Rigid but not infinitesimally rigid compact complex manifolds

joint work with I. Bauer (Bayreuth)

Roberto Pignatelli

University of Trento

Roberto.Pignatelli@unitn.it

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Overview

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   - Morrow-Kodaira's question
   - Difficulties

2. Our main results
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Two rigidities

Our title involves two different notions of rigidity of a compact complex manifold.

1. A compact complex manifold M is *rigid* if for each deformation of M, $f\colon (\mathfrak{X}, M) \to (B, b_0)$ there is an open neighbourhood $U \subset B$ of b_0 such that $M_t := f^{-1}(t) \cong M$ for all $t \in U$.
2. A compact complex manifold M is *infinitesimally rigid* if $H^1(M, \Theta) = 0$, where Θ is the sheaf of holomorphic vector fields on M.

Roughly speaking, M is rigid if every small deformation of its complex structure gives a complex manifold isomorphic to M.

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Morrow-Kodaira's question

By Kuranishi theory

$$M \text{ infinitesimally rigid} \Rightarrow M \text{ rigid}$$

so that infinitesimal rigidity provides a useful criterion for rigidity.

It is natural to ask if the converse implication also hold.

M-K Problem ([Morrow and Kodaira, 1971, page 45])

Find an example of a (compact complex manifold) M, which is rigid, but $H^1(M, \Theta) \neq 0$. (Not easy?)

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Interpretation of the M-K Problem in terms of Kuranishi family

By the work of Kuranishi, the base Def(M) of the semiuniversal deformation of M is the germ at 0 of the fibre over 0 of a holomorphic map

$$K\colon H^1(M,\Theta) \to H^2(M,\Theta),$$

called the Kuranishi map. Def(M) is the *Kuranishi family* of M.
If Def(M) is a point, *i.e.* $K^{-1}(0) = \{0\}$, then M is rigid.

The components of the Kuranishi map (once chosen a basis of $H^2(M,\Theta)$) are holomorphic functions whose Taylor expansion starts in degree 2. In particular $H^1(M,\Theta)$ is isomorphic to the Zariski tangent space of Def(M).

A solution of the M-K Problem is a manifold M such that $Def(M)$ is a *fat* point, a *singular* point.

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Nobody did it in almost 50 years. Why? Where is the difficulty?

M needs obstructed deformations: this is (now) easy.

We obviously need $\dim M \geq 2$.
Several examples of manifolds M of dimension 2 with obstructed deformations are known. Two important examples:

1. [Burns and Wahl, 1974] shows how to associate to each smooth rational curve E with $E^2 = -2$ in a complex surface M a $1-$dimensional subspace $H^1_E(M)$ of $H^1(M, \Theta)$: in particular M can't be infinitesimally rigid. [Burns and Wahl, 1974] constructs many examples of smooth surfaces with obstructed deformations by resolving the singularities of certain hypersurfaces in $\mathbb{P}^3$ with rational double points.
2. [Catanese, 1989] constructs surfaces M whose Kuranishi family $\mathrm{Def}(M)$ is *everywhere nonreduced* by resolving the singularities of certain quotients $(C_1 \times C_2)/G$ (C_i curves, G finite group) with rational double points.

M needs to be rigid: this is (still) difficult.

Theorem ([Bauer and Catanese, 2018])

Let S be a smooth compact complex surface, which is rigid. Then either

1. *S is a minimal surface of general type, or*
2. *S is a Del Pezzo surface of degree $d \geq 5$*
3. *S is an Inoue surface of type S_M or $S^{-}_{N,p,q,r}$*

Rigid Del Pezzo and Inoue surfaces are infinitesimally rigid, so every surface solving M-K Problem is of general type.

Rigid surfaces are rare. And it is hard to prove that a surface is rigid, unless you can prove that it is infinitesimally rigid.

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The main result

Then a solution of the M-K Problem is provided by any rigid surface of general type whose canonical model is singular.
We did it:

Theorem

For every even $n \geq 8$ such that $3 \nmid n$ there is a minimal regular surface S_n of general type with

$$K_{S_n}^2 = 2(n-3)^2, \quad p_g(S_n) = \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right),$$

such that S_n is rigid, but not infinitesimally rigid.

The canonical models of these surfaces have exactly 6 singular points, all of type A_1 (nodes). The hard part is proving their rigidity.

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Generalization to higher dimension

Lemma

Let M, N be compact complex manifolds, such that

$$H^1(\Theta_{M\times N}) = H^1(\Theta_M) \oplus H^1(\Theta_N).$$

Then $\mathrm{Def}(M \times N) = \mathrm{Def}(M) \times \mathrm{Def}(N)$.

Then, if M is a regular surface of general type solving the M-K Problem and N is a rigid manifold, by Künneth formula $M \times N$ is a solution too. Using some rigid manifolds in [Bauer and Catanese, 2018]

Theorem

There are rigid manifolds of dimension d and Kodaira dimension κ that are not infinitesimally rigid for all possible pairs (d, κ) with $d \geq 5$ and $\kappa \neq 0, 1, 3$ and for $(d, \kappa) = (3, -\infty), (4, -\infty), (4, 4)$.

Strategy to the proof of the main theorem

Rigid surfaces are rare. I know a short list of examples of rigid surfaces of general type, all infinitesimally rigid: ball quotients, irreducible bi-disk quotients, Beauville surfaces, Mostow-Siu surfaces, some Kodaira fibrations constructed by Catanese and Rollenske.

Example

Consider two projective curves C_1, C_2, a finite group G and two injective homomorphisms $G \subset Aut(C_i)$.
Assume that the induced action $g(x, y) = (gx, gy)$ of G on $C_1 \times C_2$ is free. Then $X := (C_1 \times C_2) / G$ is smooth.
If (C_i, G) are *triangle curves*, *i.e.* $C_i/G \cong \mathbb{P}^1$ and $p_i \colon C_i \to C_i/G$ has exactly three branching points, then X is a *Beauville surface*.

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Fabrizio's lemma

Lemma ([Catanese, 1989])

Let Z be a smooth algebraic surface and G a finite group acting on it freely in codimension 1. Set $p\colon Z \to X := X/G$.
Then $\Theta_X \cong (p_*\Theta_Z)^G$ and then $H^1(X, \Theta_X) \cong H^1(Z, \Theta_Z)^G$.

This implies that Beauville surfaces are infinitesimally rigid. Note however that here G may act not freely, and then X have isolated singularities.

Corollary

Consider two projective curves C_1, C_2, a finite group G and two injective homomorphisms $G \subset Aut(C_i)$. Set $X := (C_1 \times C_2)/G$.
If (C_i, G) are triangle curves, *then $H^1(X, \Theta_X) = 0$.*

The minimal resolution of the singularities $S \to X$ has no nontrivial deformations coming from the construction.
S may still be neither rigid nor infinitesimally rigid.

$(-2)-$curves obstruct infinitesimal rigidity

If a point $p \in C_1 \times C_2$ to has stabilizer of order 2, then X has a singular point of type A_1 (a *node*) at $[p]$, whose preimage in S is a smooth rational curve E with $E^2 = -2$.

Then, by the mentioned result in [Burns and Wahl, 1974] E produces a 1-dimensional subspace $H^1_E(S)$ of $H^1(S, \Theta_S)$ and then S is not infinitesimally rigid.

We need S to be rigid. First, we need to understand if $H^1_E(S)$ produces a true family of deformations (*smoothing the node*).

[Kas, 1977] gives a very explicit solution to this problem.

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The Kas map

Let now X be a compact complex surface with a node ν.
Let $S \to X$ be the minimal resolution of singularities of X and let E be exceptional curve mapping to ν.
Let θ be a generator of $H^1_E(\Theta_S) \subset H^1(S, \Theta_S)$.
Then $K(t\theta) = \alpha_\nu t^2 + O(3) \in H^2(S, \Theta_S)$.
By Serre duality we can see α_ν as a map $\alpha_\nu \colon H^0(S, \Omega^1_S \otimes \Omega^2_S) \to \mathbb{C}$.
A small neighbourhood U of ν in X is isomorphic to the quotient of a small disc $\Delta \subset \mathbb{C}^2$, with coordinates (z_1, z_2), by the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$.
This gives an inclusion $H^0(U, \Omega^1_U \otimes \Omega^2_U) \subset H^0(\Delta, \Omega^1_\Delta \otimes \Omega^2_\Delta)^+$.
Then every $\eta \in H^0\left(S, \Omega^1_S \otimes \Omega^2_S\right)$ can be locally written as

$$\eta = (f_1 dz_1 + f_2 dz_2) \otimes (dz_1 \wedge dz_2).$$

Then [Kas, 1977] shows

$$\alpha_\nu(\eta) = \left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right)(0,0). \qquad (1)$$

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A criterion to prove rigidity

Theorem

Let $S \to X$ be the minimal resolution of the singularities of a nodal surface X. Assume that

1. $H^1(X, \Theta_X) = 0$;
2. *the maps α_{ν_i} associated to the nodes ν_i of X locally described in (1) are linearly independent in $H^0(\Omega^1_S \otimes \Omega^2_S)^\vee$.*

Then S is rigid and $h^1(\Theta_S)$ equals the number of nodes of X.

Sketch of the proof.

By [Pinkham, 1981] $H^1(\Theta_S) = H^1(\Theta_X) \oplus \left(\bigoplus H^1_{E_i}(\Theta_S) \right)$ so, by condition 1, $H^1(\Theta_S) \cong \bigoplus H^1_{E_i}(\Theta_S)$. Choose $0 \neq \theta_i \in H^1_{E_i}(\Theta_S)$.
Write every $\theta \in H^1(\Theta_S)$ as $\theta = \sum_1^r t_i \theta_i$.
Then $[\theta, \theta] = \sum_1^r t_i^2 [\theta_i, \theta_i]$, implies $K(\theta) = \sum_1^r t_i^2 \alpha_{\nu_i} + O(3)$.
The rigidity follows now by condition 2. □

Strategy of the proof of the main theorem

Theorem

For every even $n \geq 8$ such that $3 \nmid n$ there is a minimal regular surface S_n of general type with $K^2_{S_n} = 2(n-3)^2$, $p_g(S_n) = \left(\frac{n}{2}-2\right)\left(\frac{n}{2}-1\right)$, such that S_n is rigid, but not infinitesimally rigid.

We pick two triangle curves (C_1, G), (C_2, G) for the same finite group, we set $X := (C_1 \times C_2)/G$ the quotient by the diagonal action.
We need X to be nodal and have at least one nodes. In other words, there is at least one point of $C_1 \times C_2$ whose stabilizer has order 2, and no points whose stabilizer has a higher order.
Then by Fabrizio's Lemma the first condition in the rigidity criterion $H^1(X, \Theta_X) = 0$ is fulfilled, and we only need to check the second one.
We check it using the Kas maps.

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The Fermat curve

We enter now in the details of the construction.

The Fermat curve of degree n, $C := \{\sum_{j=0}^{2} x_j^n = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ admits a natural action of the group $G \cong (\mathbb{Z}/n\mathbb{Z})^2$:

$$(a_1, a_2)(x_0 : x_1 : x_2) = (x_0 : e^{a_1 \frac{2\pi i}{n}} x_1 : e^{a_2 \frac{2\pi i}{n}} x_2).$$

This $G-$action has only three orbits of cardinality different by n^2, all of cardinality n:

- $C \cap \{x_0 = 0\}$ with stabilizer $\langle (1,1) \rangle \cong \mathbb{Z}/n\mathbb{Z}$
- $C \cap \{x_1 = 0\}$ with stabilizer $\langle (1,0) \rangle \cong \mathbb{Z}/n\mathbb{Z}$
- $C \cap \{x_2 = 0\}$ with stabilizer $\langle (0,1) \rangle \cong \mathbb{Z}/n\mathbb{Z}$

By Hurwitz formula $C/G \cong \mathbb{P}^1$ so (C, G) is a triangle curve.

The surfaces S_n

Fix $n \in \mathbb{N}$.

We choose as triangle curves (C_i, G) two copies of the Fermat curve of degree n.

For the first copy (C_1, G) we take the action as described in the previous slide, whereas for the second copy (C_2, G) we twist the action by the matrix

$$A := \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

that for all n not divisible by 3 defines an automorphism of G.

So we get, for every n not divisible by 3, a surface S_n, the minimal resolution of the singularities of $(C_1 \times C_2)/G$.

Some S_n are Beauville surfaces

The elements of G fixing some points of C_1 form, as we have seen, the set $\langle(1,0)\rangle \cup \langle(0,1)\rangle \cup \langle(1,1)\rangle$.

Acting with A we deduce that the elements of G fixing some points of C_2 form the set $\langle(1,2)\rangle \cup \langle(-2,-1)\rangle \cup \langle(-1,1)\rangle$.

For n odd, the intersection of these two sets is $\{(0,0)\}$. Then the induced action on $C_1 \times C_2$ is free: S_n is a Beauville surface, infinitesimally rigid. Then we need to assume n even.

For n even the non-trivial elements of G fixing some points of $C_1 \times C_2$ are $(n/2,0)$, $(0,n/2)$ and $(n/2,n/2)$, all of order 2 fixing n^2 points: then X is a nodal surface with $3 \cdot 2 \cdot n^2/n^2 = 6$ nodes.

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Decomposition of $H^0(S,\Omega^1_{S_n}\otimes\Omega^2_{S_n})$

We skip the computation of the invariants of S_n, that is standard.
We need to check if the six maps

$$\alpha_{\nu_i}\colon H^0(S,\Omega^1_{S_n}\otimes\Omega^2_{S_n})\to\mathbb{C}$$

associated to the nodes of X are linearly independent.
We need a basis of $H^0(S,\Omega^1_{S_n}\otimes\Omega^2_{S_n})$ as explicit as possible, in order to be able to compute their image via the Kas map. By

$$\begin{aligned}
H^0(S,\Omega^1_{S_n}\otimes\Omega^2_{S_n}) &\cong H^0(S,\Omega^1_{C_1\times C_2}\otimes\Omega^2_{C_1\times C_2})^G \cong\\
&\cong \left(H^0(\omega_{C_1}^{\otimes 2})\otimes H^0(\omega_{C_2})\right)^G \oplus \left(H^0(\omega_{C_1})\otimes H^0(\omega_{C_2}^{\otimes 2})\right)^G \cong\\
&\cong \bigoplus_{\chi\in G^*}\left(\left(H^0(\omega_{C_1}^{\otimes 2})^\chi\otimes H^0(\omega_{C_2})^{-\chi}\right)\oplus\left(H^0(\omega_{C_1})^\chi\otimes H^0(\omega_{C_2}^{\otimes 2})^{-\chi}\right)\right)\cong\\
&\cong \bigoplus_{\chi\in G^*}\left(\left(H^0(\omega_{C_1}^{\otimes 2})^\chi\otimes H^0(\omega_{C_1})^{-\chi'}\right)\oplus\left(H^0(\omega_{C_1})^\chi\otimes H^0(\omega_{C_1}^{\otimes 2})^{-\chi'}\right)\right)
\end{aligned}$$

where. writing χ,χ' as a column, $\chi':={}^tA^{-1}\chi$

Decomposition of $p_*\omega_C$

The Fermat triangle curve $p\colon C = C_1 \to \mathbb{P}^1$ is an abelian cover, with group G. We compute then the decomposition of $p_*\omega_C$ by the formulas in [Pardini, 1991].

n-1	-1	-1	-1	-1	-1	*	-1	-1	-1	-1	-1
n-2	-1	0	-1	-1	-1	*	-1	-1	-1	-1	-1
	-1	0	0	-1	-1	*	-1	-1	-1	-1	-1
	-1	0	0	0	-1	*	-1	-1	-1	-1	-1
	-1	0	0	0	0	*	-1	-1	-1	-1	-1
	*	*	*	*	*	*	*	*	*	*	*
	-1	0	0	0	0	*	0	-1	-1	-1	-1
3	-1	0	0	0	0	*	0	0	-1	-1	-1
2	-1	0	0	0	0	*	0	0	0	-1	-1
1	-1	0	0	0	0	*	0	0	0	0	-1
0	-2	-1	-1	-1	-1	*	-1	-1	-1	-1	-1
	0	1	2	3							n-1

Figure: The degrees of $(p_*\omega_C)^{(\alpha,\beta)}$

Decomposition of $p_*\omega_C^{\otimes 2}$

Similarly we compute the decomposition $p_*\omega_C^{\otimes 2}$.

n-1	-1	-1	0	0	0	*	0	0	0	0	0
n-2	0	-1	0	0	0	*	0	0	0	0	0
n-3	0	0	0	0	0	*	0	0	0	0	0
	0	0	1	0	0	*	0	0	0	0	0
	0	0	1	1	0	*	0	0	0	0	0
	*	*	*	*	*	*	*	*	*	*	*
	0	0	1	1	1	*	0	0	0	0	0
3	0	0	1	1	1	*	1	0	0	0	0
2	0	0	1	1	1	*	1	1	0	0	0
1	-1	-1	0	0	0	*	0	0	0	-1	-1
0	-1	-1	0	0	0	*	0	0	0	0	-1
	0	1	2	3							n-1

Figure: The degrees of $(p_*\omega_C^{\otimes 2})^{(\alpha,\beta)}$

Note that $\forall n \geq 4$ the degree is negative only for 10 characters.

Six good characters are enough

Lemma

Set $k_0 = k_1 = (1,0)$, $k_\infty = (0,1) \in G$. *Assume that there is a set of six characters* $\mathcal{C} := \{\chi_0, \chi'_0, \chi_1, \chi'_1, \chi_\infty, \chi'_\infty\} \subset G^*$, *such that*

1. $\chi_0 \equiv \chi'_0 \equiv (0,1)$, $\chi_1 \equiv \chi'_1 \equiv (1,1)$, $\chi_\infty \equiv \chi'_\infty \equiv (1,0)$ *mod* 2;
2. $\forall p \in \{0,1,\infty\}$, $\chi_p(k_p) \neq \chi'_p(k_p)$;
3. *if* $\chi \in \mathcal{C}$, *then* $H^0(\omega_C)^{(\chi)} \neq \{0\}$, $H^0(\omega_C^{\otimes 2})^{(-\chi')} \neq \{0\}$.

Then condition 2 in the rigidity criterion holds.

Sketch of the proof - part 1

We need to check the linear independence of the six maps α_{ν_j}.
We decomposed $H^0(S, \Omega^1_{S_n} \otimes \Omega^2_{S_n})$ obtaing addenda of the form $H^0(\omega_{C_1})^\chi \otimes H^0(\omega_{C_1}^{\otimes 2})^{-\chi'}$. When $\chi \in \mathcal{C}$, by condition 3, the addendum is not trivial. Picking one general element in each of them, we get six different elements in $H^0(S, \Omega^1_{S_n} \otimes \Omega^2_{S_n})$.

Six good characters are enough

Lemma

Set $k_0 = k_1 = (1,0)$, $k_\infty = (0,1) \in G$. *Assume that there is a set of six characters* $\mathcal{C} := \{\chi_0, \chi_0', \chi_1, \chi_1', \chi_\infty, \chi_\infty'\} \subset G^*$, *such that*

1. $\chi_0 \equiv \chi_0' \equiv (0,1)$, $\chi_1 \equiv \chi_1' \equiv (1,1)$, $\chi_\infty \equiv \chi_\infty' \equiv (1,0)$ *mod* 2;
2. $\forall p \in \{0,1,\infty\}$, $\chi_p(k_p) \neq \chi_p'(k_p)$;
3. *if* $\chi \in \mathcal{C}$, *then* $H^0(\omega_C)^{(\chi)} \neq \{0\}$, $H^0(\omega_C^{\otimes 2})^{(-\chi')} \neq \{0\}$.

Then condition 2 in the rigidity criterion holds.

Sketch of the proof - part 2.

Computing explicitely the six Kas maps (1) in them we get the following

$$
\begin{array}{lll}
(\chi_0(k_0),1,0,0,0,0) & (0,0,\chi_1(k_1),1,0,0) & (0,0,0,0,\chi_\infty(k_\infty),1) \\
(\chi_0'(k_0),1,0,0,0,0) & (0,0,\chi_1'(k_1),1,0,0) & (0,0,0,0,\chi_\infty'(k_\infty),1)
\end{array}
$$

□

End of the proof of the main theorem

We need then to find six characters such that

1. $\chi_0 \equiv \chi_0' \equiv (0,1)$, $\chi_1 \equiv \chi_1' \equiv (1,1)$, $\chi_\infty \equiv \chi_\infty' \equiv (1,0) \mod 2$;
2. $\forall p \in \{0,1,\infty\}$, $\chi_p(k_p) \neq \chi_p'(k_p)$;
3. if $\chi \in \mathcal{C}$, then $H^0(\omega_C)^{(\chi)} \neq \{0\}$, $H^0(\omega_C^{\otimes 2})^{(-\chi')} \neq \{0\}$.

S_4 is a numerical Campedelli surface with fundamental group $(\mathbb{Z}/2\mathbb{Z})^3$: its Kuranishi family has dimension 6. So we need $n \geq 8$.

End of the proof.

We pick the following six characters

$$\begin{array}{lll} \chi_0 = (2,1) & \chi_1 = (1,3) & \chi_\infty = (1,2) \\ \chi_0' = (4,1) & \chi_1' = (3,1) & \chi_\infty' = (1,4) \end{array}$$

The only check that is not trivial is that $H^0(\omega_C^{\otimes 2})^{(-\chi')} \neq \{0\}$: this indeed fails for $n = 4$ but a tedious computation shows that it holds for $n \geq 8$. □

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