Surfaces with Non-Polyhedral Effective Cone

based on a joint work with F. Fallucca and F. Polizzi

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Cox rings of surfaces with geometric genus zero

Many authors have studied Cox rings of various varieties. However it seems that not much is known about Cox rings of varieties of general type. Moreover there are few literatures discussing Mori dream spaces of general type.

In a 2019 paper ([4]) J. Keum and K-S. Lee focus on surfaces of general type with $p_g = 0$, a very important class of surfaces of general type.

They provide several examples of minimal surfaces of general type with $p_g = 0$ that are Mori dream spaces, as well as a few of non-minimal ones that are not.

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J. Keum and K-S. Lee, at the end of [4], commented

It is also interesting to find examples of minimal surfaces of general type (especially those with $p_g = 0$) which are not Mori dream.

The only answer know to us was obtained last year

Theorem ([1] P. Cascini, F. Catanese, Y. Chen, J. Keum 2025)

A Shavel type surface S is an even fake quadric which is not a Mori dream surface.

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Moreover in the introduction of [4], Keum and Lee asked the following

Question

Question 1.1 Let X be a minimal surface of general type with $p_g = 0$. When the effective cone of X is rational polyhedral? When the nef cone and the semiample cone of X are the same? When the Cox ring of X is finitely generated?

In our research we focused on the pseudo-effective cone, the closure of the effective cone, and his polyhedrality. Note that for a fake quadric $\rho = 2$, so every closed cone is polyhedral. In fact, [1] proves that a Shavel type surface has a nef class that is not semiample.

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The main theorem

Theorem (Fallucca, P, Polizzi)

Let $f: S \to B$ be a fibration of a smooth projective surface onto a smooth curve. Assume that the pseudo-effective cone of S is polyhedral. Then

$$\rho(S) = n(f) + 2$$

Here $\rho(S)$ is the usual Picard number of S. The number $n(f)$ is the number of reducible fibres of f in the following sense.

Definition

Let $f: S \to B$ be a fibration. For each $b \in B$ let k_b be the number of distinct irreducible components of the fibre $f^{-1}b$. Then

$$n(f) := \sum_{b \in B} k_b - 1.$$

Elliptic surfaces

An elliptic surface is a surface with a fibration $f\colon S \to B$ whose general fibre is an elliptic curve.

By the Shioda-Tate formula, if $q(S) = 0$ and f has a section, then

$$\rho(S) - n(f) - 2$$

equals the rank of its Mordell-Weil group, the Mordell-Weil group of the general fibre, a finitely generated abelian group.

Since every Mori dream space has a polyhedral pseudo-effective cone, we deduce:

Corollary

The Mordell-Weil group of a Mori dream elliptic surface with a section is finite.

A criterion for a surface not to be a Mori dream space

A useful corollary is

Corollary

Let $f : S \to B$ be a fibration of a projective surface onto a curve. Assume that

$$\rho(S) > n(f) + 2.$$

Then S is not a Mori dream space, since its pseudo-effective cone is non-polyhedral (in the neighbourhood of the class of a fibre).

In the spirit of [4] Question 1.1, we applied it to several families of minimal surfaces of general type, thereby showing in this way that they are not Mori dream spaces.

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The Craighero-Gattazzo surface

This is a famous minimal simply connected surface of general type with $K^2 = 1$ and $p_g = 0$ constructed by Craighero and Gattazzo in 1994, studied mainly by Dolgachev-Werner (1999) and Rana-Tevelev-Urzúa (2017). It has $\rho(S) = h^{1,1}(S) = 9$.

Dolgachev and Werner studied a base point free pencil on the Craighero-Gattazzo surface, which they call $|3K - R|$. Setting $f : S \to \mathbb{P}^1$ to be the associated fibration, we computed $n(f) = 5$:

Corollary

The Craighero-Gattazzo surface is not a Mori dream space.

This provides a further answer to Keum and Lee's request for examples of minimal surfaces of general type with $p_g = 0$ that are not Mori dream spaces, very different from the one in [1].

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If a smooth surface $S \subset \mathbb{P}^3$ contains a line l, then the pencil of the planes containing l cuts out a base-point-free pencil $|H_S - l|$ on S, to which is natural to apply our result.
Setting $f\colon S \to \mathbb{P}^1$ for the fibration induced by $|H_S - l|$ we proved $n(f) \leq 2\frac{(d-2)^2(d+2)}{d-1}$ and then

Proposition

Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 2$ containing a line l.

If $\rho(S) > 2\frac{(d-2)^2(d+2)}{d-1} + 2$ then S is not a Mori dream space, as its pseudo-effective cone is non-polyhedral.

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Natural examples are the Fermat surfaces

$$x_0^d + x_1^d + x_2^d + x_3^d = 0$$

whose Picard numbers have been computed by Shioda.
We obtain

Corollary

All Fermat surfaces of degree $d \geq 4$ are not Mori dream spaces.

To our knowledge, this was previously known only for $d = 4$.
Those with $d = 2, 3$ are del Pezzo surfaces, which are well known to be Mori dream spaces.

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Proof of the main theorem: the easy part

Let's recall the statement:

Theorem (Fallucca, P, Polizzi)

Let $f : S \to B$ a fibration of a smooth projective surface onto a smooth curve. Assume that the pseudo-effective cone of S is polyhedral. Then

$$\rho(S) = n(f) + 2$$

The easy part is the following rather straightforward statement

Lemma

Let $f : S \to B$ a fibration of a smooth projective surface onto a smooth curve. Then $\rho(S) \geq n(f) + 2$.

Proof.

The subspace W of $H^{1,1}(S)$ generated by the curves contracted by f has dimension $n(f) + 1$. But no ample class is in W, so $\rho(S) > \dim W$. $\square$

Proof of the main theorem: the other part

We now only need to prove the following proposition.

Proposition

Let $f\colon S \to B$ be a fibration of a smooth projective surface onto a smooth curve. Assume that

$$\rho(S) > n(f) + 2.$$

Then the pseudo-effective cone $\overline{Eff}(S)$ of S is non-polyhedral.

The kernel of the map $f_*\colon N^1(S) \cong N_1(S) \to N_1(B)$ has dimension $\rho(S) - 1 > n(f) + 1 = \dim W$. So we have a class $v \in \ker f_*$, $v \notin W$ and we may choose it with $vh = 0$ for a fixed ample class h.

By a result of Debarre-Jiang-Voisin, $\ker f_* \cap \overline{Eff}(S)$ is contained in W. So, if e is the class of the general fibre, the line $e + \mathbb{R}v$ intersects $\overline{Eff}(S)$ only at e. This implies the result by a Lemma of Rabindranath.

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Rabindranath's Lemma

Lemma

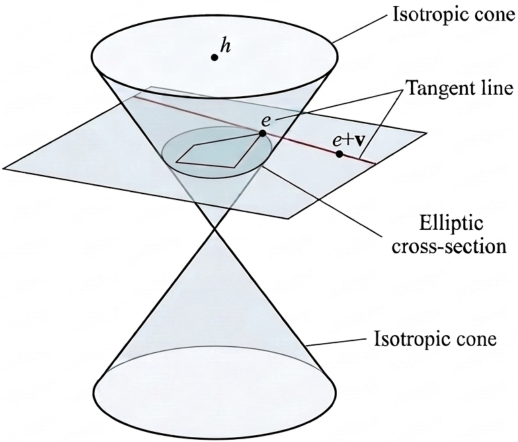
Let S be a smooth projective surface. Suppose that one can find three non-zero classes $e, v, h \in N^1(S)$, where e is pseudo-effective and h is ample, such that $e^2 = ev = hv = 0$ and moreover

$$(e + \mathbb{R}v) \cap \overline{Eff}(S) = \{e\}. \tag{1}$$

Then $\overline{Eff}(S)$ is non-polyhedral.

Sketch of proof.

By the index theorem, the unique affine plane π through e, v, h cuts the isotropic cone $\{x | x^2 = 0\}$ in an ellipse C. The classes in the interior of C have positive self-intersection and intersect h positively, so they are nef. This implies $C \subset \overline{Eff}(S)$. If $\overline{Eff}(S)$ is polyhedral, then $\overline{Eff}(S) \cap \pi$ is a convex polygon with, by (1), e as a vertex. Thus, we would have an ellipse contained in a polygon and passing through one of its vertices, absurd. $\square$


Isotropic cone
h
Tangent line
e
e+v
Elliptic
cross-section
Isotropic cone

Related results

1. Rabindranath, in his Ph.D. thesis (2018), proved an upper bound for $\rho(S)$ by the same strategy: in fact we obtained our result by discussing his proof;
2. Exactly 7 days ago, C. Casagrande informed us that in a paper of 2022 she had found the equality $\rho(S) = n(f) + 2$ under the further assumption that S is a Mori dream surface;
3. Exactly 7 days ago, A. Laface informed us that in a paper of 2023 with Castravet, Tevelev and Ugaglia they had found the equality $\rho(S) = n(f) + 2$ under the further assumption that $B \cong \mathbb{P}^1$.

References for the talk

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Thank you for your attention to this matter!

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