Computer Aided Algebraic Geometry

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Birational Invariants

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- the Euler characteristic $\chi = \chi(\mathcal{O}_S) = 1 - q + p_g$.
- The plurigenera $P_n(S) = h^0(\mathcal{O}_S(nK_S))$.
- The Kodaira dimension $\kappa(S)$ is the rate of growth of the plurigenera: it is the smallest number κ such that P_n/n^κ is bounded from above.

Surfaces of general type

The Enriques-Kodaira classification provides a relatively good understanding of the surfaces of *special* type, which are those with $\kappa(S) < 2$.

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A surface is *minimal* if K_S is *nef*, that is if the intersection of K_S with any curve is nonnegative.

In every birational class of surfaces of general type there is exactly one *minimal* surface. If S is a surface of general type, S is obtained by the only minimal surface $\overline{S}$ in its birational class by a sequence of $K^2_{\overline{S}} - K^2_S$ blow ups.

M. Noether conjecture

At the end of the XIX^{th} century M. Noether conjectured that every surface with $p_g = 0$ be birational to $\mathbb{P}^2$.

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Surfaces of general type with $p_g = 0$ have been very useful to construct counterxamples in many different fields (Kähler-Einsten metrics, complex structures on real manifolds, Chow groups, Phantoms, ...).

Still, they are very very difficult to construct; ten years ago we had only a couple of dozens of constructions.

Inequalities for surfaces of general type

If S is of general type, then the Riemann-Roch formula computes all $P_n(S)$ from χ and K_S^2.

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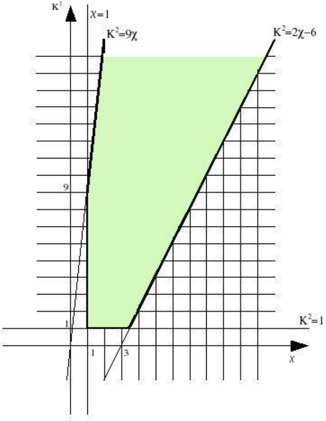
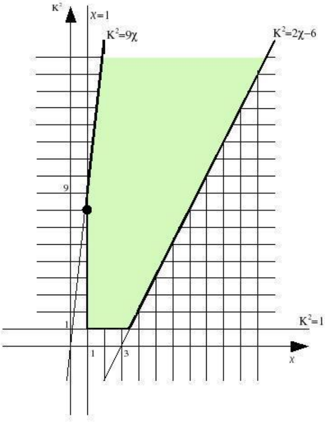


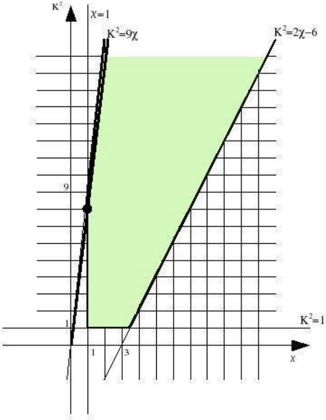
Beauville construction

Beauville suggestion: take $S = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group of order $(g_1 - 1)(g_2 - 1)$ acting freely; S is automatically minimal of general type with $\chi = 1$ and $K^2 = 8$.



Surfaces isogenous to a product

A surface is isogenous to a (higher) product if $S = (C_1 \times C_2)/G$ where C_i are Riemann Surfaces of genus $g_i \geq 2$ and G is a group acting freely; S is automatically minimal of general type with $K^2 = 8\chi$.



Quasi-étale quotients

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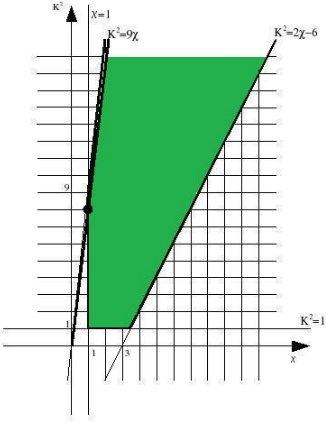
This gives a powerful tool to answer (positively) existence conjectures.



Mixed and unmixed structures

We know that

- either $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$,
- or $C_1 \cong C_2 \cong C$ and $\mathrm{Aut}(C^2) \cong (\mathrm{Aut}(C))^2 \rtimes \mathbb{Z}_{/2\mathbb{Z}}$.

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$\forall G < \mathrm{Aut}(C_1 \times C_2)$ we define

$$G^{(0)} = G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)).$$

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Theorem (Frapporti)

π is not quasi-étale if and only if $G^{(0)} \ncong G$ and $(\#)$ splits.

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- a *suitable* system of generators of $G^{(0)}$ (one for each loop).

For later use: to each *suitable* system of generators we associate its *signature*, which is the unordered list of the orders of some of these generators. By the Hurwitz formula the genus of C is a function of $|G|$, the signature and the genus of $C/G^{(0)}$.

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There is a simple formula for the irregularity:

$$\begin{cases} q(S) = g(C_1/G) + g(C_2/G) & \text{in the unmixed case} \\ q(S) = g(C/G^{(0)}) & \text{in the mixed case} \end{cases}$$

K^2 and χ

We know that

$$K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} - \sum_{x \in \mathrm{Sing} X} k_x$$

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Then we are able to construct every quasi-étale surface and compute its invariants p_g, q and K_S^2.

The algorithms

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5. check the singularities of the surfaces in the output.

We implemented the algorithm in MAGMA in the unmixed case for $p_g = 0$. Recall that if a surface of general type S is minimal, then K_S^2 is positive.

Theorem (Bauer, Catanese, Grunewald, -)

Unmixed quasi-étale surfaces of g. t. with $p_g = 0$ form

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A similar classification for $p_g = q \geq 1$ has been obtained by Carnovale, Mistretta, Penegini, Polizzi, Zucconi.

In the mixed case the algorithm is implemented in the case $q = 0$.

Theorem (Bauer, Catanese, Grunewald, Frapporti)

Mixed quasi-étale surfaces of general type with $p_g = 0$ form

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Similar results have been obtained by the same authors mentioned before for $p_g = q \geq 1$ only in the étale case.

Algorithmic problems

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Extend the programs to the irregular case.

Theoretical problems

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and, related to it

Problem

Can we find all quasi-étale surfaces of general type with $p_g = 0$, or, more generally, with given p_g and q?

We need an explicit bound $K_S^2 \geq k(p_g, q)$. We need for that a better understanding of the rational curves on X.

Literature

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