Simple fibrations in (1,2) surfaces and 3-folds near the Noether line

Joint work with S. Coughlan 2207.06845
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Explicit Algebraic Geometry

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For projective varieties of general type, the volume measures the asymptotic growth of the plurigenera

$$\mathrm{vol}(X) = n! \limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^n} \qquad n := \dim X$$

If K_X is ample, it equals the selfintersection K_X^n

$n=1$

If X is a smooth curve of genus $g \geq 2$

$$2\mathbb{N} \ni \mathrm{vol}(X) = 2(g-1)$$

From now on: $p_g(X) = h^0(X, K_X)$

$n=2$ Classical Noether Inequality

If X is a surface

$$\mathbb{N} \ni \mathrm{vol}(X) \geq 2p_g - 4$$

since the canonical model of a surface is Gorenstein

$n=3$ 3-dimensional Noether inequality

After a long work with contributions of several scholars we have [J. Chen, M. Chen, C. Jiang '20]

Theorem 1. *Let X be a projective 3-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 11$. Then*

$$\mathbb{Q}_+ \ni \mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$

because the canonical model of a 3-fold is $\mathbb{Q}$-Gorenstein

To my knowledge, the inequality $\mathrm{vol}(X) \geq \frac{4}{3}p_g - \frac{10}{3}$ was first conjectured by M. Reid.

One interpretation of the Noether inequality in dim 2

X canonical surface that is
"canonical model of a surface S of general type": $\mathrm{Proj}\,(\bigoplus H^0(S, mK_S))$
its unique birational model with at most canonical singularities and K_X ample

then $\quad K_X^2 \geq 2p_g - 4$

WLOG $p_g \geq 2 \Rightarrow$ the canonical image Σ of X is a nondegenerate subvariety of $\mathbb{P}^{p_g-1}$ of dimension 1 or 2

If $\dim \Sigma = 1 \quad \Rightarrow K_X^2 \geq 4p_g - 6$ $\quad$ Xiao's inequality

If $\dim \Sigma = 2 \quad \Rightarrow K_X^2 \geq \deg(\varphi_K) \cdot \deg \Sigma$ $\quad$ φ_K is the canonical map

If $\deg \varphi_K = 1$, then Σ is of general type and then $\deg \Sigma \geq 3p_g - 7$ so $K^2 \geq 3p_g - 7$

Otherwise $\deg \Sigma \geq p_g - 2 \Rightarrow K^2 \geq 2p_g - 4$

Surfaces "on the Noether line." M. Noether – Enriques – Horikawa

Annals of Mathematics, **104** (1976), 357–387

Algebraic surfaces of general type with small c_1^2, I

By Eiji Horikawa*

In Section 1 we shall prove that the canonical map Φ_K induces a holomorphic map of degree 2 onto a surface of degree $n-1$ in $\mathbf{P}^n$, $n = p_g - 1$. We shall classify our surfaces according to their canonical images.

In Sections 3 and 5 we shall prove that minimal algebraic surfaces with given p_g and c_1^2 satisfying $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$ have one and the same deformation type provided that c_1^2 is not divisible by 8.

In Section 7 we shall study the case in which c_1^2 is divisible by 8. If we fix c_1^2, these surfaces are divided into two deformation types. They are homotopically equivalent or not according to whether c_1^2 is divisible by 16 or not. It is not known whether these two deformation types are diffeomorphic or not when c_1^2 is divisible by 16.

Recently Rana-Rollenske proved that the intersection of the closures of these two components in the KSBA moduli space is not empty.

Horikawa proves that, if $K^2 = 2p_g - 4$ and $p_g \geq 7$, then Σ is a

Segre-del Pezzo - Hirzebruch surface $\mathbb{F}_e$ embedded in $\mathbb{P}^{p_g-1}$

by $|\Delta_0 + \frac{(e+p_g-2)}{2}\Gamma|$ $p_g \geq 2(e-1)$

Here Δ_0 is the section with $\Delta_0^2 = -e$

Γ is the class of the ruling $\Gamma^2 = 0$

X is a double cover of $\mathbb{F}_e$ branched on $B \in |6\Delta_0 + (3e+p_g+2)\Gamma|$

$$(B - \Delta_0) \cdot \Delta_0 = 5(-e) + 3e + p_g + 2 = p_g + 2 - 2e \geq 0$$

The exceptional family happens when $(B - \Delta_0) \cdot \Delta_0 = 0$. Then $p_g = 2(e-1)$

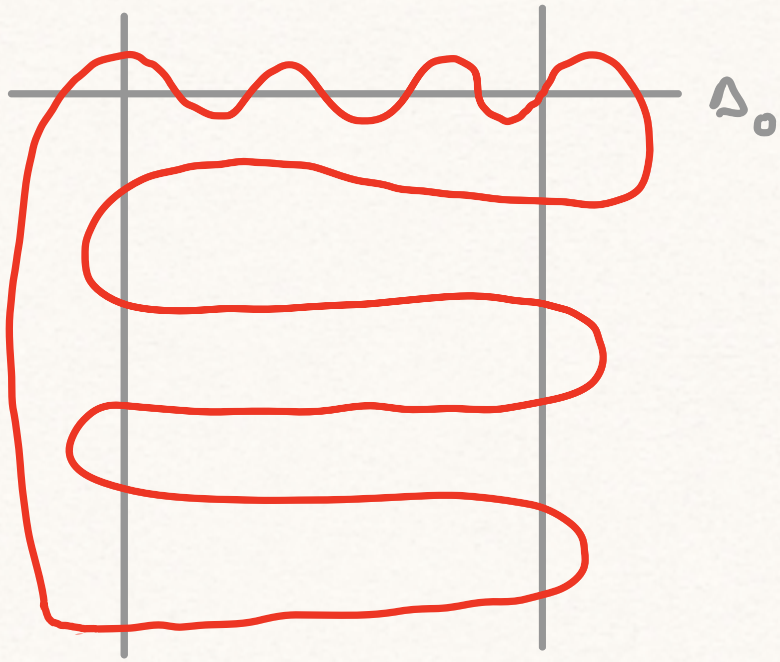
Then p_g, $e + p_g - 2$ are even $\Rightarrow$ e is even

$K^2 = 2p_g - 4 = 4e - 8$ is divisible by 8

Double cover description of the general surface in each family

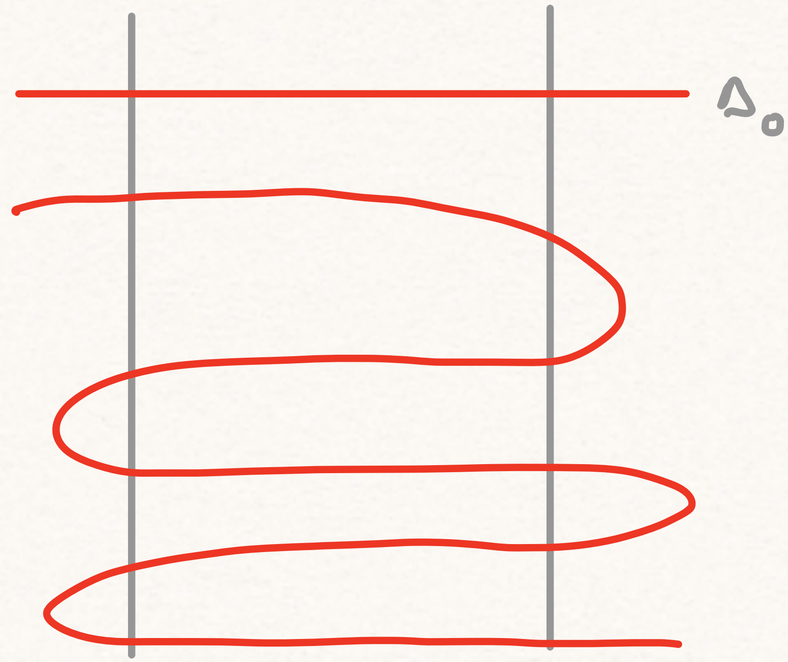
Main family

$\mathbb{F}_e$ $\quad e = 0, 1$



Exceptional family with K^2 div. by 8

$\mathbb{F}_e$ $\quad e = \frac{p_g}{2} + 1$



The ruling induces a genus 2 fibration on X

If $X \longrightarrow \mathbb{P}^1$ is a relatively minimal genus 2 fibration then (see [Horikawa '77 / Xiao Gang '85 / Catanese,- '06])

$$K_X^2 = 2\chi(\mathcal{O}_X) - 6 + \deg \tau = 2p_g - 4 + \deg \tau - 2q$$

for τ effective divisor on $\mathbb{P}^1$ supported on the image of the 2-disconnected fibers : $F = A + B$ with $A \cdot B = 1$

The 2-disconnected fibres are those with special canonical rings.

The general fibre are hypersurfaces $C_6 \subset \mathbb{P}(1,1,3)$

The 2-disconnected fibres are complete intersections $C_{2,6} \subset \mathbb{P}(1,1,2,3)$

So the surfaces on the Noether line with $p_g \geq 7$ are genus 2 fibrations with all fibres of type $C_6 \subset \mathbb{P}(1,1,3)$ and base $\mathbb{P}^1$

Toric description (also in Rana-Rollenske '22)

$$\mathbb{F} \qquad \begin{array}{ccccc} t_0 & t_1 & x_0 & x_1 & z \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & e & b \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}$$

and irrelevant ideal $I = (t_0, t_1) \cap (x_0, x_1, z)$

So we get $\mathbb{C}^5 \setminus V(I) \big/ (\mathbb{C}^*)^2$

$$(\lambda, \mu)(t_0, t_1, x_0, x_1, z) \to (\lambda t_0, \lambda t_1, \mu x_0, \lambda^e \mu x_1, \lambda^b \mu^3 z)$$

$$\begin{array}{c} \mathbb{F} \\ \downarrow \\ \mathbb{P}^1 \end{array}$$ is a fibration in $\mathbb{P}(1,1,3)$ singular along a section s_3

$[D_{t_0}] = [D_{t_1}] = F \quad [D_{x_0}] = H \qquad [D_{x_1}] = H + eF \qquad [D_z] = 3H + bF$

Take $X \in |6H + 2bF|$. X is defined by a polynomial of the form

$$\underline{z}^2 + x_0^6 f_{2b}(t_0,t_1) + \ldots\ldots + x_0 x_1^5 f_{2b-5e}(t_0,t_1) + x_1^6 f_{2b-6e}(t_0,t_1)$$

with $f_d(t_0,t_1)$ homogeneous of degree d. Note $X \cap s_3 = \emptyset$

$K_X = |H + (b-e-2)F| \quad H^0(X, K_X) = \langle f_{b-e-2}x_0, f_{b-2e-2}x_1 \rangle$

$p_g = 2b - 3e - 2 \qquad K^2 = 2p_g - 4$ and the canonical image is $\mathbb{F}_e$

$$z^2 + x_0^6 f_{2b}(t_0,t_1) + \ldots\ldots + x_0 x_1^5 \underline{f_{2b-5e}}(t_0,t_1) + x_1^6 f_{2b-6e}(t_0,t_1)$$

Note that if $5e > 2b$ both the coefficients of x_1^6 and $x_0 x_1^5$ vanish so X is not normal having a double line on $x_1 = z = 0$.

So $5e \leq 2b \iff$ Horikawa's $p_g \geq 2e-2$

The exceptional family comes then when $5e = 2b$ and X is

$$z^2 + x_0\left(x_1^5 + f_e(t_0,t_1)\, x_0 x_1^4 + f_{2e}(t_0,t_1) x_0^2 x_1^3 + \ldots + f_{5e} x_0^5\right)$$

Threefolds

This is the best result that I know on the Noether inequality of 3-folds

Theorem 1. *Let X be a projective 3-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 11$. Then*

$$\mathrm{vol}(X) \geq \frac{4}{3} p_g(X) - \frac{10}{3}.$$

It is published on '20, by J. Chen, M. Chen, C. Jiang.
It builds on several previous similar results by the same authors and their collaborators.

In all these proofs it appears evident that the threefolds "near the Nöther line" admit a fibration $X \dashrightarrow \mathbb{P}^1$ with general fibre a "(1,2)-surface".

Definition A $(1,2)$-surface is a canonical surface with
$K^2 = 1$ and $p_g = 2$

They are classified (by Horikawa?).

They are hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$.
In other words, their canonical ring is of the form $\mathbb{C}[\overset{1}{x_0},\overset{1}{x_1},\overset{2}{y},\overset{5}{z}]/_{z^2+\dots}$

Theorem [Franciosi, Pardini, Rollenske '17]

All Gorenstein stable surfaces with $K^2=1$ and $p_g=2$ are hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$

Our idea [Coughlan, –]

If fibrations in (1,2) surfaces play the same role for 3-folds as genus 2 fibrations for surfaces, those on the Noether line should be those with "no special fibres" whatever that means...

Then we decided to study fibrations such that ALL fibres are hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$

Definition 4.1. A *simple fibration in* $(1,2)$*-surfaces* is a morphism $\pi\colon X \to B$ between compact varieties of respective dimension 3 and 1 such that

(1) B is smooth;
(2) X has canonical singularities;
(3) K_X is π-ample;
(4) for all $p \in B$, the canonical ring $R(X_p, K_{X_p}) := \bigoplus_d H^0(X_p, K_{X_p})$ of the surface $X_p := \pi^{-1}p$ is generated by four elements of respective degree $1, 1, 2$ and 5 and related by a single equation of degree 10.

Definition 3.1. Let B be an algebraic variety, a_i positive integers. A *weighted symmetric algebra* $\mathcal{S}$ on B with weights $(a_1, \ldots, a_n)$ is a sheaf of graded $\mathcal{O}_B$-algebras $\mathcal{S} := \bigoplus_{d \geq 0} \mathcal{S}_d$ such that $\mathcal{S}_0 \cong \mathcal{O}_B$ and B is covered by open sets U with the property

$$\mathcal{S}_{|U} \cong \mathcal{O}_U[x_1, \ldots, x_n] \tag{8}$$

where $\mathcal{O}_U[x_1, \ldots, x_n]$ is graded by $\deg x_i = a_i$.

We consider, for all k, the truncation $S[k] \subset S$ to be the subalgebra generated in degree k

Definition 3.9. Let $\mathcal{S}$ be a weighted symmetric algebra with weights $(a_1^{r_1}, \ldots, a_n^{r_n})$ where $a_1 < \cdots < a_n$. For every $1 \leq j \leq n$, the *characteristic sheaf* of degree a_j is the cokernel $\mathcal{E}_{a_j}(\mathcal{S})$ of the natural inclusion

$$\sigma_{a_j} \colon \mathcal{S}[a_j - 1]_{a_j} \hookrightarrow \mathcal{S}_{a_j}.$$

Since $\mathcal{S}_{|U} \cong \mathcal{O}_U[x_1, \ldots, x_r]$, this is a locally free sheaf of rank r_j. We denote the projection maps by $\epsilon_{a_j} \colon \mathcal{S}_{a_j} \to \mathcal{E}_{a_j}(\mathcal{S})$.

Definition 3.12. Let $\mathcal{S}$ be a weighted symmetric algebra with weights $(a_1, \ldots, a_n)$. Then $\mathbb{F} := \mathbf{Proj}_B(\mathcal{S})$ is called a $\mathbb{P}(a_1, \ldots, a_n)$-bundle over B.

The following is a generalization of a famous result of Grothendieck

Proposition 3.19. *Let $\mathbb{F} = \mathbf{Proj}_B(\mathcal{S})$ be a well-formed $\mathbb{P}(a_1^{r_1}, \ldots, a_n^{r_n})$-bundle, $a_1 < a_2 < \cdots < a_n$.*

The relative dualising sheaf of $\mathbb{F}$ is

$$\omega_{\mathbb{F}/B} \cong \pi^* \left(\bigotimes_k \det \mathcal{E}_{a_k}\right) \left(-\sum_k r_k a_k\right),$$

where $\mathcal{E}_{a_k}$ is the characteristic sheaf of $\mathcal{S}$ in degree a_k.

Here π is the natural map $\pi : \mathbb{F} \to B$

We first proved the following natural result

Theorem 4.6. *Let $\pi\colon X \to B$ be a simple fibration in $(1,2)$-surfaces. Then there is a weighted symmetric algebra $\mathcal{S}(X)$ with weights $(1^2, 2, 5)$ such that X is isomorphic to a hypersurface of relative degree 10 in the $\mathbb{P}(1,1,2,5)$-bundle $\mathbb{F}(X) := \mathbf{Proj}_B(\mathcal{S}) \to B$.*

Then we started studying these hypersurfaces

Definition 4.8. The singular locus of a $\mathbb{P}(1,1,2,5)$-bundle over B is the disjoint union of two sections $\mathfrak{s}_2$ and $\mathfrak{s}_5$, where $\mathfrak{s}_k$ has Gorenstein index k.

Proposition 4.9. *Let $\pi\colon X \to B$ be a simple fibration in $(1,2)$-surfaces and suppose that $X \subset \mathbb{F}(X)$ where $\mathbb{F}(X)$ is the $\mathbb{P}(1,1,2,5)$-bundle constructed in Thm 4.6. Then*

(1) $X \cap \mathfrak{s}_5 = \emptyset$;

(2) $\mathfrak{s}_2 \not\subset X$.

In particular X is 2-Gorenstein, and Gorenstein iff $X \cap S_2 = \emptyset$

Proposition 4.21. *Let $\pi\colon X \to B$ be a simple fibration in (1,2)-surfaces. Then*

$$K_X^3 = \frac{4}{3}\chi(\omega_X) - 2\chi(\mathcal{O}_B) + \frac{N}{6} = \frac{4}{3}(p_g - q_2) + \frac{10}{3}(q_1 - 1) + \frac{N}{6}.$$

Here $q_3 = h^3(X, \mathcal{O}_X)$ and N is the degree of the coefficient of y^5.

Geometrically N is the expected cardinality of $X \cap S_2$.

So $N, q_1, q_2 \geq 0$. In fact q_1 equals the genus of B, and we proved $q_2 \leq 2$

Corollary 4.22. *Let $\pi\colon X \to B$ be a simple fibration in $(1,2)$-surfaces. Then $K_X^3 \geq \frac{1}{3}(4(p_g - q_2) - 10(1 - q_1))$ with equality holding if and only if X is Gorenstein.*

Finally $K^3 = \frac{4}{3}p_g - \frac{10}{3} \iff N = q_1 = q_2 = 0$

The classification

1.1. **Toric bundles.** Choose integers d, d_0 and define $\mathbb{F} = \mathbb{F}(d; d_0)$ to be the toric 4-fold with weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & d - d_0 & d_0 - 2d & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix} \tag{1}$$

and irrelevant ideal $I = (t_0, t_1) \cap (x_0, x_1, y, z)$. In other words $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^6$ with coordinates t_0, t_1, x_0, x_1, y, z via (1):

$$(\lambda, \mu) \cdot (t_0, t_1, x_0, x_1, y, z) = (\lambda t_0, \lambda t_1, \lambda^{d-d_0} \mu x_0, \lambda^{d_0 - 2d} \mu x_1, \mu^2 y, \mu^5 z)$$

and $\mathbb{F}$ is the quotient $(\mathbb{C}^6 \setminus V(I))/(\mathbb{C}^*)^2$.

Up to exchanging the x_j we may and do assume without loss of generality any of the following equivalent conditions:

$$d - d_0 \geq d_0 - 2d \iff d_0 \leq \tfrac{3}{2} d \iff e := 3d - 2d_0 \geq 0.$$

and let $X(d; d_0)$ be a Cartier divisor with canonical singularities of the form

$$z^2 + y^5 + \sum_{\substack{a_0 + a_1 + 2a_2 = 10 \\ a_2 \neq 5}} c_{a_0, a_1, a_2}(t_0, t_1) x_0^{a_0} x_1^{a_1} y^{a_2}$$

$X(d;d_0)$ are regular Gorenstein simple fibrations in $(1,2)$ surfaces.
We will say that they are "of type $(d;d_0)$"

Theorem Every Gorenstein simple fibration in $(1,2)$ surfaces is a $X(d;d_0)$

Proposition 1.6. *Gorenstein regular simple fibrations in $(1,2)$-surfaces of type (d, d_0) exist if and only if $d_0 \geq \frac{1}{4}d$. The singular locus of the general $X(d;d_0)$ is contained in the torus invariant section $\mathfrak{s}_0 := D_{x_1} \cap D_y \cap D_z$. More precisely*

(a) *X is nonsingular iff $d \leq d_0 \leq \frac{3}{2}d$ or $d_0 = \frac{7}{8}d$;*
(b) *X has $8d_0 - 7d$ terminal singularities iff $\frac{7}{8}d < d_0 < d$;*
(c) *X has canonical singularities along $\mathfrak{s}_0$ iff $\frac{1}{4}d \leq d_0 < \frac{7}{8}d$.* □

Those with $d_0 = \frac{7}{8}d$ have equation of the form

$$z^2 + y^5 + x_1(x_0^9 + g(t_0, t_1, x_0, x_1, y))$$

Canonical models

Theorem 1.11. *Gorenstein regular simple fibrations in* $(1,2)$*-surfaces of type* (d, d_0) *are canonical 3-folds if and only if* $\min(d, d_0) \geq 3$*. In these cases*

$$p_g = 3d - 2, \quad q_1 = q_2 = 0, \quad K_X^3 = 4d - 6 = \frac{4p_g - 10}{3}.$$

Their canonical image is the Hirzebruch surface $\mathbb{F}_e$, $e = 3d - 2d_0$.

$X(3;2)$ and $X(2;d)$, $2 \leq d \leq 8$, have K_X big and nef but not ample.

Their canonical models have the same invariants as above and then they lie on the Noether line.

The few $X(d;d_0)$ with $\min(d, d_0) \leq 1$ are

- either not of general type
- or their canonical model is above the Noether line (small p_g, small volume).

Theorem 5.4. *For every $p_g \geq 7$ of the form $3d-2$ let $\mathcal{N}^0_{p_g}$ be the subset of the moduli space of canonical threefolds with geometric genus p_g and $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ given by smooth simple fibrations in $(1,2)$-surfaces. Then $\mathcal{N}^0_{p_g}$ has*

one connected component if d is not divisible by 8
two connected components if d is divisible by 8.

All these components are unirational.

One component is formed by those 3-folds with canonical image $\mathbb{F}_e$, $0 \leq e \leq d$. This is an open subset of the moduli space of canonical 3-folds.

When d is divisible by 8 there is a second component of the moduli space of canonical 3-folds including smooth 3-folds whose canonical image is $\mathbb{F}_{\frac{5}{4}d}$. The intersection of the closures of the components in the moduli space of canonical 3-folds is not empty.

In particular the moduli space of canonical 3-folds with given $p_g = 3d-2$, $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ is reducible when d is divisible by 8.

Yong Hu and Tong Zhang proved the following (2022)

Theorem 1.5. *Let X be a minimal 3-fold of general type with $p_g(X) \geq 11$.*

(1) *Suppose that $K_X^3 > \frac{4}{3}p_g(X) - \frac{10}{3}$. Then we have the optimal inequality*

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{19}{6}. \tag{1.3}$$

If the equality holds, then $p_g(X) \equiv 2 \pmod 3$. Moreover, X has only one non-Gorenstein terminal singularity, and it is of type $\frac{1}{2}(1, -1, 1)$.

(2) *Suppose that $K_X^3 > \frac{4}{3}p_g(X) - \frac{19}{6}$. Then we have the optimal inequality*

$$K_X^3 \geq \frac{4}{3}p_g(X) - 3. \tag{1.4}$$

If the equality holds, then $p_g(X) \equiv 0 \pmod 3$. Moreover, one of the following two cases occurs:

(i) *X has two non-Gorenstein terminal singularities, and they are of the same type $\frac{1}{2}(1, -1, 1)$;*

(ii) *X has only one non-Gorenstein terminal singularity, and it is of type cA_1/μ_2.*

A similar analysis can be done for non Gorenstein simple fibrations in $(1,2)$-surfaces producing lots of 2-Gorenstein 3-folds with

$$K^3 = \frac{4}{3}p_g - \frac{10}{3} + \frac{N}{6} \qquad N \in \mathbb{N} \quad N \geq 1$$

For $N = 1,2$ we are exactly in the situation of Hu and Zhang

This motivates the following

Conjecture [Coughlan, P] There exist a $\varepsilon > 0$ such that all canonical 3-folds with $K^3 \leq \frac{4}{3}p_g - \frac{10}{3} + \varepsilon$ are simple fibrations in $(1,2)$ surfaces.

Theorem [Coughlan, Hu, P, Zhang]

The conjecture is true for $\varepsilon = \frac{1}{6}$.

All canonical 3-folds with $K^3 \leq \frac{4}{3}p_g - \frac{10}{3} + \frac{1}{6}$ with $p_g \geq 22$ are $X(d; d_0)$

What is the maximal ε? We know $\varepsilon < \frac{2}{3}$

There are not Gorenstein 3-Gorenstein canonical 3-folds (so not simple fibrations...)

with $K^3 = \frac{4}{3}p_g - \frac{8}{3}$

Take $\mathbb{F}$ toric with weight matrix
$$\begin{array}{ccccccc} t_0 & t_1 & x_0 & x_1 & y & u & z \end{array}$$
$$\begin{pmatrix} 1 & 1 & -a & -a & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \end{pmatrix}$$

and a general c.i of

$z^2 + y^5 + \dots \qquad\qquad t_0 u + \dots$

then $p_g = 6a \qquad K^3 = 8a - \frac{8}{3} = \frac{4}{3}p_g - \frac{8}{3}$

What's next?

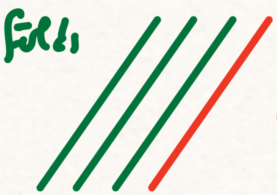
We are working on

- complete description of the moduli space of canonical 3-folds with $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ and $p_g \geqslant$ something

- find the optimal ε for which the conjecture is true

In the "best" case $\varepsilon = \frac{2}{3}$ the canonical 3-folds with $K^3 \lneq 4p_g - \frac{8}{3}$ would form 4 lines in the geography: $K^3 = \frac{4}{3}p_g - \frac{10}{3} + \frac{N}{6}$ $N = 0, 1, 2, 3$ filling all points (K^3, p_g) with $K^3 + \frac{N}{2} \in \mathbb{N}$

Canonical 3-folds of Gorenstein index 2



Noether line Gorenstein canonical 3-folds