Quasi-étale quotients of products of two curves

Roberto Pignatelli

Department of Mathematics
University of Trento

8 June 2012

Definition

A surface is a projective compact complex manifold of dimension 2*.*

Definition

A surface is a projective compact complex manifold of dimension 2*.*

The goal

Our goal is to construct surfaces.

Definition

A surface is a projective compact complex manifold of dimension 2*.*

The goal

Our goal is to construct interesting surfaces.

Definition

A surface is a projective compact complex manifold of dimension 2.

The goal

Our goal is to construct interesting surfaces.

How do we decide that a surface is "interesting"?

Definition

A surface is a projective compact complex manifold of dimension 2.

The goal

Our goal is to construct new interesting surfaces.

How do we decide that a surface is "interesting"?

Definition

A surface is a projective compact complex manifold of dimension 2*.*

The goal

Our goal is to construct new interesting surfaces.

How do we decide that a surface is "interesting"?

How do we check that a surface is "new"?

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

The classical birational invariants are

- the geometric genus
  $p_g(S) = h^{2,0}(S) = h^0(\Omega^2_S) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$.

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

The classical birational invariants are

- the geometric genus $p_g(S) = h^{2,0}(S) = h^0(\Omega_S^2) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$.
- the irregularity $q(S) = h^{1,0}(S) = h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$.

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

The classical birational invariants are

- the geometric genus $p_g(S) = h^{2,0}(S) = h^0(\Omega_S^2) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$.
- the irregularity $q(S) = h^{1,0}(S) = h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$.
- the Euler characteristic $\chi = \chi(\mathcal{O}_S) = 1 - q + p_g$.

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

The classical birational invariants are

- the geometric genus $p_g(S) = h^{2,0}(S) = h^0(\Omega^2_S) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$.
- the irregularity $q(S) = h^{1,0}(S) = h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$.
- the Euler characteristic $\chi = \chi(\mathcal{O}_S) = 1 - q + p_g$.
- The plurigenera $P_n(S) = h^0(\mathcal{O}_S(nK_S))$.

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

The classical birational invariants are

- the geometric genus $p_g(S) = h^{2,0}(S) = h^0(\Omega_S^2) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$.
- the irregularity $q(S) = h^{1,0}(S) = h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$.
- the Euler characteristic $\chi = \chi(\mathcal{O}_S) = 1 - q + p_g$.
- The plurigenera $P_n(S) = h^0(\mathcal{O}_S(nK_S))$.
- The Kodaira dimension $\kappa(S)$ is the rate of growth of the plurigenera: it is the smallest number κ such that P_n/n^κ is bounded from above.

Birational Invariants

We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.

The classical birational invariants are

- the geometric genus $p_g(S) = h^{2,0}(S) = h^0(\Omega_S^2) = h^0(\mathcal{O}_S(K_S)) = h^2(\mathcal{O}_S)$.
- the irregularity $q(S) = h^{1,0}(S) = h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$.
- the Euler characteristic $\chi = \chi(\mathcal{O}_S) = 1 - q + p_g$.
- The plurigenera $P_n(S) = h^0(\mathcal{O}_S(nK_S))$.
- The Kodaira dimension $\kappa(S)$ is the rate of growth of the plurigenera: it is the smallest number κ such that P_n/n^κ is bounded from above.
- The (topological or algebraic) fundamental group.

Surfaces of general type

The Enriques-Kodaira classification provides a relatively good understanding of the surfaces of *special* type, which are those with $\kappa(S) < 2$.

Definition

A surface is *of general type* if $\kappa(S) = 2$ (equiv.$\kappa(S) \geq 2$) .

Surfaces of general type

The Enriques-Kodaira classification provides a relatively good understanding of the surfaces of *special* type, which are those with $\kappa(S) < 2$.

Definition

A surface is *of general type* if $\kappa(S) = 2$ (equiv.$\kappa(S) \geq 2$) .

Definition

A surface is *minimal* if K_S is *nef*, that is if the intersection of K_S with any curve is nonnegative.

Surfaces of general type

The Enriques-Kodaira classification provides a relatively good understanding of the surfaces of *special* type, which are those with $\kappa(S) < 2$.

Definition

A surface is *of general type* if $\kappa(S) = 2$ (equiv.$\kappa(S) \geq 2$) .

Definition

A surface is *minimal* if K_S is *nef*, that is if the intersection of K_S with any curve is nonnegative.

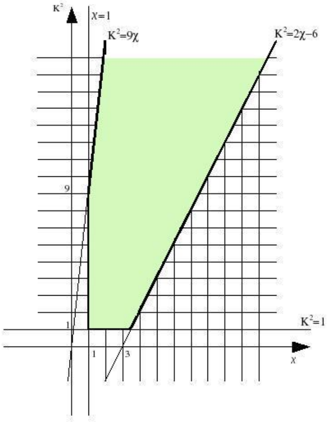
In every birational class of surfaces of general type there is exactly one *minimal* surface. If S is a surface of general type, S is obtained by the only minimal surface in its birational class $\bar{S}$ by a sequence of $K^2_{\bar{S}} - K^2_S$ blow ups.

Inequalities for surfaces of general type

If S is of general type, then the Riemann-Roch formula computes all $P_n(S)$ from χ and K_S^2.

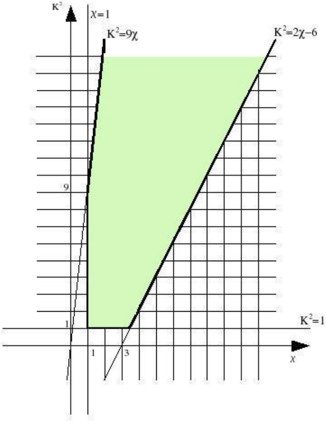
Inequalities for surfaces of general type

If S is of general type, then the Riemann-Roch formula computes all $P_n(S)$ from χ and $K_{\bar{S}}^2$.The possible values of the pair $(\chi, K_{\bar{S}}^2)$ are almost all the integral points of the unbounded green region below.



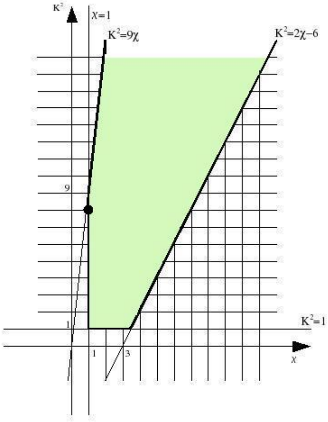
Inequalities for surfaces of general type

In general, the surfaces with the most interesting geometry are the ones when "the inequalities are equalities", as for the boundary of the picture. This includes the surfaces with $\chi = 1$ and, among those, the surfaces with $p_g = 0$.



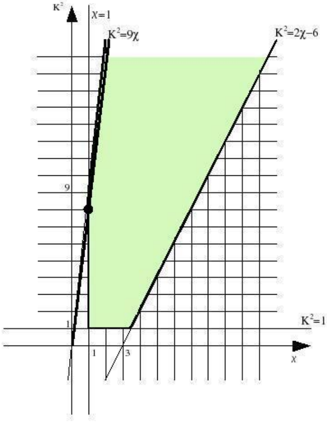
Beauville's idea

Beauville suggestion: take $S = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a free group of automorphisms of order $(g_1 - 1)(g_2 - 1)$; S is automatically minimal of general type with $\chi = 1$ and $K^2 = 8$.



Surfaces isogenous to a product

A surface is isogenous to a (higher) product if $S = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a free group of automorphisms; S is automatically minimal of general type, with $K^2 = 8\chi$.



Quasi-étale quotients

Definition

A *quasi-étale surface* is $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

Quasi-étale quotients

Definition

A *quasi-étale surface* is $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi\colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).

Quasi-étale quotients

Definition

A *quasi-étale surface* is $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi\colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).This implies $\pi^* K_X = K_{C_1 \times C_2}$, so K_X is nef .

Quasi-étale quotients

Definition

A *quasi-étale surface* is $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi \colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).This implies $\pi^* K_X = K_{C_1 \times C_2}$, so K_X is nef .

Disadvantages:

- X is singular,

Quasi-étale quotients

Definition

A *quasi-étale surface* is $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi\colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).This implies $\pi^* K_X = K_{C_1 \times C_2}$, so K_X is nef .

Disadvantages:

- X is singular,

Quasi-étale quotients

Definition

A *quasi-étale surface* is $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi\colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).This implies $\pi^* K_X = K_{C_1 \times C_2}$, so K_X is nef .

Disadvantages:

- X is singular, we need to consider a resolution of its singularities S.

Quasi-étale quotients

Definition

A *quasi-étale surface* is the min. res. S of the sings of $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi \colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).This implies $\pi^* K_X = K_{C_1 \times C_2}$, so K_X is nef .

Disadvantages:

- X is singular, we need to consider a resolution of its singularities S.

Quasi-étale quotients

Definition

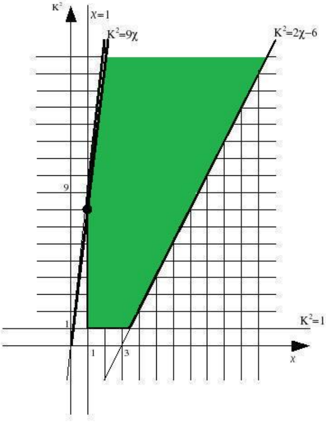
A *quasi-étale surface* is the min. res. S of the sings of $X = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of automorphisms acting freely out of a finite set of points.

If $\pi\colon C_1 \times C_2 \to X$ is the quotient map, we are assuming π quasi-étale (instead of étale).This implies $\pi^* K_X = K_{C_1 \times C_2}$, so K_X is nef .

Disadvantages:

- X is singular, we need to consider a resolution of its singularities S.
- We lose every rigidity property.

Advantage: it may be $K^2 < 8\chi$. We may in principle fill most of the picture.
This gives a powerful tool to answer (positively) existence conjectures.



Mixed and unmixed structures

We know that

- either $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$,
- or $C_1 \cong C_2 \cong C$ and $\mathrm{Aut}(C^2) \cong (\mathrm{Aut}(C))^2 \rtimes \mathbb{Z}_{/2\mathbb{Z}}$.

Mixed and unmixed structures

We know that

- either $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$,
- or $C_1 \cong C_2 \cong C$ and $\mathrm{Aut}(C^2) \cong (\mathrm{Aut}(C))^2 \rtimes \mathbb{Z}_{/2\mathbb{Z}}$.

Following Catanese, for $G < \mathrm{Aut}(C_1 \times C_2)$ we define $G^{(0)} = G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$. There are two possibilities

- either $G = G^{(0)}$ (the *unmixed case*, the case of the *product-quotient surfaces*, the *standard isotrivial fibrations*);

Mixed and unmixed structures

We know that

- either $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$,
- or $C_1 \cong C_2 \cong C$ and $\mathrm{Aut}(C^2) \cong (\mathrm{Aut}(C))^2 \rtimes \mathbb{Z}_{/2\mathbb{Z}}$.

Following Catanese, for $G < \mathrm{Aut}(C_1 \times C_2)$ we define $G^{(0)} = G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$. There are two possibilities

- either $G = G^{(0)}$ (the *unmixed case*, the case of the *product-quotient surfaces*, the *standard isotrivial fibrations*);
- or (*mixed case*) there is an exact sequence

$$(\#) \quad 1 \to G^{(0)} \to G \to \mathbb{Z}_{/2\mathbb{Z}} \to 1.$$

Mixedness and quasi-étaleness

Theorem (Frapporti)

π is not quasi-étale if and only if $G^{(0)} \not\cong G$ and

$$(\#) \quad 1 \to G^{(0)} \to G \to \mathbb{Z}_{/2\mathbb{Z}} \to 1$$

splits.

Mixedness and quasi-étaleness

Theorem (Frapporti)

π is not quasi-étale if and only if $G^{(0)} \ncong G$ and

$$(\#) \quad 1 \to G^{(0)} \to G \to \mathbb{Z}_{/2\mathbb{Z}} \to 1$$

splits.

Assuming quasi-étale we consider a class larger than the class of the standard isotrivial fibrations (=the unmixed quasi étale surfaces). The *quotient surfaces* we are excluding are dominated by the symmetric product of a curve.

Constructing curves with group actions

By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve C is equivalent to give

- the curve $C/G^{(0)}$;

Constructing curves with group actions

By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve C is equivalent to give

- the curve $C/G^{(0)}$;
- a choice of some points on $C/G^{(0)}$;

Constructing curves with group actions

By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve C is equivalent to give

- the curve $C/G^{(0)}$;
- a choice of some points on $C/G^{(0)}$;
- a *suitable* choice of loops in the complement of these points.

Constructing curves with group actions

By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve C is equivalent to give

- the curve $C/G^{(0)}$;
- a choice of some points on $C/G^{(0)}$;
- a *suitable* choice of loops in the complement of these points.
- a *suitable* system of generators $G^{(0)}$: a generator for each loop, defining a surjection from the fundamental group of the complement of the chosen points to $G^{(0)}$).

Constructing curves with group actions

By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve C is equivalent to give

- the curve $C/G^{(0)}$;
- a choice of some points on $C/G^{(0)}$;
- a *suitable* choice of loops in the complement of these points.
- a *suitable* system of generators $G^{(0)}$: a generator for each loop, defining a surjection from the fundamental group of the complement of the chosen points to $G^{(0)}$).

Constructing curves with group actions

By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve C is equivalent to give

- the curve $C/G^{(0)}$;
- a choice of some points on $C/G^{(0)}$;
- a *suitable* choice of loops in the complement of these points.
- a *suitable* system of generators $G^{(0)}$: a generator for each loop, defining a surjection from the fundamental group of the complement of the chosen points to $G^{(0)}$).

For later use: to each *suitable* system of generators we associate its *signature*, which is the unordered list of the orders of some of these generators. The genus of C is a function (Hurwitz' formula) of $|G|$, the signature, and the genus of $C/G^{(0)}$.

Constructing quotients

To construct an unmixed surface I need two curves with an action of the same group, so two systems of generators of the same group $G = G^{(0)}$.

Constructing quotients

To construct an unmixed surface I need two curves with an action of the same group, so two systems of generators of the same group $G = G^{(0)}$.

In the mixed case, we only need one system of generators of $G^{(0)}$ ("twice"), and a degree 2 extension G of $G^{(0)}$.

Constructing quotients

To construct an unmixed surface I need two curves with an action of the same group, so two systems of generators of the same group $G = G^{(0)}$.

In the mixed case, we only need one system of generators of $G^{(0)}$ ("twice"), and a degree 2 extension G of $G^{(0)}$. Indeed, fixed a $G^{(0)}$ action on C and $\tau' \in G \setminus G^{(0)}$, then a mixed action of G on $C \times C$ is given by

$$\begin{cases} g(x,y) = (gx, \tau' g \tau'^{-1} y) & \forall g \in G^{(0)} \\ \tau' g(x,y) = (\tau' g \tau'^{-1} y, \tau'^2 gx) & \forall g \in G^{(0)} \end{cases}$$

and all mixed actions come in this way.

Constructing quotients

To construct an unmixed surface I need two curves with an action of the same group, so two systems of generators of the same group $G = G^{(0)}$.

In the mixed case, we only need one system of generators of $G^{(0)}$ ("twice"), and a degree 2 extension G of $G^{(0)}$. Indeed, fixed a $G^{(0)}$ action on C and $\tau' \in G \setminus G^{(0)}$, then a mixed action of G on $C \times C$ is given by

$$\begin{cases} g(x,y) = (gx, \tau' g \tau'^{-1} y) & \forall g \in G^{(0)} \\ \tau' g(x,y) = (\tau' g \tau'^{-1} y, \tau'^2 g x) & \forall g \in G^{(0)} \end{cases}$$

and all mixed actions come in this way.
Moreover, different choices of τ' give isomorphic constructions.

Computing the invariants: the irregularity

From now on, we assume quasi-étaleness, running in parallel both the mixed and the unmixed case.

Computing the invariants: the irregularity

From now on, we assume quasi-étaleness, running in parallel both the mixed and the unmixed case.

The easier formula is for the irregularity:

$$\begin{cases} q(S) = g(C_1/G) + g(C_2/G) & \text{in the unmixed case} \\ q(S) = g(C/G^{(0)}) & \text{in the mixed case} \end{cases}$$

Computing the invariants: the irregularity

From now on, we assume quasi-étaleness, running in parallel both the mixed and the unmixed case.

The easier formula is for the irregularity:

$$\begin{cases} q(S) = g(C_1/G) + g(C_2/G) & \text{in the unmixed case} \\ q(S) = g(C/G^{(0)}) & \text{in the mixed case} \end{cases}$$

To compute the other invariants we need a better understanding of the singularities of $X = (C_1 \times C_2)/G$.

The singularities

The singularities of X are the image of the points of $C_1 \times C_2$ with non trivial stabilizer.

The singularities

The singularities of X are the image of the points of $C_1 \times C_2$ with non trivial stabilizer.

- in the unmixed case X has only *cyclic quotient singularities*, locally biregular to the quotient of $\mathbb{C}^2$ by the automorphism $\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$ where ω is a n-th primitive root of 1, $0 < q < n$ and $(q, n) = 1$. We say that these singularities are of type $C_{n,q}$.

The singularities

The singularities of X are the image of the points of $C_1 \times C_2$ with non trivial stabilizer.

- in the unmixed case X has only *cyclic quotient singularities*, locally biregular to the quotient of $\mathbb{C}^2$ by the automorphism $\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$ where ω is a n-th primitive root of 1, $0 < q < n$ and $(q, n) = 1$. We say that these singularities are of type $C_{n,q}$. We know how to compute them from the systems of generators.

The singularities

The singularities of X are the image of the points of $C_1 \times C_2$ with non trivial stabilizer.

- in the unmixed case X has only *cyclic quotient singularities*, locally biregular to the quotient of $\mathbb{C}^2$ by the automorphism $\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$ where ω is a n-th primitive root of 1, $0 < q < n$ and $(q, n) = 1$. We say that these singularities are of type $C_{n,q}$. We know how to compute them from the systems of generators.
- In the mixed case we have an intermediate unmixed quotient $Y = C^2/G^{(0)}$ and an involution i on Y with $Y/i = X$. $\mathrm{Sing} X$ is determined by $\mathrm{Sing} Y$ and the action of i on it:

The singularities

The singularities of X are the image of the points of $C_1 \times C_2$ with non trivial stabilizer.

- in the unmixed case X has only *cyclic quotient singularities*, locally biregular to the quotient of $\mathbb{C}^2$ by the automorphism $\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$ where ω is a n-th primitive root of 1, $0 < q < n$ and $(q, n) = 1$. We say that these singularities are of type $C_{n,q}$. We know how to compute them from the systems of generators.
- In the mixed case we have an intermediate unmixed quotient $Y = C^2/G^{(0)}$ and an involution i on Y with $Y/i = X$. $\mathrm{Sing}X$ is determined by $\mathrm{Sing}Y$ and the action of i on it: when i exchanges two singular points of Y we get a point of type $C_{n,q}$ on X;

The singularities

The singularities of X are the image of the points of $C_1 \times C_2$ with non trivial stabilizer.

- in the unmixed case X has only *cyclic quotient singularities*, locally biregular to the quotient of $\mathbb{C}^2$ by the automorphism $\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$ where ω is a n-th primitive root of 1, $0 < q < n$ and $(q, n) = 1$. We say that these singularities are of type $C_{n,q}$. We know how to compute them from the systems of generators.
- In the mixed case we have an intermediate unmixed quotient $Y = C^2/G^{(0)}$ and an involution i on Y with $Y/i = X$. $\mathrm{Sing}X$ is determined by $\mathrm{Sing}Y$ and the action of i on it: when i exchanges two singular points of Y we get a point of type $C_{n,q}$ on X; when i fixes a point of type $C_{n,q}$, then we get a point of type $D_{n,q}$ on X.

$C_{n,q}$

The exceptional divisor of the minimal resolution of a singularity $C_{n,q}$ is a chain of rational curves $A_1, \ldots, A_k$ with self intersections $-b_1, \ldots, -b_k$ given by the continued fraction:

$$\frac{n}{q} = [b_1, \ldots, b_k] = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}.$$

The dual graph is

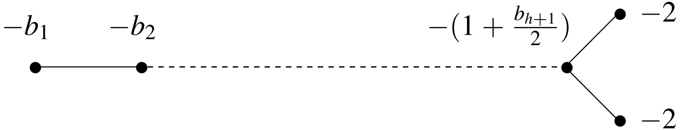
If $qq' \equiv 1 \mod n$, then $\frac{n}{q'} = [b_k, \ldots, b_1]$ and therefore $C_{n,q} \cong C_{n,q'}$

$D_{n,q}$

Proposition (Frapporti)

If $P \in Y$ is a fixed point of Y, then P is a singular point of type $C_{n,q}$ with $q^2 \equiv 1$ mod n, and the lift of i to a resolution of the singularity exchanges the ends of the string

The resolution graph of a singularity of type $D_{n,q}$ is $(k = 2h+1)$



K^2 and χ

There are explicit formulas

$$K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} - \sum_{x \in \mathrm{Sing}X} k_x$$

$$e(S) = \frac{4(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \mathrm{Sing}X} e_x = 12\chi - K_S^2$$

where k_x and e_x are positive rational numbers depending only on the type of the singularity. it follows

$$K_S^2 = 8\chi - \sum_x \frac{2e_x + k_x}{3} \leq 8\chi.$$

The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants p_g, q and K_S^2 (which is often but not always equal to $K_{\bar{S}}^2$).

The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants p_g, q and K_S^2 (which is often but not always equal to $K_{\overline{S}}^2$). We can compute $\pi_1(S)$ by a lemma of Armstrong.

The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants p_g, q and K_S^2 (which is often but not always equal to $K_{\bar{S}}^2$). We can compute $\pi_1(S)$ by a lemma of Armstrong.

We are interested in the inverse procedure: if we are interested in constructing surfaces with certain p_g, q and K^2, what can we do?

Reversing the above formula we can compute by them

- the possible $g(C_i/G^{(0)})$ (by q);

The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants p_g, q and K_S^2 (which is often but not always equal to $K_{\bar{S}}^2$). We can compute $\pi_1(S)$ by a lemma of Armstrong.

We are interested in the inverse procedure: if we are interested in constructing surfaces with certain p_g, q and K^2, what can we do?
Reversing the above formula we can compute by them

- the possible $g(C_i/G^{(0)})$ (by q);
- $\sum_x 2e_x + k_x = 24\chi - 3K_S^2$: there are finitely many possible configurations ("baskets") of singularities for each value of this;

The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants p_g, q and K_S^2 (which is often but not always equal to $K_{\bar{S}}^2$). We can compute $\pi_1(S)$ by a lemma of Armstrong.

We are interested in the inverse procedure: if we are interested in constructing surfaces with certain p_g, q and K^2, what can we do?
Reversing the above formula we can compute by them

- the possible $g(C_i/G^{(0)})$ (by q);
- $\sum_x 2e_x + k_x = 24\chi - 3K_S^2$: there are finitely many possible configurations ("baskets") of singularities for each value of this;
- Hurwitz formula yields an equation involving $|G|$, K_S^2, $\sum k_x$, $g(C_i/G^{(0)})$ and the "signatures" of the actions of $G^{(0)}$ on the C_i.

The algorithms: procedure

We proved some inequalities to bound the possible signatures.

The algorithms: procedure

We proved some inequalities to bound the possible signatures.This gives an algorithm that computes all quasi-étale surfaces S with fixed p_g, q and K_S^2.

The algorithms: procedure

We proved some inequalities to bound the possible signatures.This gives an algorithm that computes all quasi-étale surfaces S with fixed p_g, q and K_S^2.

1. find all (fin. many) possible pairs of genera $(g(C_1/G^{(0)}), g(C_2/G^{(0)}))$ (equal in the mixed case) and configurations ("baskets") of singularities with $\sum_x (2e_x + k_x) = 24\chi - 3K^2$;

The algorithms: procedure

We proved some inequalities to bound the possible signatures.This gives an algorithm that computes all quasi-étale surfaces S with fixed p_g, q and K_S^2.

1. find all (fin. many) possible pairs of genera $(g(C_1/G^{(0)}), g(C_2/G^{(0)}))$ (equal in the mixed case) and configurations ("baskets") of singularities with $\sum_x(2e_x + k_x) = 24\chi - 3K^2$;
2. for every "basket" and pair of genera, list all "signatures" satisfying those inequalities;

The algorithms: procedure

We proved some inequalities to bound the possible signatures.This gives an algorithm that computes all quasi-étale surfaces S with fixed p_g, q and K_S^2.

1. find all (fin. many) possible pairs of genera $(g(C_1/G^{(0)}), g(C_2/G^{(0)}))$ (equal in the mixed case) and configurations ("baskets") of singularities with $\sum_x (2e_x + k_x) = 24\chi - 3K^2$;
2. for every "basket" and pair of genera, list all "signatures" satisfying those inequalities;
3. to each (pair of) signature(s), search all groups ($G^{(0)}$) of the order prescribed by the Hurwitz formula for set of generators of the prescribed signatures;

The algorithms: procedure

We proved some inequalities to bound the possible signatures.This gives an algorithm that computes all quasi-étale surfaces S with fixed p_g, q and K_S^2.

1. find all (fin. many) possible pairs of genera $(g(C_1/G^{(0)}), g(C_2/G^{(0)}))$ (equal in the mixed case) and configurations ("baskets") of singularities with $\sum_x (2e_x + k_x) = 24\chi - 3K^2$;
2. for every "basket" and pair of genera, list all "signatures" satisfying those inequalities;
3. to each (pair of) signature(s), search all groups $(G^{(0)})$ of the order prescribed by the Hurwitz formula for set of generators of the prescribed signatures;
4. in the mixed case, consider all the unsplit degree 2 extensions of $G^{(0)}$;

The algorithms: procedure

We proved some inequalities to bound the possible signatures.This gives an algorithm that computes all quasi-étale surfaces S with fixed p_g, q and K_S^2.

1. find all (fin. many) possible pairs of genera $(g(C_1/G^{(0)}), g(C_2/G^{(0)}))$ (equal in the mixed case) and configurations ("baskets") of singularities with $\sum_x (2e_x + k_x) = 24\chi - 3K^2$;
2. for every "basket" and pair of genera, list all "signatures" satisfying those inequalities;
3. to each (pair of) signature(s), search all groups $(G^{(0)})$ of the order prescribed by the Hurwitz formula for set of generators of the prescribed signatures;
4. in the mixed case, consider all the unsplit degree 2 extensions of $G^{(0)}$;
5. check the singularities of the surfaces in the output.

We implemented the algorithm in MAGMA in the unmixed case for $p_g = 0$. Recall that if a surface of general type S is minimal, then K_S^2 is positive.

Theorem (Bauer, Catanese, Grunewald, -)

Unmixed quasi-étale surfaces of g. t. with $p_g = 0$ form

1. *exactly 13 irreducible families of surfaces for the case in which G acts freely: they form 13 irreducible connected components of the moduli space;*

We implemented the algorithm in MAGMA in the unmixed case for $p_g = 0$. Recall that if a surface of general type S is minimal, then K_S^2 is positive.

Theorem (Bauer, Catanese, Grunewald, -)

Unmixed quasi-étale surfaces of g. t. with $p_g = 0$ form

1. *exactly 13 irreducible families of surfaces for the case in which G acts freely: they form 13 irreducible connected components of the moduli space;*
2. *exactly 72 irreducible families of minimal surfaces;*

We implemented the algorithm in MAGMA in the unmixed case for $p_g = 0$. Recall that if a surface of general type S is minimal, then K_S^2 is positive.

Theorem (Bauer, Catanese, Grunewald, -)

Unmixed quasi-étale surfaces of g. t. with $p_g = 0$ form

1. *exactly 13 irreducible families of surfaces for the case in which G acts freely: they form 13 irreducible connected components of the moduli space;*
2. *exactly 72 irreducible families of minimal surfaces;*
3. *there is exactly one unmixed quasi-étale surface with $p_g = 0$ and $K_S^2 > 0$ which is not minimal, the "fake Godeaux": it has $K_S^2 = 1$, whereas $K_S^2 = 3$.*

We implemented the algorithm in MAGMA in the unmixed case for $p_g = 0$. Recall that if a surface of general type S is minimal, then K_S^2 is positive.

Theorem (Bauer, Catanese, Grunewald, -)

Unmixed quasi-étale surfaces of g. t. with $p_g = 0$ form

1. *exactly 13 irreducible families of surfaces for the case in which G acts freely: they form 13 irreducible connected components of the moduli space;*
2. *exactly 72 irreducible families of minimal surfaces;*
3. *there is exactly one unmixed quasi-étale surface with $p_g = 0$ and $K_S^2 > 0$ which is not minimal, the "fake Godeaux": it has $K_S^2 = 1$, whereas $K_S^2 = 3$.*

A similar classification for $p_g = q \geq 1$ has been obtained by Carnovale, Mistretta, Penegini and Polizzi.

In the mixed case the algorithm is implemented in the case $q = 0$.

Theorem (Bauer, Catanese, Grunewald, Frapporti)

Mixed quasi-étale surfaces of general type with $p_g = 0$ form

1. *exactly 5 irreducible families of surfaces for the case in which G acts freely: they form 5 irreducible connected components of the moduli space;*

In the mixed case the algorithm is implemented in the case $q = 0$.

Theorem (Bauer, Catanese, Grunewald, Frapporti)

Mixed quasi-étale surfaces of general type with $p_g = 0$ form

1. *exactly 5 irreducible families of surfaces for the case in which G acts freely: they form 5 irreducible connected components of the moduli space;*
2. *exactly 17 irreducible families of minimal surfaces;*

In the mixed case the algorithm is implemented in the case $q = 0$.

Theorem (Bauer, Catanese, Grunewald, Frapporti)

Mixed quasi-étale surfaces of general type with $p_g = 0$ form

1. *exactly 5 irreducible families of surfaces for the case in which G acts freely: they form 5 irreducible connected components of the moduli space;*
2. *exactly 17 irreducible families of minimal surfaces;*
3. *all mixed quasi-étale surface with $p_g = 0$ and $K_S^2 > 0$ are minimal.*

In the mixed case the algorithm is implemented in the case $q = 0$.

Theorem (Bauer, Catanese, Grunewald, Frapporti)

Mixed quasi-étale surfaces of general type with $p_g = 0$ form

1. *exactly 5 irreducible families of surfaces for the case in which G acts freely: they form 5 irreducible connected components of the moduli space;*
2. *exactly 17 irreducible families of minimal surfaces;*
3. *all mixed quasi-étale surface with $p_g = 0$ and $K_S^2 > 0$ are minimal.*

Similar results have been obtained by the same authors mentioned before for $p_g = q \geq 1$ only in the étale case.

Some corollaries

The Campedelli surfaces are the min. surf. of g. t. with $p_g = 0$, $K^2 = 2$.

Conjecture

The possible π_1 of the Campedelli surfaces are all abelian groups of order ≤ 9 and the quaternion group.

This is now proved for π_1^{alg} (Reid+...).

Some corollaries

The Campedelli surfaces are the min. surf. of g. t. with $p_g = 0, K^2 = 2$.

Conjecture

The possible π_1 of the Campedelli surfaces are all abelian groups of order ≤ 9 and the quaternion group.

This is now proved for π_1^{alg} (Reid+...). By our constructions:

Corollary (1)

There are Campedelli surfaces with π_1 equal $\mathbb{Z}_{/3\mathbb{Z}}$ and $\mathbb{Z}_{/4\mathbb{Z}}$.

Park, Park and Shin found similar results for π_1^{alg}.

Some corollaries

The Campedelli surfaces are the min. surf. of g. t. with $p_g = 0$, $K^2 = 2$.

Conjecture

The possible π_1 of the Campedelli surfaces are all abelian groups of order ≤ 9 and the quaternion group.

This is now proved for π_1^{alg} (Reid+...). By our constructions:

Corollary (1)

There are Campedelli surfaces with π_1 equal $\mathbb{Z}_{/3\mathbb{Z}}$ and $\mathbb{Z}_{/4\mathbb{Z}}$.

Park, Park and Shin found similar results for π_1^{alg}.

Corollary (2)

Minimal surfaces of general type with $p_g = 0$, $3 \leq K_S^2 \leq 6$ realize at least 47 topological types.

Algorithmic problems

Problem

We have to run a search on all groups of a given order: sometimes there are too many even for a computer, sometimes we do not have a complete list of them. We used some group theory to exclude the cases that the computer could not do.

Algorithmic problems

Problem

We have to run a search on all groups of a given order: sometimes there are too many even for a computer, sometimes we do not have a complete list of them. We used some group theory to exclude the cases that the computer could not do.

Problem

The algorithm is very time and memory consuming. We need some help in computational algebra to get results for different values of the invariants.

Algorithmic problems

Problem

We have to run a search on all groups of a given order: sometimes there are too many even for a computer, sometimes we do not have a complete list of them. We used some group theory to exclude the cases that the computer could not do.

Problem

The algorithm is very time and memory consuming. We need some help in computational algebra to get results for different values of the invariants.

Problem

Extend the programs to the irregular case.

Theoretical problems

Problem

How do we determine the minimal model of S?

Theoretical problems

Problem

How do we determine the minimal model of S?

And, related to it is

Problem

Can we find all quasi-étale surfaces of general type with $p_g = 0$, or, more generally, with given p_g and q?

If we could find an explicit bound $K_S^2 \geq k(p_g, q)$...

Searching rational curves

To answer the last questions we need to study the rational curves on a quasi-étale surface.

We would like to be able to locate all exceptional curves of the first kind (if any).

Searching rational curves

To answer the last questions we need to study the rational curves on a quasi-étale surface.

We would like to be able to locate all exceptional curves of the first kind (if any).

Remark

Rational curves on S are

- either exceptional for the resolution $S \to X$

Searching rational curves

To answer the last questions we need to study the rational curves on a quasi-étale surface.

We would like to be able to locate all exceptional curves of the first kind (if any).

Remark

Rational curves on S are

- either exceptional for the resolution $S \to X$
- or pass through the singular points of X at least three times.

Mistretta-Polizzi's example

In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\left\{\frac{1}{7}, 2 \times \frac{2}{7}\right\}$.

Mistretta-Polizzi's example

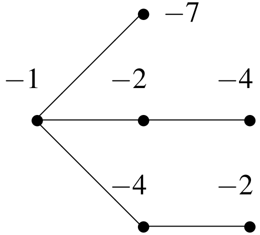
In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\left\{\frac{1}{7}, 2 \times \frac{2}{7}\right\}$. Since S is irregular, the Albanese map α contracts all rational curves.

Mistretta-Polizzi's example

In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\left\{\frac{1}{7}, 2 \times \frac{2}{7}\right\}$. Since S is irregular, the Albanese map α contracts all rational curves.
In this case, all exceptional divisors for $S \to X$ are mapped to the same point $p \in \alpha(S)$, so all rational curves are in $\alpha^{-1}(p)$.

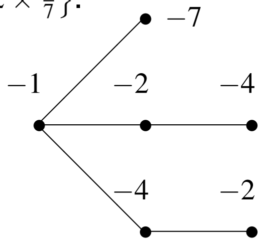
Mistretta-Polizzi's example

In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\{\frac{1}{7}, 2 \times \frac{2}{7}\}$. Since S is irregular, the Albanese map α contracts all rational curves.
In this case, all exceptional divisors for $S \to X$ are mapped to the same point $p \in \alpha(S)$, so all rational curves are in $\alpha^{-1}(p)$. $\alpha^{-1}(p)$ is made of rational curves, with dual graph



Mistretta-Polizzi's example

In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\{\frac{1}{7}, 2 \times \frac{2}{7}\}$.



Then the minimal model has $K_S^2 = 3$. This strategy works in every irregular case.

Question

Can we use this argument to get an inequality $K^2 \geq k(p_g, q)$ for the irregular case?

How to prove the minimality in the regular case

Assume that the singularities are mild, for example just k nodes. Then S has k (-2) curves, every further rational curve should meet them at least three times.

How to prove the minimality in the regular case

Assume that the singularities are mild, for example just k nodes. Then S has k (-2) curves, every further rational curve should meet them at least three times.

If we had a (-1) curve, contracting them I would get either two (-1) curves intersecting, or a singular rational curve intersecting negatively the canonical system. This implies that the surface is rational.

How to prove the minimality in the regular case

Assume that the singularities are mild, for example just k nodes. Then S has k (-2) curves, every further rational curve should meet them at least three times.

If we had a (-1) curve, contracting them I would get either two (-1) curves intersecting, or a singular rational curve intersecting negatively the canonical system. This implies that the surface is rational.

If the surface is not simply connected, I have a contradiction, and the surface is minimal.

The fake Godeaux surface

- $G = PSL(2,7)$;
- $\Psi_1 \colon \mathbb{T}(7,3,3) \to G$, $\Psi_2 \colon \mathbb{T}(7,4,2) \to G$;
- $p_g(S) = 0$, $K_S^2 = 1$, $\pi_1(S) = \mathbb{Z}_{/6\mathbb{Z}}$;
- $\mathfrak{B}(X) = \{\frac{1}{7}, 2 \times \frac{2}{7}\}$.

How do we find the $(-1)-$curves?

The first exceptional curve

$$
\begin{array}{ccccc}
\textit{branch pts of } \hat{f}_1 & \quad & \hat{C}_1 & \xrightarrow{\hat{\xi}} & C_1 & \quad & \textit{branch pts of } f_1 \\
\downarrow & & \downarrow \hat{f}_1 & & \downarrow f_1 & & \downarrow \\
(7,7,7) & & \mathbb{P}^1 & \underset{(3:1)}{\xrightarrow{\xi}} & \mathbb{P}^1 & & (7, {\color{red}3}, {\color{red}3})
\end{array}
$$

The first exceptional curve

$$\begin{array}{ccccccc} \textit{branch pts of } \hat f_1 & & \hat C_1 & \xrightarrow{\hat\xi} & C_1 & & \textit{branch pts of } f_1 \\ \downarrow & & \downarrow{\scriptstyle \hat f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow \\ (7,7,7) & & \mathbb{P}^1 & \xrightarrow[(3:1)]{\xi} & \mathbb{P}^1 & & (7,3,3) \end{array}$$

$$\begin{array}{ccccccccc} & & & & \hat C_2 \xrightarrow{\hat\eta} C_2 & & & & \\ \textit{branch pts of } \hat f_2 & & \hat C_2 & \xrightarrow{\hat\eta_2} & \bar C_2 & \xrightarrow{\hat\eta_1} & C_2 & & \textit{branch pts of } f_2 \\ \downarrow & & \downarrow{\scriptstyle \hat f_2} & & \downarrow{\scriptstyle \bar f_2} & & \downarrow{\scriptstyle f_2} & & \downarrow \\ (7,7,7) & & \mathbb{P}^1 & \xrightarrow[(2:1)]{\eta_2} & \mathbb{P}^1 & \xrightarrow[(2:1)]{\eta_1} & \mathbb{P}^1 & & (7,4,2) \\ & & & & \mathbb{P}^1 \xrightarrow{\eta} \mathbb{P}^1 & & & & \end{array}$$

Proposition

1. *$(\hat{C}_1, \hat{f}_1)$ and $(\hat{C}_2, \hat{f}_2)$ are isomorphic as G-covers of $\mathbb{P}^1$ (hence we write $\hat{C} := \hat{C}_1 = \hat{C}_2$).*
2. *The curve*

$$C' := (\hat{\xi}, \hat{\eta})(\hat{C}) \subset C_1 \times C_2,$$

is G-invariant and the quotient is a rational curve $D' \subset X$.

Proposition

1. $(\hat{C}_1, \hat{f}_1)$ *and* $(\hat{C}_2, \hat{f}_2)$ *are isomorphic as* G*-covers of* $\mathbb{P}^1$ *(hence we write* $\hat{C} := \hat{C}_1 = \hat{C}_2$*).*
2. *The curve*
$$C' := (\hat{\xi}, \hat{\eta})(\hat{C}) \subset C_1 \times C_2,$$
is G*-invariant and the quotient is a rational curve* $D' \subset X$.
3. *The strict transform* E' *of* D' *is a* (-1)*-curve on* S.

The second (-1)-curve on S

$$(7,7,7,7) \longrightarrow ({\color{red}7},{\color{red}7},7) \longrightarrow (7,{\color{red}3},{\color{red}3})$$

$$\mathbb{P}^1 \xrightarrow{(2:1)} \mathbb{P}^1 \xrightarrow{(3:1)} \mathbb{P}^1$$

The second (-1)-curve on S

$$(7,7,7,7) \longrightarrow (7,7,7) \longrightarrow (7,3,3)$$

$$\mathbb{P}^1 \xrightarrow{(2:1)} \mathbb{P}^1 \xrightarrow{(3:1)} \mathbb{P}^1$$

$$(7,7,7,7) \longrightarrow (7,4,2)$$

$$\mathbb{P}^1 \xrightarrow{(4:1)} \mathbb{P}^1$$

The second (-1)-curve on S

$$(7,7,7,7) \longrightarrow (7,7,7) \longrightarrow (7,3,3)$$

$$\mathbb{P}^1 \xrightarrow{(2:1)} \mathbb{P}^1 \xrightarrow{(3:1)} \mathbb{P}^1$$

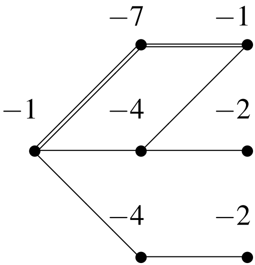
$$(7,7,7,7) \longrightarrow (7,4,2)$$

$$\mathbb{P}^1 \xrightarrow{(4:1)} \mathbb{P}^1$$

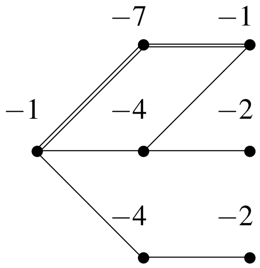
Proposition

The two G-coverings (with branching indices $(7,7,7,7)$) of $\mathbb{P}^1$ are isomorphic, and give a further (-1)-curve on S.

The rational curves we have found on S (5 from the resolution, 2 from the above construction) have dual graph



The rational curves we have found on S (5 from the resolution, 2 from the above construction) have dual graph



Exercise: the surface obtained by contracting the two (-1)-curves is minimal.

Work in progress (with I. Bauer)

We have 73 families of unmixed quasi-étale surfaces with $p_g = 0$ and $K^2 > 0$; 72 families of minimal surfaces, and the fake Godeaux.

Work in progress (with I. Bauer)

We have 73 families of unmixed quasi-étale surfaces with $p_g = 0$ and $K^2 > 0$; 72 families of minimal surfaces, and the fake Godeaux.

By inspecting the list, we noticed that all the minimal surfaces have $H^2(X) \cong \mathbb{C}^2$, generated by the classes of the fibres of the two fibrations.

Work in progress (with I. Bauer)

We have 73 families of unmixed quasi-étale surfaces with $p_g = 0$ and $K^2 > 0$; 72 families of minimal surfaces, and the fake Godeaux.

By inspecting the list, we noticed that all the minimal surfaces have $H^2(X) \cong \mathbb{C}^2$, generated by the classes of the fibres of the two fibrations. On the contrary, the fake Godeaux surface has $H^2(X) \cong \mathbb{C}^4$, generated by the classes of the two fibres and of the two (-1)-curves.

Work in progress (with I. Bauer)

We have 73 families of unmixed quasi-étale surfaces with $p_g = 0$ and $K^2 > 0$; 72 families of minimal surfaces, and the fake Godeaux.

By inspecting the list, we noticed that all the minimal surfaces have $H^2(X) \cong \mathbb{C}^2$, generated by the classes of the fibres of the two fibrations. On the contrary, the fake Godeaux surface has $H^2(X) \cong \mathbb{C}^4$, generated by the classes of the two fibres and of the two (-1)-curves.

Question

Is there a reason for that?

Hodge theoretic information

For sake of simplicity we assume from now on $q = 0$ and unmixedness.

Proposition

Let $X := (C_1 \times C_2)/G$ *be the quotient model of an unmixed quasi-étale surface. Then*

- $\dim H^2(X) \geq 2$,

Hodge theoretic information

For sake of simplicity we assume from now on $q = 0$ and unmixedness.

Proposition

Let $X := (C_1 \times C_2)/G$ be the quotient model of an unmixed quasi-étale surface. Then

- $\dim H^2(X) \geq 2$,
- $\dim H^2(X) \equiv 0 \mod 2$,

Hodge theoretic information

For sake of simplicity we assume from now on $q = 0$ and unmixedness.

Proposition

Let $X := (C_1 \times C_2)/G$ be the quotient model of an unmixed quasi-étale surface. Then

- $\dim H^2(X) \geq 2$,
- $\dim H^2(X) \equiv 0 \mod 2$,

Let $\sigma\colon S \to X$ be the minimal resolution of the singularities of X. Then $H^2(S, \mathbb{C}) \cong H^2(X, \mathbb{C}) \oplus \mathbb{C}^l$, where $l =$ numb. of irr. comp.s of $Exc(\sigma)$.

Global definition of γ

Let $X := (C_1 \times C_2)/G$ be the quotient model of an unmixed quasi-étale surface with $q = 0$. We set

$$\gamma(X) := \frac{h^2(S, \mathbb{C}) - l}{2} - 1 - 2p_g(S) \in \mathbb{Z}$$

Global definition of γ

Let $X := (C_1 \times C_2)/G$ be the quotient model of an unmixed quasi-étale surface with $q = 0$. We set

$$\gamma(X) := \frac{h^2(S,\mathbb{C}) - l}{2} - 1 - 2p_g(S) \in \mathbb{Z}$$

$\gamma \geq -p_g$. Indeed $\gamma + p_g$ is half of the codimension in $H^{1,1}(S)$ of the subspace generated by the classes we know (fibres + exceptional).

Local definition of γ

Lemma

γ depends only on the basket of X. More precisely $\gamma(X) = \sum_{x \in \mathfrak{B}(X)} \gamma_x$ where, for a singular point of type $C_{n,q}$ with $\frac{n}{q} = [b_1, \ldots, b_l]$,

$$\gamma_x = \frac{1}{6}\left(\frac{q+q'}{n} + \sum_{i=1}^{l}(b_i - 3)\right),$$

where $1 \leq q' \leq n-1$ and $qq' \equiv 1 \bmod n$.

Local definition of γ

Lemma

γ depends only on the basket of X. More precisely $\gamma(X) = \sum_{x \in \mathfrak{B}(X)} \gamma_x$ where, for a singular point of type $C_{n,q}$ with $\frac{n}{q} = [b_1, \ldots, b_l]$,

$$\gamma_x = \frac{1}{6}\left(\frac{q+q'}{n} + \sum_{i=1}^{l}(b_i - 3)\right),$$

where $1 \le q' \le n-1$ and $qq' \equiv 1$ mod n.

Remark

$K_S^2 = 8\chi - 2\gamma - l$. We have implemented a similar algorithm constructing all product-quotient surfaces with $q = 0$, given p_g, and γ (and looks much quicker than the other one!)

The dual surface

Let S be a product-quotient surface with quotient model

$$X = (C_1 \times C_2)/G.$$

We assume furthermore that S is *regular*, i.e., $q(S) = 0$.

The dual surface

Let S be a product-quotient surface with quotient model

$$X = (C_1 \times C_2)/G.$$

We assume furthermore that S is *regular*, i.e., $q(S) = 0$. Suppose that S is given by a pair of spherical systems of generators: $(a_1, \ldots, a_s)$, $(b_1, \ldots, b_t)$ of G.

Definition

The dual surface S' *is the product-quotient surface given by the pair of spherical systems of generators:* $(a_1, \ldots, a_s)$, $(b_t^{-1}, \ldots, b_1^{-1})$.

The invariants of S and S'

Remark

$$C_{n,q} \in \mathfrak{B}(X) \iff C_{n,n-q} \in \mathfrak{B}(X').$$

The invariants of S and S'

Remark

$$C_{n,q} \in \mathfrak{B}(X) \iff C_{n,n-q} \in \mathfrak{B}(X').$$

Proposition

1. $\gamma' := \gamma(S') = -\gamma(S) = -\gamma;$
2. $q(S') = q(S)$
3. $p_g(S') = p_g(S) + \gamma;$

Back to the original problem: bounding K^2 or equivalently, γ.
Can we find an explicit function $C(p_g, q)$ such that for all unmixed quasi-étale surfaces of general type, $\gamma \leq C(p_g, q)$?
We have

$$H^2(S) = H^2(X) \oplus L,$$

where $L = \langle A_1, \ldots, A_l \rangle \cong \mathbb{C}^l$ is the subspace generated by the classes of the l irreducible rational curves of the exceptional locus of σ.

Back to the original problem: bounding K^2 or equivalently, γ.
Can we find an explicit function $C(p_g, q)$ such that for all unmixed quasi-étale surfaces of general type, $\gamma \leq C(p_g, q)$?
We have

$$H^2(S) = H^2(X) \oplus L,$$

where $L = \langle A_1, \ldots, A_l \rangle \cong \mathbb{C}^l$ is the subspace generated by the classes of the l irreducible rational curves of the exceptional locus of σ.
It is easy to show that the exceptional divisors of the first kind do not belong to $H^2(X)$.

Consider the subspace $W \subset H^2(S, \mathbb{C})$ generated by the exceptional divisors of the first kind.

Conjecture

$W \cap H^2(X, \mathbb{C}) = \{0\}$.

Consider the subspace $W \subset H^2(S, \mathbb{C})$ generated by the exceptional divisors of the first kind.

Conjecture

$W \cap H^2(X, \mathbb{C}) = \{0\}$.

Assume the conjecture to be true. Then:

$$l = \dim L \geq dimW \geq 2\chi(S) - 6 - K_S^2 = l + 2\gamma - 6(\chi(S) + 1),$$

whence

$$\gamma(S) \leq 3(\chi(S) + 1).$$

(and, with a similar argument $\gamma < 4\chi$).

Literature

- [BCG]
- I. Bauer, F. Catanese, F. Grunewald, R. Pignatelli *Quotients of products of curves, new surfaces with* $p_g = 0$ *and their fundamental groups*, arXiv:0809.3420, to appear in Am. J. Math.
- I. Bauer, R. Pignatelli *The classification of minimal product-quotient surfaces with* $p_g = 0$ arXiv:1006.3209, to appear in Math. of Computations.
- D. Frapporti *Mixed quasi-étale surfaces and new surfaces of general type* Ph. D. Thesis Univ. Trento
- D. Frapporti *Mixed quasi-étale surfaces, new surfaces of general type with* $p_g = 0$ *and their fundamental group* arXiv:1105.1259v2.
- I. Bauer, R. Pignatelli (unpublished notes)