Fibrations of low genus and surfaces with $q = p_g = 1$

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joint work with F. Catanese

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Outline

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1. Introduction
   - Situation and motivation
   - The geometrical interpretation
2. The structure theorems
   - The relative canonical model
   - $g = 2$
   - $g = 3$
3. Minimal surfaces of general type with $p_g = q = 1$
   - Known results
   - The classification of the case $K^2 = 3$

Fibrations

Definition

In this talk a genus g fibration is a surjective morphism $f: S \to B$ of a smooth compact complex surface onto a smooth curve

- with connected fibres of genus g
- relatively minimal

- $g = 0$: $S = \mathbb{P}(V)$ for V rank 2 vector bundle over B
- $g = 1$: elliptic fibrations, central in classification theory
- $g \geq 2$: the case we are interested in.

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Goal

- Find numerical conditions for the existence of genus g fibrations (cf. Horikawa, Xiao, ...)
- Develop constructive methods to classify minimal surfaces of general type with low values of the invariants
  - K^2,
  - $q = h^1(\mathcal{O}_S)$,
  - $p_g = h^2(\mathcal{O}_S)$,
  - $\chi := \chi(\mathcal{O}_S) = 1 - q + p_g$

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$g = 2$: classical approach

Every smooth genus 2 curve is hyperelliptic: its canonical map is a double cover of $\mathbb{P}^1$ branched on 6 points

Glueing these maps one gets a double cover of a ruled surface

More precisely we get a *rational map* of degree 2

$$S \dashrightarrow \mathbb{P}(V_1)$$

where $V_1 = f_* \omega_{S|B}$ with $\omega_{S|B} := \omega_S \otimes f^* \omega_B^{-1}$

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The relative canonical map in genus 2

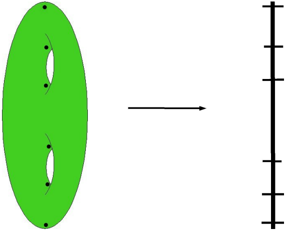

Figure: The canonical map of a genus 2 curve

Classical costruction: first construct the ruled surface $\mathbb{P}(V_1)$ (easy) and then a relative sextic with prescribed singularities

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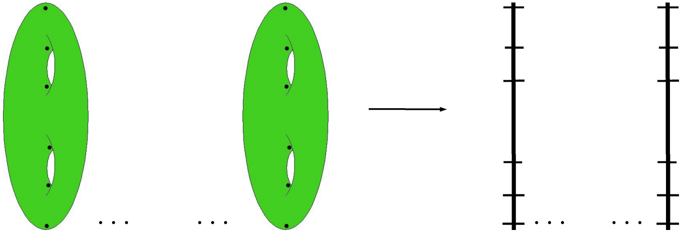

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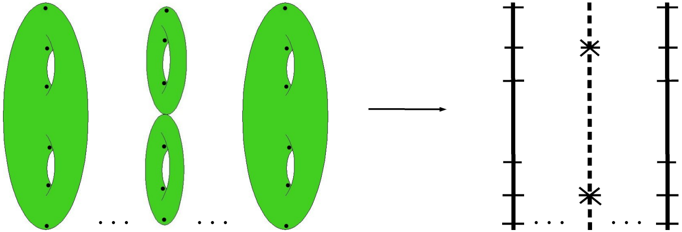

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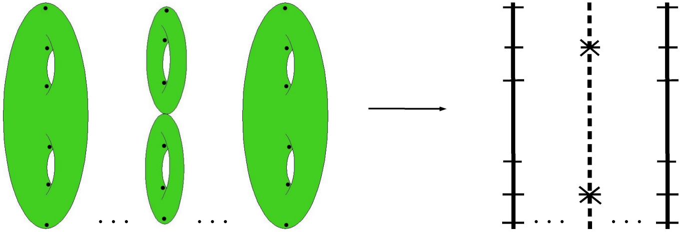

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$g = 2$: alternative approach

Geometric idea: the bicanonical map is nicer

The bicanonical map of a smooth genus 2 curve is a double cover of a conic in $\mathbb{P}^2$ branched on 6 points: it is simply the canonical map composed with the Segre embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

Glueing these maps we get a *morphism* of degree 2

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where $\mathcal{Q}$ is a conic subbundle of $\mathbb{P}(V_2)$, and $V_2 = f_* \omega_{S|B}^{\otimes 2}$

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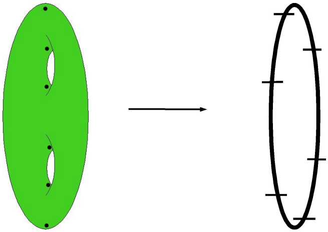

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Alternative costruction: first construct the (singular) conic bundle and then a relative cubic section without essential singularities

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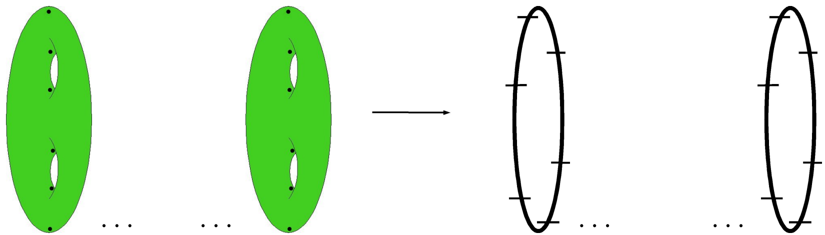

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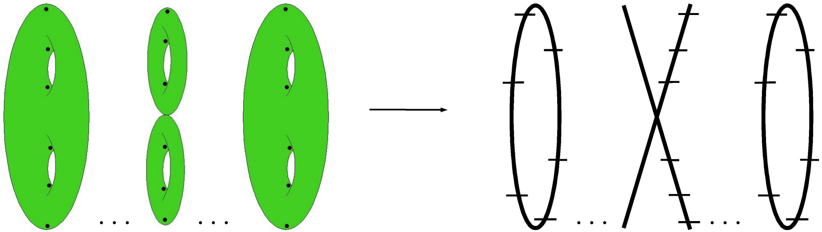

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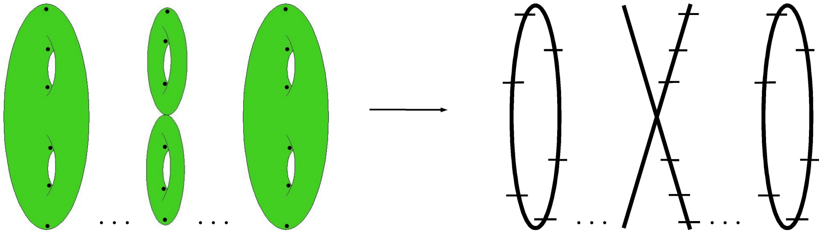

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$g = 3$ and nonhyperelliptic general fibre

The canonical map of a smooth genus 3 curve is

- either an embedding as a plane quartic
- or a double cover of a plane conic

The relative canonical map is a birational map

$$S \dashrightarrow \Sigma_1 \subset \mathbb{P}(V_1)$$

where $V_1 = f_* \omega_{S|B}$ and Σ_1 is a relative quartic singular along double conics corresponding to the hyperelliptic fibres

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The bicanonical map of a smooth genus 3 curve is an embedding in $\mathbb{P}^5$ as a quadric section

- of the Veronese surface in the nonhyperelliptic case
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The relative canonical algebra

The relative canonical algebra of a fibration f is the sheaf of $\mathcal{O}_B$-algebras $\mathcal{R}(f) := \bigoplus_{i=0}^{\infty} V_i$, where $V_i = f_*\omega_{S|B}^i$

The fibre of $\mathcal{R}(f)$ over a point p is the canonical ring $R(f^{-1}(p), K_{f^{-1}(p)})$

$\mathcal{R}(f)$ determines f since there is a birational morphism $S \to \mathbf{Proj}(\mathcal{R})$ commuting with the projections on B

The algebra structure induces a map $\sigma_2 : Sym^2 V_1 \to V_2$

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Constructing $\mathcal{Q}$

Lemma

There is a divisor τ on B such that there is an exact sequence

$$0 \to Sym^2(V_1) \stackrel{\sigma_2}{\to} V_2 \to \mathcal{O}_\tau \to 0$$

The above exact sequence induces a

$$\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, Sym^2 V_1)/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$$

B, V_1, τ, ξ determine the conic bundle $\mathcal{Q} \subset \mathbb{P}(V_2)$
In fact, if $\mathcal{A}$ is the sheaf of algebras generated by V_1, V_2 and σ_2 then $\mathcal{Q} \cong \mathbf{Proj}(\mathcal{A})$

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We say that (B, V_1, τ, ξ, w) is the 5-tuple associated to f

genus 2 5-tuples

Definition

(B, V_1, τ, ξ, w) is an admissible genus 2 $5-$tuple if B is a smooth curve, V_1 is a rank 2 vector bundle on B, $\tau \in Div_+(B)$, $\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, Sym^2 V_1)/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ yielding a v.b. (V_2), $w \in \mathbb{P}(H^0(\tilde{A}_6))$ and they verify three (easy to check) open conditions

Theorem

Associating to every genus 2 fibration its associated 5-tuple gives a bijection between genus 2 fibrations and admissible genus 2 $5-$tuples

Moreover $\chi(\mathcal{O}_S) = \deg V_1 + (g(B) - 1)$,
$K_S^2 = 2\deg V_1 + \deg \tau + 8(g(B) - 1)$

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Moreover $\chi(\mathcal{O}_S) = \deg V_1 + (g(B) - 1)$,
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The associated 5−tuple of a genus 3 fibration

In this case we assume that every fibre of $f\colon S \to B$ is 2−connected

Lemma

There is a divisor τ on B such that there is an exact sequence

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Outline

1. Introduction
   - Situation and motivation
   - The geometrical interpretation
2. The structure theorems
   - The relative canonical model
   - $g = 2$
   - $g = 3$
3. Minimal surfaces of general type with $p_g = q = 1$
   - **Known results**
   - The classification of the case $K^2 = 3$

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Minimal surfaces of general type with $p_g = q = 1$ have $\chi(\mathcal{O}_S) = 1$, the minimal possible value

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The case $g = 2$ exists

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The number of f_i equal to zero is one more than the number fo indecomposable factors of V_2.

We stratify our moduli space as $\mathcal{M}_I \cup \mathcal{M}_{II} \cup \mathcal{M}_{III}$ according to the number of indecomposable factors of V_2: B, V_1, τ, ξ give respectively $4, 3$ and 2 parameters in each case

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$w \in \mathbb{P}(H^0(\tilde{A}_6))$

By a result of Clemens the expected dimension is 5

- For $\mathcal{M}_I$ we get $H^0(\tilde{\mathcal{A}}_6) = 2$: a family of dimension $4 + 1 = 5$. This family exists and that its general element is a double cover of $\mathcal{Q}$ with irreducible branch divisor
- For $\mathcal{M}_{II}$ we get $H^0(\tilde{\mathcal{A}}_6) = 2$: a family of dimension $3 + 1 = 4$, too small
- For $\mathcal{M}_{III}$ we get $H^0(\tilde{\mathcal{A}}_6) = 4$ unless
  - $\tau = [0]$ and $H^0(\tilde{\mathcal{A}}_6) = 5$
  - τ is a nontrivial $2-$torsion and $H^0(\tilde{\mathcal{A}}_6) = 5$

  three families of dimension 5. The first case (τ general) do not fullfil our conditions. The other two cases exist, double covers of $\mathcal{Q}$ with disconnected branch divisor

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For Further Reading I

F. Catanese, R. Pignatelli,
Fibrations of low genus, I
at http://xxx.lanl.gov as math.AG/0503294
or
at http://www.science.unitn.it/ pignatel/papers.html

Thanks for your attention