-7(r^2+r+1)(C_1C_2+C_0C_3)-(11r^2+16r+6)C_1C_3]+\\
&+Q_0Q_1^2[7(r+1)(C_0^2+C_2^2)+7(r^2+r+1)(C_0C_1+C_2C_3)+7r(C_1^2+C_3^2)+\\
&+(11r^2+16r+6)C_0C_2+14(r+1)(C_1C_2+C_0C_3)-(r^2+4r-9)C_1C_3]+\\
&\qquad\qquad\qquad\qquad+(3r^2+5r+1)Q_1^3C_1C_3
\end{aligned}$$

We can combine the results of our calculations above with the previous results of Craighero and Gattazzo ([**CG**]) and Dolgachev and Werner ([**DW**]),

THEOREM 5.2.1. *The Craighero Gattazzo surface is a simply connected numerical Godeaux surface with ample canonical bundle. The bicanonical system has exactly* 4 *distinct base points and contains exactly two hyperelliptic fibres with multiplicity* 1.

Proof.

We need only to verify the last two assertions.

Recall that by [**DW**], S does not contain $(-2)-$curves; so, by lemma 2.1.1, $2K_S$ has no fixed part.

Restricting to the line $x = y = 0$ the equation F_5, we get the polynomial $az^2t^3 + z^3t^2$. So the smooth point of X of coordinates $(0, 0, -a, 1)$ is a base point of the bicanonical system of $2K_S$; since its orbit by the $\mathbb{Z}/4\mathbb{Z}$ action consists of four distinct points, we have gotten 4 distinct base points. These build up the whole base locus because $(2K_S)^2 = 4$.

We have shown before that the minimal m such that the kernel of the map $S^m(H^0(2K)) \otimes S^2(H^0(3K)) \to H^0((2m+6)K)$ is not trivial, is 3. This allows us to conclude, by theorem 2.4.1, that there are two hyperelliptic bicanonical divisors (counted with multiplicity).

But the $\mathbb{Z}/4\mathbb{Z}$ action on X induces a $\mathbb{Z}/2\mathbb{Z}$ action on the bicanonical system (since the bicanonical sections are invariant by σ^2). So, if there were only one hyperelliptic bicanonical divisor (with multiplicity two), it would be cut by a $\sigma-$ invariant quadric in the pencil generated by Q_0 and Q_1, i.e. by $Q_0 + Q_1$ or $Q_0 - Q_1$.

But we have written down explicitly the tricanonical system, so we can explicitly write the tricanonical images of these two divisors. We can prove that neither of them is hyperelliptic, because otherwise we would find three quadrics containing the image of one of them, whereas we have checked with the program Macaulay2 that in both cases there is only one such a quadric.

□

5.3. The local moduli space of the Craighero Gattazzo surface

It is known that the local moduli space of the Barlow surface is smooth of dimension 8 (cf. [**CL**], and also [**Lee**]). The main scope of this section is to prove that the same holds for the Craighero Gattazzo surface:

THEOREM 5.3.1. *The local moduli space of the Craighero Gattazzo surface is smooth of dimension* 8.

Proof.

Let X be the quintic constructed by Craighero and Gattazzo, and $\pi : S \to X$ its minimal resolution.

By Kodaira and Spencer's first main result in deformation theory (cf. [**KS**], also [**KM**]) our claim will be stablished if we show that $h^1(\Theta_S) = 8$, $h^2(\Theta_S) = 0$.

In fact, $h^0(\Theta_S) = 0$, since S is of general type, and moreover $h^1(\Theta_S) - h^2(\Theta_S) = -\chi(\Theta_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 = 8$. Therefore, it suffices to prove that $h^1(\Theta_S) = 8$.

Applying to the standard exact sequence

$$0 \to \mathcal{O}_X(-5) \to \Omega^1_{\mathbb{P}|X} \to \Omega^1_X \to 0$$

the functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{O}_X)$, we get the standard long exact sequence

$$H^0(\Theta_{\mathbb{P}^3|X}) \to H^0(\mathcal{O}_X(5)) \to$$

$$\to Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^1(\Theta_{\mathbb{P}^3|X}) \to H^1(\mathcal{O}_X(5)) \to$$

$$\to Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^2(\Theta_{\mathbb{P}^3|X}).$$

However, taking the restriction to X of the Euler exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^4 \to \Theta_{\mathbb{P}^3|X} \to 0$$

we can easily compute that $H^1(\Theta_{\mathbb{P}^3|X}) = H^2(\Theta_{\mathbb{P}^3|X}) = 0$.

Therefore, keeping also in mind that $H^1(\mathcal{O}_X(5)) = 0$, we find that the map $H^0(\mathcal{O}_X(5)) \xrightarrow{f} Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is surjective and that $Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$.

In turn, applying the Ext spectral sequence, we obtain the following exact sequence:

$$0 \to H^1(\Theta_X) \to Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \xrightarrow{g} H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \to H^2(\Theta_X) \to 0.$$

We are now going to show the vanishing of $H^1(\Theta_X)$.

Let us denote by p the natural projection $H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{O}_X(5))$, and consider the map $g \circ f \circ p : H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$.

The $\mathbb{Z}/4\mathbb{Z}$ action on X allows us to choose a basis in $H^0(\mathcal{O}_{\mathbb{P}^3}(5))$, say $v_1, \ldots, v_{56}$, s.t., if σ is the generator of the action given in the previous section,

$$\sigma(v_j) = \begin{cases} v_j & \text{if } 1 \le v_j \le 14 \\ iv_j & \text{if } 15 \le v_j \le 28 \\ -v_j & \text{if } 29 \le v_j \le 42 \\ -iv_j & \text{if } 43 \le v_j \le 56 \end{cases}$$

(notice in fact that σ acts freely on the set of monomials of degree 5).

We observe that $H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$, as a representation of $\mathbb{Z}/4\mathbb{Z}$, is isomorphic to the direct sum of the quotients of $\mathcal{O}_X$ by the jacobian ideal in the

4 singular points of X, and these addenda are permuted by σ, since the 4 singular points are an orbit for σ.

Thus the map $H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$ is given via a matrix of the following form:

$$\begin{pmatrix} A & B & C & D \\ A & iB & -C & -iD \\ A & -B & C & -D \\ A & -iB & -C & iD \end{pmatrix}$$

where every block is a matrix of size 10×14. We observe immediately that the above matrix has the same rank of the matrix

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix}.$$

We have explicitly checked with the program Macaulay2 that the matrices A, B, C, D have maximal rank, so that g is a surjective map; since

$$\dim\ Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \dim\ H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) = 40,$$

it follows that g is an isomorphism and therefore $H^1(\Theta_X) = H^2(\Theta_X) = 0$.

By [**BW**] $\pi_*(\Theta_S) = \Theta_X$. So, by the Leray spectral sequence we get $H^1(\Theta_S) \cong H^0(R^1\pi_*\Theta_S)$, and the last vector space equals, by the theorem on formal functions ([**H**])

$$\lim_{\leftarrow} H^1(\Theta_{S|nD}),$$

where D is the exceptional locus of π.

Since D consists of the sum of the four elliptic curves $D_1, \ldots, D_4$, corresponding to the 4 singular points of X, we can conclude that

$$h^1(\Theta_S) = 4\ \dim\ \lim_{\leftarrow} H^1(\Theta_{S|nC}),$$

where C is a smooth elliptic curve with $C^2 = -1,\ K_S C = 1$.

So we are left with proving the following lemma:

LEMMA 5.3.2. *Let S a smooth surface containing a smooth elliptic curve with normal bundle of degree -1. Then*

$$dim\ \lim_{\leftarrow} H^1(\Theta_{|nC}) = 2.$$

Proof.

Since a simple elliptic singularity is analytically isomorphic to the blow down of the 0-section in the normal bundle to the exceptional curve (cf. [**R1**], [**Lau**]), we can assume, w.l.o.g., that S the total space of a line bundle over C of degree -1, that is, $\mathcal{O}_C(-p)$ for some $p \in C$.

By the exact sequence

$$0 \to \Theta_C \to \Theta_{S|C} \to \mathcal{O}_C(-p) \to 0,$$

where C is a smooth elliptic curve (thus $\Theta_C = \mathcal{O}_C$), we get $h^1(\Theta_{S|C}) = 2$.

Tensoring this exact sequence by $\mathcal{O}_C(mp)$, we obtain, $\forall m > 0$, that $h^1(\Theta_{S|C}(mp)) = h^1(\mathcal{O}_C((m-1)p))$, whence we get 0 if $m \geq 2$, 1 for $m = 1$.

Applying this result to the exact sequence

$$0 \to \Theta_{S|C}(-(n-1)C) \to \Theta_{S|nC} \to \Theta_{S|(n-1)C} \to 0$$

we get that for $n \geq 3$, the restriction map $H^1(\Theta_{S|nC}) \to H^1(\Theta_{S|(n-1)C})$ is an isomorphism, therefore

$$\lim_{\leftarrow} H^1(\Theta_{|nC}) \cong H^1(\Theta_{|2C})$$

and $2 \leq h^1(\Theta_{S|2C}) \leq 3$.

Let us now consider the canonical projection $q : S \to C$; the fibre of q on every point is contractible, so for every line bundle L on S, $h^0(R^1 q_* L) = 0$, then $h^1(q_* L) = h^1(L)$. Moreover, $q_*(\mathcal{O}_S) \cong \bigoplus_{n \geq 0} \mathcal{O}_C(np)$.

Consider the exact sequence

$$(\#) \qquad 0 \to q^*\mathcal{O}_C(-p) \to \Theta_S \to q^*\Theta_C(\cong \mathcal{O}_S) \to 0.$$

Tensoring this sequence by $\mathcal{O}_S(-2C) \cong q^*\mathcal{O}_C(2p)$, since

$$H^i(q^*\Theta_C \otimes \mathcal{O}_S(-2C)) = H^i(q_*\mathcal{O}_S \otimes \mathcal{O}_C(2p)) =$$

$$= H^i((\bigoplus_{n \geq 0} \mathcal{O}_C(np)) \otimes \mathcal{O}_C(2p)) = \bigoplus_{n \geq 2} H^i(\mathcal{O}_C(np)),$$

we get $h^1(\Theta_S)(-2C) = h^2(\Theta_S)(-2C) = 0$, so $h^1(\Theta_{S|2C}) = h^1(\Theta_S)$.

Again by (#), since

$$h^1(q^*\Theta_C) = h^1(q_*\mathcal{O}_S) = \sum_{n \geq 0} h^1(\mathcal{O}_C(np)) = 1$$

$$h^1(q^*\mathcal{O}_C(-p)) = \sum_{n \geq -1} h^1(\mathcal{O}_C(np)) = 2$$

remembering that we have shown that $2 \leq h^1(\Theta_{S|2C}) = h^1(\Theta_S) \leq 3$, we see that we have to prove that the projection map

$$H^0(\Theta_S) \to H^0(q^*\Theta_C)$$

is not surjective.

We claim that we can write $S = (\mathbb{C}^* \times \mathbb{C})/\sim$, where $\sim$ is the equivalence relation generated by $(z,w) \sim (\mu^2 z, \mu w z)$. $C \subset S$ is defined by the equation $w = 0$, so that $C \cong \mathbb{C}^*/ < z \sim \mu^2 z >$.

In fact we can assume the point p to be the origin of the elliptic curve C, and we observe that every elliptic curve occurs as a quotient of $\mathbb{C}^*$ as above. Since the functional equation of the Riemann theta function is then $f(\mu^2 z) = \mu^{-1} z^{-1} f(z)$, we obtain the desired assertion.

We shall prove now that the global holomorphic never vanishing section of $q^*\Theta_C$ defined by $z\frac{\partial}{\partial z}$ is not a projection of a global section of Θ_S.

In fact, a global holomorphic vector field on S can be written as $a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w}$ with a, b global holomorphic functions on $\mathbb{C}^* \times \mathbb{C}$ satisfying the following functional equations: $\forall z, w \in \mathbb{C}^* \times \mathbb{C}$

$$a(z,w) = \mu^2 a(\frac{z}{\mu^2}, \mu\frac{w}{z})$$

$$b(z,w) = \mu w a(\frac{z}{\mu^2}, \mu\frac{w}{z}) + \mu z b(\frac{z}{\mu^2}, \mu\frac{w}{z}).$$

If there were a global holomorphic vector field on S whose projection on $q^*\Theta_C$ is $z\frac{\partial}{\partial z}$, then there would be a global holomorphic function b in $\mathbb{C}^* \times \mathbb{C}$ s.t.

$$b(z,w) = \mu^{-1} w z + \mu z b(\frac{z}{\mu^2}, \mu\frac{w}{z}).$$

Let us write b as a power series

$$b(z,w) = \sum_{n\in\mathbb{Z}, i\in\mathbb{N}} b_{ni} z^n w^i.$$

Then our condition can be written as :

$$\mu^{-1} wz = b(z,w) - \mu z b(\frac{z}{\mu^2}, \mu\frac{w}{z}) = \sum_{n,i} b_{ni}(z^n w^i - \mu z (\frac{z}{\mu^2})^n (\mu \frac{w}{z})^i) =$$

$$= \sum_{n,i} b_{ni}(z^n w^i - \mu^{i+1-2n} z^{n+1-i} w^i);$$

looking at the coefficient of wz we get $\mu^{-1} = b_{11}(1 - \mu^0) = 0$, a contradiction.

□

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