Threefolds near the Noether line - Part 1

joint work with S. Coughlan [C], Y. Hu [H], T. Zhang [Z]

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Overview

1. Volume, genus and Noether inequalities
2. Fibrations
3. Threefolds on the Noether line
4. Their moduli spaces

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Volume and genus

Let X be a complex projective variety whose singularities are at worst canonical. Let n be the dimension of X.

Definition

The genus of X is $p_g(X) := h^0(X, K_X)$.
The volume of X is

$$\mathrm{vol}(X) := n! \limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^n}$$

A variety is of general type if and only if its volume is positive.
X is a canonical model if the singularities of X are canonical and K_X is ample.

Both genus and volume are birational invariants.
If X is a canonical model then

$$\mathrm{vol}(X) = K_X^n.$$

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Volume and genus: Noether inequality

If X is a curve ($n = 1$) then $2\mathbb{N} \ni \mathrm{vol}(X) = 2p_g(X) - 2$.
If X is a surface ($n = 2$) we have the Noether inequality

$$\mathbb{N} \ni \mathrm{vol}(X) \geq 2p_g(X) - 4$$

If X is a threefold ($n = 3$) we have the Noether inequality[1]

Theorem 1. *Let X be a projective 3-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 11$. Then*

$$\mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$

[M. Chen, Y. Hu, C. Jiang 2024]: The inequality holds also for $p_g = 5$. Moreover $\mathrm{vol}(X) \in \mathbb{Q}$ may be not integral if the canonical model of X is not Gorenstein.

[1] Chen, Jungkai A.; Chen, Meng; Jiang, Chen; *The Noether inequality for algebraic 3-folds*. With an appendix by János Kollár, Duke Math. J. **169** (2020), no.9, 1603–1645.

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Horikawa surfaces

The canonical surfaces X with $K_X^2 = 2p_g(X) - 4$ are usually referred to as Horikawa surfaces because Horikawa described[2] their moduli space. Horikawa writes

> In Sections 3 and 5 we shall prove that minimal algebraic surfaces with given p_g and c_1^2 satisfying $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$ have one and the same deformation type provided that c_1^2 is not divisible by 8.

> In Section 7 we shall study the case in which c_1^2 is divisible by 8. If we fix c_1^2, these surfaces are divided into two deformation types. They are homo-

By this classification, if $p_g \geq 7$, then there is a fibration $f\colon X \to \mathbb{P}^1$ with fibres of genus 2.

[2]Horikawa, Eiji; *Algebraic surfaces of general type with small C_1^2. I.* Ann. of Math. (2) 104 (1976), no.2, 357–387.

Genus 2 fibrations

Let $f\colon X \to B$ be a genus 2 fibration, that is a surjective morphism onto a projective curve B whose general fibre is a curve of genus 2. Let F be a fibre.

By the theory of genus 2 fibrations, the canonical ring of F, that is the ring $\bigoplus_d H^0(F, dK_F)$ is of one of the following two forms:

If F is 2−connected

$$\frac{\mathbb{C}[x_0, x_1, z]}{f_6(x_0, x_1, z)}$$

$\deg x_j = 1$ $\quad$ $\deg z = 3$

$\deg f_6 = 6$

If $F = C_1 + C_2$ with $C_1 C_2 = 1$

$$\frac{\mathbb{C}[x_0, x_1, y, z]}{f_2(x_0, x_1), f_6(x_0, x_1, y, z)}$$

$\deg x_j = 1$ $\quad$ $\deg y = 2$ $\quad$ $\deg z = 3$

$\deg f_2 = 2$ $\quad$ $\deg f_6 = 6$

Invariants of genus 2 fibrations

If X is moreover a canonical surface, then[3]

$$\mathrm{vol}(X) = 2p_g(X) - 4 + 6b + \deg \tau - 2h^1(X, \mathcal{O}_X)$$

where b is the genus of the base curve B, and τ is an effective divisor on B supported on the image of the *2-disconnected* fibres, those of the form $C_1 + C_2$ with $C_1 C_2 = 1$.

By an inequality of Debarre, if $K_X^2 < 2p_g$, then $h^1(X, \mathcal{O}_X) = 0$, which in turn implies $b = 0$.

In particular

$$K_X^2 = 2p_g(X) - 4 \Leftrightarrow h^1(X, \mathcal{O}_X) = 0 \text{ and } \tau = 0.$$

[3] Catanese, F., Pignatelli, R.; *Fibrations of low genus, I.* Annales Scientifiques de l'École Normale Supérieure, **39**, 6 (2006), 1011–1049

Simple genus 2 fibrations

We can reformulate more precisely the previous remark as follows:

Definition

A simple genus 2 fibration is a morphism $f\colon X \to B$ between projective varieties of respective dimension 2 and 1 such that

1. B is smooth;
2. all singularities of X are canonical;
3. K_X is $f-$ample;
4. for **all** $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form $\frac{\mathbb{C}[x_0,x_1,z]}{f_6(x_0,x_1,z)}$, with $\deg x_i = 1$, $\deg z = 3$, $\deg f_6 = 6$

Theorem

Let X be a canonical surface with $p_g \geq 7$. Then $K_X^2 = 2p_g - 4$ if and only if X is a regular[a] simple genus 2 fibration.

[a] regular means $h^1(X, \mathcal{O}_X) = 0$

Fibrations in (1,2)-surfaces

The proof of the Noether inequality for threefolds shows that, if the volume is not far from $\frac{4}{3}p_g - \frac{10}{3}$ then there exists a fibration

$$f\colon X \dashrightarrow \mathbb{P}^1$$

such that the general fibre of f is a $(1,2)-$surface.

Definition

A $(1,2)$-surface is a canonical surface with $K^2 = 1$ and $p_g = 2$.
They are the hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$ with at worst canonical singularities.

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Simple fibrations in $(1,2)$-surfaces

However, the previous analysis suggests the following definition

Definition (CP 2023[a])

[a] Coughlan, S.; Pignatelli, R.; *Simple fibrations in (1,2)-surfaces*, Forum of Mathematics, Sigma, Volume **11**, 2023, e43

A simple fibration in $(1,2)-$surfaces is a morphism $f\colon X \to B$ between projective varieties of respective dimension 3 and 1 such that

1. B is smooth;
2. all singularities of X are canonical;
3. K_X is $f-$ample;
4. for **all** $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form[a]

$$\frac{\mathbb{C}[x_0, x_1, y, z]}{f_{10}(x_0, x_1, y, z)}$$

[a] $\deg x_i = 1$, $\deg y = 2$, $\deg z = 5$, $\deg f_{10} = 10$

Regular and/or Gorenstein simple fibrations in $(1,2)$-surfaces

A simple fibration in $(1,2)$-surfaces is **regular** if and only if $h^1(X, \mathcal{O}_X) = 0$.
In fact

$$h^1(X, \mathcal{O}_X) = 0 \Leftrightarrow B \cong \mathbb{P}^1$$

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$$h^1(X, \mathcal{O}_X) = 0 \Leftrightarrow B \cong \mathbb{P}^1$$

A simple fibration in $(1,2)$-surfaces is **Gorenstein** if and only if K_X is Cartier.
Recall that each fibre is a double cover of a quadric cone $\mathbb{P}(1,1,2)$ branched on the vertex v and a curve Γ.
Then

$$K_X \text{ is Cartier} \Leftrightarrow \text{ on each fibre of } f,\ v \notin \Gamma$$

In other words, X is Gorenstein if and only if any fibre F does not contain the point $(0,0,1,0)$ of $\mathbb{P}(1,1,2,5)$.

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Simple fibrations vs Noether inequality

Theorem (CP 2023)

Let $f : X \to B$ *be a simple fibration in* $(1,2)-$*surfaces. Suppose that* X *is* **Gorenstein** *and* **regular***. Then* $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$.

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Theorem (CP 2023)

Let $f\colon X \to B$ be a simple fibration in $(1,2)-$surfaces. Suppose that X is **Gorenstein** *and* **regular**. *Then $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$.*

Conversely

Theorem (CHPZ[a] 2024)

[a] Coughlan, S.; Hu, Y.,; Pignatelli, R.; Zhang, T.; *Threefolds on the Noether line and their moduli spaces*, arXiv:2409.17847

Suppose that X is a canonical threefold with $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$.

1. *If $p_g \geq 23$ then X is a Gorenstein regular simple fibration in $(1,2)$-surfaces.*
2. *If $p_g \geq 11$ then there is a Gorenstein regular simple fibration in $(1,2)$-surfaces $f\colon Y \to \mathbb{P}^1$ with canonical model X and either the map $Y \to X$ is an isomorphism or it contracts a section of f to a point.*

Strategy of the proof of the last theorem (CHPZ 2024)

Suppose that X is a canonical threefold with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$. Assume that $p_g(X) \geq 11$.
[HZ2022][4]: We can choose a minimal model X_1 of X so that X_1 admits a fibration $X_1 \to \mathbb{P}^1$ whose general fiber is a smooth $(1,2)$-surface. We know that X_1 is Gorenstein.

[4] Hu, Y.; Zhang, T.: *Algebraic threefolds of general type with small volume* Mathematische Annalen, to appear, arXiv: 2204.02222.

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Now replace $X_1 \to \mathbb{P}^1$ with its relative canonical model $X_0 \to \mathbb{P}^1$. Let F_p be the fibre over **any** point p. Then F_p is Gorenstein, and by the use of standard exact sequences it has $p_g = 2$, $K^2 = 1$.

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If F_p is stable, by a Theorem of Franciosi, Pardini and Rollenske, F_p is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$.

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Now replace $X_1 \to \mathbb{P}^1$ with its relative canonical model $X_0 \to \mathbb{P}^1$. Let F_p be the fibre over **any** point p. Then F_p is Gorenstein, and by the use of standard exact sequences it has $p_g = 2$, $K^2 = 1$.

If F_p is stable, by a Theorem of Franciosi, Pardini and Rollenske, F_p is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$.

Not all of our fibres are stable. However we could apply their argument: key steps are showing that $h^1(F_p, nK_{F_p}) = 0$ for all $n \in \mathbb{N}$ and that there is an integral curve $C \in \left|K_{F_p}\right|$.

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[HZ2022][4]: We can choose a minimal model X_1 of X so that X_1 admits a fibration $X_1 \to \mathbb{P}^1$ whose general fiber is a smooth $(1,2)$-surface. We know that X_1 is Gorenstein.

Now replace $X_1 \to \mathbb{P}^1$ with its relative canonical model $X_0 \to \mathbb{P}^1$. Let F_p be the fibre over **any** point p. Then F_p is Gorenstein, and by the use of standard exact sequences it has $p_g = 2$, $K^2 = 1$.

If F_p is stable, by a Theorem of Franciosi, Pardini and Rollenske, F_p is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$.

Not all of our fibres are stable. However we could apply their argument: key steps are showing that $h^1(F_p, nK_{F_p}) = 0$ for all $n \in \mathbb{N}$ and that there is an integral curve $C \in \left|K_{F_p}\right|$.

Then X_0 is a simple fibration in $(1,2)$-surfaces and we can apply the results on them in [CP2023].

[4] Hu, Y.; Zhang, T.: *Algebraic threefolds of general type with small volume* Mathematische Annalen, to appear, arXiv: 2204.02222.

$\mathbb{P}(1,1,2,5)$-bundles

Every simple fibration in $(1,2)$-surfaces $X \to B$ is naturally embedded as a divisor in a $4-$fold $\mathbb{F}$ having a morphism on B whose fibres are all weighted projective spaces $\mathbb{P}(1,1,2,5)$. X has relative degree 10.

Some of these $\mathbb{F}$ are toric varieties.
Consider $\mathbb{C}^6$ with coordinates t_0, t_1, x_0, x_1, y, z.
Consider the toric 4-fold $\mathbb{F} = \mathbb{C}^6//(\mathbb{C}^*)^2$ defined by the weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & a_0 & a_1 & b & c \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$.
Then (t_0, t_1) defines a bundle $f : \mathbb{F} \to \mathbb{P}^1$ in weighted projective spaces $\mathbb{P}(1,1,2,5)$.

Divisors in $\mathbb{P}(1,1,2,5)$-bundles

Theorem (CP 2023)

Let $f\colon X \to \mathbb{P}^1$ be a Gorenstein regular simple fibration in $(1,2)-$surfaces. Then $X =: X(d;d_0)$ is a divisor in the toric $4-$fold $\mathbb{C}^6//(\mathbb{C}^*)^2$ with weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & d-d_0 & d_0-2d & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0,t_1)\cap(x_0,x_1,y,z)$, given by a bihomogeneous (of bidegree $(0,10)$) equation of the form

$$z^2 = y^5 + \dots$$

So we know that all canonical 3-folds on the Noether line, at least for $p_g \geq 23$ (or $p_g \geq 11$ if we allow a small modification), are $X(d;d_0)$ for some integer $d, d_0 \in \mathbb{Z}$.

$X(d; d_0)$

Fix d, d_0. The threefolds $X(d; d_0)$ form an unirational family of threefolds. Direct computations show:

1. The general $X =: X(d; d_0)$ has only canonical singularities iff

$$\frac{1}{4}d \leq d_0 \leq \frac{3}{2}d$$

2. If[5] $\min(d, d_0) \geq 1$, then[6]

$$p_g(X) = 3d - 2 \quad h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0 \quad K_X^3 = 4d - 6$$

3. K_X is big and nef[7] iff $\min(d; d_0) \geq 2$ and ample iff $\min(d; d_0) \geq 3$

[5] Else $d = d_0 = 0$ and $X = S \times \mathbb{P}^1$.

[6] Note that this shows that d is a deformation invariant

[7] The cases with $d_0 = 2$ are those where we need the small contraction

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The canonical image of $X(d; d_0)$

The image Σ of the canonical map of $X(d; d_0)$, a subvariety of $\mathbb{P}^{3d-1}$, depends from d_0.

In fact, a basis from $H^0(X, K_X)$ is given by the monomials

$$t_0^{d_0-2}x_0, t_0^{d_0-3}t_1x_0, \ldots, t_1^{d_0-2}x_0, t_0^{3d-d_0-2}x_1, t_0^{3d-d_0-3}t_1x_1, \ldots, t_1^{3d-d_0-2}x_1$$

So Σ is

- a rational normal curve if $d_0 = 1$.
- the cone over a rational normal curve if $d_0 = 2$
- the Hirzebruch surface $\mathbb{F}_e$ with $e = 3d - 2d_0$ if $d_0 \geq 3$

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The singularities of the general $X(d;d_0)$

Recall that the general $X(d;d_0)$ has at worst canonical singularities iff $\frac{1}{4}d \leq d_0 \leq \frac{3}{2}d \Leftrightarrow \frac{5}{2}d \geq e \geq 0$. More precisely the singular locus of the general $X(d;d_0)$ is

1. empty if
$$d \leq d_0 \leq \frac{3}{2}d \qquad \Leftrightarrow d \geq e \geq 0$$
2. $8d_0 - 7d$ terminal singularities[8] if
$$\frac{7}{8}d < d_0 < d \qquad \Leftrightarrow \frac{5}{4}d > e > d$$
3. empty if
$$d_0 = \frac{7}{8}d \qquad \Leftrightarrow e = \frac{5}{4}d$$
4. a section of $f\colon X \to \mathbb{P}^1$ if
$$\frac{1}{4}d \leq d_0 < \frac{7}{8}d \qquad \Leftrightarrow \frac{5}{2}d \leq e < \frac{5}{4}d$$

[8] locally of the form $z^2 + y^5 + x_1 t = 0$

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Components of the moduli space containing smooth 3-folds

Theorem (CP 2023)

Assume $d \geq 3$ (equiv. $p_g \geq 7$).
The threefolds $X(d; d_0)$ with $d \leq d_0 \leq \frac{3}{2}d$ belongs all to the same irreducible component of the moduli space of canonical threefolds that I call the KCH[a] component.
Moreover $X\left(d; \left\lfloor \frac{3}{2}d \right\rfloor\right)$ is an open subset of the KCH component.

The threefolds $X(d; d_0)$ with $d_0 = \frac{7}{8}d$ belong to a different irreducible component.

[a] Threefolds in this component, say the KCH component, had been found first by Y. Chen and Y. Hu, generalizing work of Kobayashi.

The second component shows up only when d is divisible by 8. This is a 3-dimensional version of the "second component" in the moduli space of the Horikawa surfaces, showing up only when K^2 is divisible by 8.

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Mori Dream Spaces

The explicit algebraic description of these threefolds allowed us to prove:

Theorem (CP 2023)

The general element of the KCH component is a Mori Dream Space.

In fact we proved that the general $X(d; d_0)$ is a Mori Dream Space for all $d \leq d_0 \leq \frac{3}{2}d$.

We still do not know if an analogous statement holds for $d_0 < d$.

Components of the moduli space not containing smooth 3-folds

Theorem (CHPZ 2024)

Assume $\frac{d}{4} \leq d_0 \leq \frac{25d-3}{26}$. *Then the threefolds* $X(d; d_0)$ *form an irreducible component of the moduli space of threefolds of general type.*

Components of the moduli space not containing smooth 3-folds

Theorem (CHPZ 2024)

Assume $\frac{d}{4} \leq d_0 \leq \frac{25d-3}{26}$. Then the threefolds $X(d; d_0)$ form an irreducible component of the moduli space of threefolds of general type.

The idea of the proof is that, if a threefold $X(d, d_0)$ is a degeneration of a family of threefolds $X(d, d_0')$, then their canonical image $\mathbb{F}_{3d-2d_0}$ is a degeneration of a family of surfaces $\mathbb{F}_{3d-2d_0'}$, which implies $d_0 \leq d_0'$.
A direct computation shows that, when $d_0 < d$, the modular dimension of the family of the threefolds $X(d; d_0)$ is strictly **decreasing** as a function of d_0, so the general $X(d; d_0)$ cannot be specialization of any $X(d; d_0')$ with $d_0' < d$.
If $d_0 \leq \frac{25d-3}{26}$ this dimension is also bigger than the dimension of the KCH component, completing the proof.

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The moduli space

Even if we are not able to determine if the threefolds $X(d; d_0)$ with $\frac{25d-3}{26} < d_0 < d$ form an irreducible component of the moduli space or are specialisation of threefolds in the KCH component, we obtain

Corollary (CHPZ 2024)

The number of irreducible components of the moduli space of canonical threefolds with $p_g \geq 11$, $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ *is at most* $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ *and at least* $\left\lfloor \frac{p_g+6}{4} \right\rfloor - \left\lfloor \frac{p_g+8}{78} \right\rfloor$.

This in contrast with Horikawa's result in dimension 2. However, it is analogous to the huge number[9] of irreducible components of the moduli space of stable surfaces with $K^2 = 2p_g - 4$.

[9] Rana, J., Rollenske, S. *Standard stable Horikawa surfaces*, Algebraic Geometry **11** (4), 2024, 569–592

The case $\min(d, d_0) = 1$

We also studied the threefolds $X(d, d_0)$ with $\min(d, d_0) = 1$,
$\frac{1}{4}d \le d_0 \le \frac{3}{2}d$.
So $d_0 = 1$, $1 \le d \le 4$.

We have found that $X(1;1)$ has Kodaira dimension zero, whereas $X(2;1)$, $X(3;1)$ and $X(4;1)$ are of general type.

Set then $X^+(d;1)$ for the canonical model of $X(d;1)$, $d = 2, 3, 4$. Then

d	p_g	$K^3_{X^+}$	Singularities of the general $X^+(d;1)$
2	4	$2 + \frac{1}{4}$	$2 \times \frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,3,3)$
3	7	$6 + \frac{1}{14}$	$\frac{1}{2}(1,1,1)$, $\frac{1}{7}(3,4,6)$
4	10	$10 + \frac{1}{30}$	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,2,2)$, $\frac{1}{5}(1,4,4)$

The threefolds $X^+(3;1)$ and $X^+(4;1)$ were already in literature[10].

[10] Chen, M., Jiang, C., Li, B.; *On minimal varieties growing from quasi- smooth weighted hypersurfaces*. J. Differential Geom. **127** (1), 2024, 35–76

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